NUMBER & ALGEBRA

Chapter 17

Complex Numbers

17

SYLLABUS 1.12–1.14 HL only

In the sixteenth century, Italian mathematicians learned to solve cubic equations. Sometimes the working of their formula contained the square root of a negative number, even when the equation had three ordinary real answers. No number they knew could give a negative square. Yet if they wrote the impossible quantity down and kept calculating, it disappeared before the end, and the answers were correct.

This chapter begins with that impossible quantity. Define a number i with $i^2 = -1$. No real number does this, so i is new. For a long time many mathematicians distrusted it, and the name it was given, "imaginary", records that distrust. The name was unfair. Once i is allowed, every non-constant polynomial has all its zeros in the complex numbers, a result now called the Fundamental Theorem of Algebra.

These new numbers are also geometric. Each one is a point in a plane, with the real numbers along one axis. Adding a complex number moves a point across the plane. Multiplying by one turns the point about 0 and stretches its distance from 0. Multiplying by i rotates by a quarter-turn. In ordinary coordinates a rotation takes several steps; here it is one multiplication. This is one reason complex numbers are used widely in physics and engineering.

By the end of this chapter you will be able to write a complex number in three forms, $a + bi$, $r\,\mathrm{cis}\,\theta$ and $re^{i\theta}$, and move between them. You will add, multiply and divide in each form and read each operation on the Argand diagram. You will prove De Moivre's theorem by induction and use it to find powers, nth roots and trigonometric identities. You will also use conjugate pairs to factorise polynomials with real coefficients.

Three forms may seem like more than you need. Each one makes a different calculation easy. Adding is simplest in $a + bi$ form. A power such as $(1 + i)^8$ takes seven multiplications in Cartesian form and one step in polar form. You will practise choosing the form before you start.

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17.1 Cartesian Form

Every real number has a place on the number line, and the line has no gaps: every point on it is a real number. Now look on that line for a number whose square is -1. A positive number squared is positive, a negative number squared is positive, and $0^2 = 0$. So no point of the line solves $x^2 = -1$.

So extend the numbers. Write i for a new number with $i^2 = -1$, and calculate with it as with any other letter, replacing i^2 by -1 wherever it appears. For example, $(2i)^2 = 4i^2 = -4$ and $(3i)(-i) = -3i^2 = 3$.

The number i cannot lie on the real line, so where is it? Multiplying by -1 turns the line through half a turn about 0: it sends 1 to -1. Suppose multiplying by i also turns the line about 0. Multiplying by i twice is multiplying by $i^2 = -1$, so multiplying by i once is half of that turn: **a quarter-turn**, taken anticlockwise by convention. That quarter-turn takes 1 to the point one unit above 0, and that point is i.

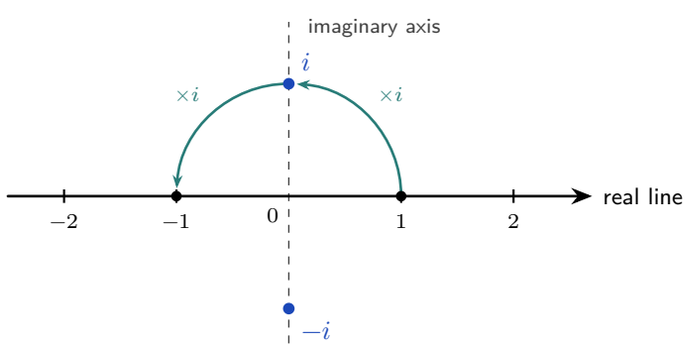


Figure 17.1 No point of the real line has square -1. Multiplying by i is a quarter-turn about 0. Two quarter-turns take 1 to -1, so $i^2 = -1$.

The vertical line through 0 and i is the **imaginary axis**. The numbers $2i$, $3i$, $-i$ and every other real multiple of i lie on it, just as the real numbers lie on the horizontal **real axis**. The two axes together make the **complex plane**, and every point of it is a number: the point a units along and b units up is $a + bi$.

DEFINITION 17.1 **Complex number**

The **imaginary unit** i satisfies $i^2 = -1$. A **complex number** is any $z = a + bi$ with $a, b \in \mathbb{R}$, and this is its **Cartesian form**; $a = \operatorname{Re} z$ is the **real part** and $b = \operatorname{Im} z$ the **imaginary part**.

The formula booklet prints only the form $z = a + bi$.

The imaginary part is the real number b, not bi: for example, $\operatorname{Im}(3 - 2i) = -2$.

Complex numbers are added, subtracted and multiplied exactly like binomials in i. There is one extra rule: replace i^2 by -1 wherever it appears.

That one rule is enough to find every power of i, because the powers repeat in a cycle of four:

$$i^1 = i, \qquad i^2 = -1, \qquad i^3 = i^2 \cdot i = -i, \qquad i^4 = (i^2)^2 = 1,$$

and $i^5 = i^4 \cdot i = i$ starts the cycle again. To simplify any power, divide the exponent by 4 and keep only the remainder: $i^{2026} = i^{4(506)+2} = i^2 = -1$.

EXAMPLE 17.1

Given $z = 2 + i$ and $w = 5 - 3i$, find (a) $z + w$, (b) zw.

SOLUTION

(a) Add part by part: $z + w = (2 + 5) + (1 - 3)i = \boxed{7 - 2i}$

(b) Expand, and then use $i^2 = -1$:

$$zw = (2 + i)(5 - 3i) = 10 - 6i + 5i - 3i^2 = 10 - i + 3 = \boxed{13 - i}$$

17.1.1 The conjugate and division

Dividing by a complex number needs one new number, its conjugate, found by changing the sign of the imaginary part.

DEFINITION 17.2 **Complex conjugate**

The **complex conjugate** of $z = a + bi$ is $z^* = a - bi$.

Division: make the denominator real. The conjugate matters because a number times its own conjugate is always real: $zz^* = (a + bi)(a - bi) = a^2 + b^2$. This is the tool for division, just as a surd is rationalised. To divide, multiply the top and the bottom by the conjugate of the denominator, which makes the denominator real.

EXAMPLE 17.2

Write $\dfrac{3 + 2i}{1 - i}$ in the form $a + bi$.

SOLUTION

Multiply numerator and denominator by the conjugate $1 + i$:

$$\begin{aligned} \frac{3 + 2i}{1 - i} \cdot \frac{1 + i}{1 + i} &= \frac{3 + 3i + 2i + 2i^2}{1^2 + 1^2} \\ &= \frac{3 + 5i - 2}{2} \\ &= \frac{1 + 5i}{2} \\ &= \boxed{\tfrac{1}{2} + \tfrac{5}{2}i} \end{aligned}$$

Equating parts: turn one complex equation into two real equations. Two complex numbers are equal exactly when their real parts agree *and* their imaginary parts agree. So one complex equation gives two real equations, one for the real parts and one for the imaginary parts. Writing them down is called **equating parts**, and the two equations can be solved for two unknowns.

EXAMPLE 17.3

Find the complex number z satisfying $z + 2z^* = 9 + 2i$.

SOLUTION

Write $z = a + bi$, so $z^* = a - bi$:

$$(a + bi) + 2(a - bi) = 3a - bi = 9 + 2i.$$

Equate real parts: $3a = 9 \implies a = 3$. Equate imaginary parts: $-b = 2 \implies b = -2$, so $\boxed{z = 3 - 2i}$

17.1.2 The Argand diagram

When numbers are drawn as points on the complex plane, the plane is called the **Argand diagram**. The number $z = a + bi$ is the point (a, b): a along the real axis and b up the imaginary axis.

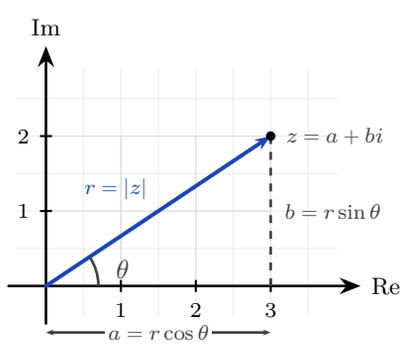


Figure 17.2 A complex number $z = a + bi$ drawn as the point (a, b) of the Argand diagram. Its distance from the origin is the modulus r, and the angle its arrow makes with the positive real axis is the argument θ (both defined in §17.2). The right triangle gives $a = r\cos\theta$ and $b = r\sin\theta$.

The arithmetic can be read from the picture. **Addition is vector addition.** To add w to z, draw w's arrow starting at the tip of z's arrow, tip to tail as in §9.1. Then $z + w$ is the far corner of the parallelogram formed by the two arrows.

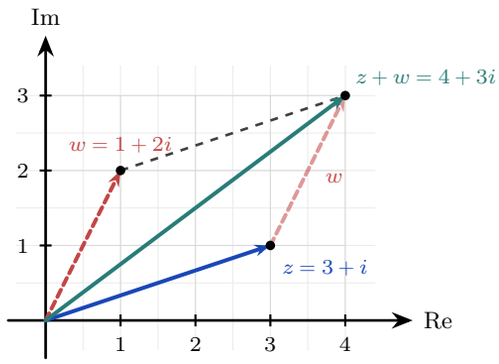


Figure 17.3 $(3 + i) + (1 + 2i) = 4 + 3i$. Place w's arrow on the tip of z's: the real parts add along the real axis and the imaginary parts up the imaginary axis.

Conjugation is reflection: z^* is the reflection of z in the real axis. So z^* is the same distance from 0 as z, and its arrow makes the same angle with the positive real axis, measured on the other side. With the modulus and argument of §17.2, $|z^*| = |z|$ and $\arg z^* = -\arg z$, except for a negative real number, where both arguments are π.

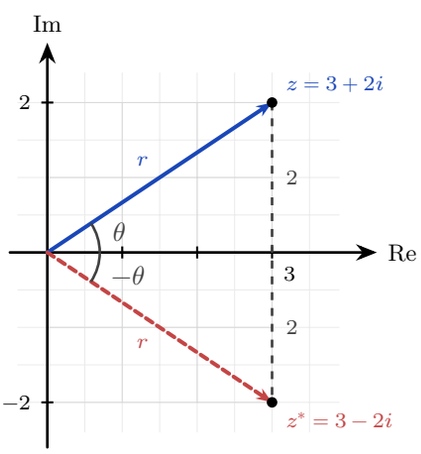


Figure 17.4 The conjugate $z^* = 3 - 2i$ is the reflection of $z = 3 + 2i$ in the real axis: the same distance r from 0, and the angle θ measured the other way.

YOUR TURN 17.1 Cartesian Form

Arithmetic in Cartesian form

1. Evaluate each expression, giving your answer in the form $a + bi$.

(a) $(3 + 2i) + (1 - 4i)$
(b) $(5 - i) - (2 + 3i)$
(c) $(2 + 3i)(1 - i)$
(d) $(2 + i)(2 - i)$
(e) $(1 + 2i)^2(3 - i)$

2. Simplify each power of i.

(a) i^3
(b) i^4

(c) i^{15} (d) i^{100}

3. Let $z = 7 - 4i$.

(a) Write down $\mathrm{Re}\, z$ and $\mathrm{Im}\, z$. (b) Write down z^*.

(c) Evaluate zz^*. (d) Evaluate $z + z^*$.

Division

4. Write each quotient in the form $a + bi$.

(a) $\dfrac{4+3i}{2-i}$ (b) $\dfrac{1}{1+i}$

(c) $\dfrac{2-3i}{i}$ (d) $\dfrac{(1+i)^2}{2-i}$

Equations involving z^*

5. Find the complex number z in each case.

(a) $z + 3z^* = 8 - 4i$

(b) $2z - z^* = 3 + 6i$

(c) $zz^* + 2z = 12 + 4i$ (there are two answers)

The Argand diagram

6. Let $z = 2 + i$ and $w = -1 + 3i$.

(a) Plot z, w and $z + w$ on an Argand diagram, and state the shape formed by the points 0, z, $z + w$ and w.

(b) Plot z^* and w^*, and describe the single transformation that maps each number to its conjugate.

(c) Find $z - w$, and describe how it appears on the diagram in relation to the points z and w.

Concept check

7. A student writes $(3 + 2i)^2 = 9 + 4i^2 = 5$. Identify the error, and find $(3 + 2i)^2$ correctly.

17.2 Modulus and Argument

A non-zero complex number can be located by two measurements:

- its distance from the origin, called the **modulus**;
- its direction from the positive real axis, described by an **argument**.

For $z = a + bi$ the modulus is given by Pythagoras' theorem: $r = |z| = \sqrt{a^2 + b^2}$.

To find the direction, sketch the point first and use the signs of a and b to identify its quadrant. Then calculate the angle.

Anticlockwise angles are positive and clockwise angles negative. Adding a full turn, 2π, gives the same direction, so one direction has many arguments. In these notes $\arg z$ means the **principal argument**, the one with $-\pi < \arg z \leq \pi$.

The number 0 has modulus 0 but no argument, because it has no direction from the origin.

KEY 17.1 **Modulus and argument**

For $z = a + bi$: $\quad |z| = \sqrt{a^2 + b^2}, \qquad \tan(\arg z) = \dfrac{b}{a}$ for $a \neq 0$.

CAUTION $\tan\theta = \frac{b}{a}$ alone cannot tell which quadrant z is in, because tan repeats every π. Find the quadrant from a sketch before you calculate the angle.

On the axes no calculation is needed. For $a > 0$: $\arg a = 0$, $\arg(-a) = \pi$, $\arg(ai) = \frac{\pi}{2}$ and $\arg(-ai) = -\frac{\pi}{2}$.

Off the axes, find the acute angle α with $\tan\alpha = \left|\frac{b}{a}\right|$. This is the **reference angle** of §8.6, the angle between the arrow and the real axis. The quadrant from the sketch then gives the principal argument:

Quadrant	Signs of a and b	$\arg z$
first	$a > 0,\ b > 0$	α
second	$a < 0,\ b > 0$	$\pi - \alpha$
third	$a < 0,\ b < 0$	$-\pi + \alpha$
fourth	$a > 0,\ b < 0$	$-\alpha$

The numbers $1 + i$ and $-1 - i$ show why the sketch is needed. Both give $\frac{b}{a} = 1$, so a calculator returns $\arctan 1 = \frac{\pi}{4}$ for both, but this is correct only for $1 + i$.

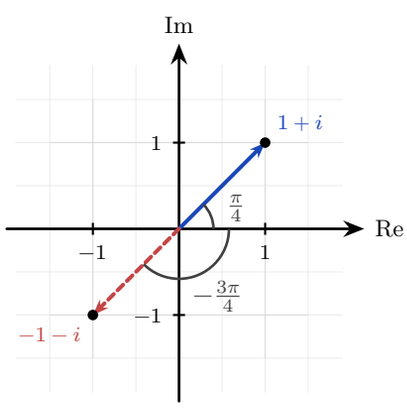


Figure 17.5 The sketch shows the difference: $1 + i$ has argument $\frac{\pi}{4}$, but $-1 - i$ is in the third quadrant, with argument $-\frac{3\pi}{4}$.

The modulus also measures the distance between two points, not only the distance from the origin. Since $z - w$ is the vector running from w to z §9.2, its length $|z - w|$ is the distance between the points representing z and w.

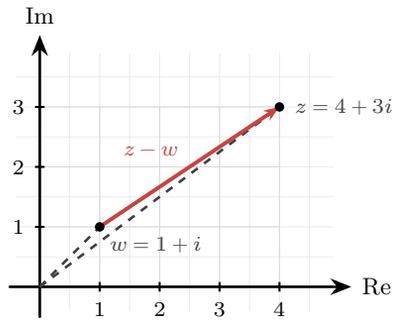


Figure 17.6 The arrow from w to z is $z - w$, so the distance between the two points is $|z - w|$. Here $|(4 + 3i) - (1 + i)| = |3 + 2i| = \sqrt{13}$.

Knowing r and θ is enough to rebuild the number, since $a = r\cos\theta$ and $b = r\sin\theta$. So $z = r(\cos\theta + i\sin\theta)$, which is shortened to $r\,\mathrm{cis}\,\theta$: the letters are the initials of $\cos$, i and $\sin$. Both are called the **modulus–argument form**, or **polar form**. A third way of writing the same number, $re^{i\theta}$, is the **Euler form**, explained in §17.2.1.

EXAMPLE 17.4

Find the modulus and the principal argument of $z = -1 - \sqrt{3}\,i$.

SOLUTION

The number z lies in the third quadrant. Its modulus is $\sqrt{(-1)^2 + (-\sqrt{3})^2} = \boxed{2}$, and its reference angle is $\frac{\pi}{3}$, since $\tan\frac{\pi}{3} = \sqrt{3}$. Its principal argument is therefore

$$\theta = -\pi + \frac{\pi}{3} = \boxed{-\frac{2\pi}{3}}$$

The same number can be written in three ways:

$$z = 2\left(\cos\left(-\tfrac{2\pi}{3}\right) + i\sin\left(-\tfrac{2\pi}{3}\right)\right) = 2\,\mathrm{cis}\left(-\tfrac{2\pi}{3}\right) = 2e^{-2\pi i/3}.$$

FORMULA 17.2 **Polar and Euler form** FORMULA BOOKLET · 1.13

$$z = r(\cos\theta + i\sin\theta) = r\,\mathrm{cis}\,\theta = re^{i\theta},$$

where $r = |z|$ and $\theta = \arg z$.

17.2.1 Euler form

The conversion above used $e^{i\theta} = \cos\theta + i\sin\theta$. This may look surprising, but it follows from the Maclaurin series for e^x. For complex inputs, e^z is defined by the same power series that e^x has for real x (§16.4). So take the Maclaurin series for e^x and substitute $x = i\theta$, using

the cycle of powers of i from §17.1:

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \cdots = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \cdots\right).$$

The even terms (blue), where i has become ± 1, form the cosine series. The odd terms (red), which still carry an i, form the sine series. So once complex inputs are allowed, the exponential function contains cosine and sine: at $i\theta$ its real part is $\cos\theta$ and its imaginary part is $\sin\theta$. Setting $\theta = \pi$ gives $e^{i\pi} + 1 = 0$, which links e, i, π, 1 and 0 in a single line.

Converting from polar or Euler form back to Cartesian form is easier, because no quadrant decision is needed. Substitute the angle and evaluate $\cos\theta$ and $\sin\theta$.

EXAMPLE 17.5

Write (a) $4\,\mathrm{cis}\,\dfrac{2\pi}{3}$ and (b) $2e^{-i\pi/2}$ in Cartesian form.

SOLUTION

(a) Expand the cis and evaluate the two exact values:

$$4\left(\cos\tfrac{2\pi}{3} + i\sin\tfrac{2\pi}{3}\right) = 4\left(-\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) = \boxed{-2 + 2\sqrt{3}\,i}$$

(b) Euler form gives the modulus and argument directly, so read $r = 2$ and $\theta = -\frac{\pi}{2}$:

$$2\left(\cos(-\tfrac{\pi}{2}) + i\sin(-\tfrac{\pi}{2})\right) = 2(0 - i) = \boxed{-2i}$$

YOUR TURN 17.2 Modulus and Argument

Modulus and argument

8. Find the modulus of each number.

(a) $3 + 4i$
(b) $5 - 12i$
(c) $1 + i$
(d) $-2i$

9. Find the principal argument of each number, with $-\pi < \theta \leq \pi$.

(a) $1 + i$
(b) i
(c) -3
(d) $-1 - i$
(e) $-\sqrt{3} + i$
(f) $-4 - 4\sqrt{3}\,i$

Converting between forms

10. Write each number in modulus–argument form and in Euler form.

(a) $2i$
(b) -3

(c) $1 - i$ (d) $-1 + \sqrt{3}\,i$

11. Write each number in Cartesian form.

(a) $4 \operatorname{cis} \frac{\pi}{3}$ (b) $2e^{-i\pi/4}$

(c) $3 \operatorname{cis} \pi$ (d) $e^{i\pi/2}$

Distance on the Argand diagram

12. Let $z = 3 + 4i$ and $w = -1 + 2i$.

(a) Find $|z - w|$, giving an exact answer.

(b) State what $|z - w|$ measures on the Argand diagram.

(c) Verify that $zz^* = |z|^2$.

Concept check

13. A student writes $\arg(-1 - \sqrt{3}\,i) = \arctan \dfrac{-\sqrt{3}}{-1} = \arctan \sqrt{3} = \dfrac{\pi}{3}$. Identify the error, and find the principal argument.

14. Explain why $\arg 0$ is undefined, although $|0| = 0$.

17.3 Products, Quotients and De Moivre's Theorem

Polar form needs extra notation, but it makes multiplication easy. When two complex numbers are multiplied, their moduli multiply and their arguments *add*, which is the law of exponents applied to $re^{i\theta}$. The next paragraph checks this without using exponentials.

KEY 17.3 **Multiplying and dividing in polar form**

To multiply, multiply the moduli and add the arguments: $|zw| = |z|\,|w|$ and $\arg(zw) = \arg z + \arg w$. To divide, divide the moduli and subtract the arguments: $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ and $\arg \dfrac{z}{w} = \arg z - \arg w$. Then add or subtract 2π if necessary to bring the argument back into $-\pi < \theta \leq \pi$.

These hold for non-zero z and w; division needs $w \neq 0$.

The rule for a product can be checked directly, without the exponential. Expand the product of $\operatorname{cis}\alpha$ and $\operatorname{cis}\beta$, using $i^2 = -1$:

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta).$$

By the compound-angle identities of Ch. 8, the two brackets are $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$, so $\operatorname{cis}\alpha\,\operatorname{cis}\beta = \operatorname{cis}(\alpha+\beta)$. The two moduli are real factors, so they multiply separately. The

rule for a quotient follows by writing $z = \dfrac{z}{w} \times w$ and applying the rule for a product to the right side.

Read geometrically, multiplying z by w turns the arrow of z about 0 through the angle $\arg w$ and multiplies its length by $|w|$. So multiplying by $i = 1 \cdot e^{i\pi/2}$, whose modulus is 1, is a quarter-turn anticlockwise with no stretching at all.

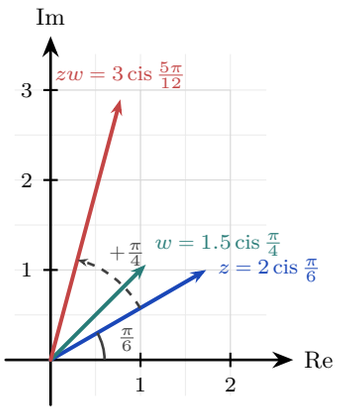


Figure 17.7 Multiplying $z = 2\,\mathrm{cis}\,\frac{\pi}{6}$ by $w = 1.5\,\mathrm{cis}\,\frac{\pi}{4}$. The arguments add, $\frac{\pi}{6} + \frac{\pi}{4} = \frac{5\pi}{12}$, and the moduli multiply, $2 \times 1.5 = 3$.

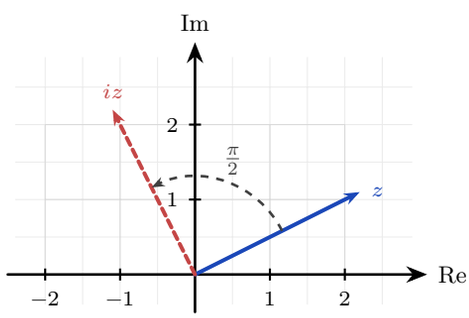


Figure 17.8 Multiplication by i: a quarter-turn anticlockwise.

The rule for a quotient, with $z = 1 = 1\,\mathrm{cis}\,0$, gives the reciprocal: for $r > 0$,

$$\frac{1}{r\,\mathrm{cis}\,\theta} = \frac{1}{r}\,\mathrm{cis}(-\theta).$$

Taking the reciprocal replaces the modulus by its reciprocal and changes the sign of the argument, so for $|z| = 1$ the reciprocal is the conjugate z^*.

EXAMPLE 17.6

Given $z = 6\,\mathrm{cis}\,\dfrac{2\pi}{3}$ and $w = 3\,\mathrm{cis}\,\dfrac{\pi}{6}$, find (a) zw, (b) $\dfrac{z}{w}$.

SOLUTION

(a) For the product, multiply the moduli and add the arguments:

$$zw = (6)(3)\,\mathrm{cis}\left(\tfrac{2\pi}{3} + \tfrac{\pi}{6}\right) = \boxed{18\,\mathrm{cis}\,\tfrac{5\pi}{6}}$$

(b) For the quotient, divide the moduli and subtract the arguments:

$$\frac{z}{w} = \frac{6}{3}\operatorname{cis}\left(\tfrac{2\pi}{3} - \tfrac{\pi}{6}\right) = 2\operatorname{cis}\tfrac{\pi}{2} = \boxed{2i}$$

Compare this with doing the same multiplication in Cartesian form, which would need four products and a collection of terms. Once the numbers are in polar form the work is one multiplication and one addition.

If multiplying adds arguments, then raising to a power *multiplies* the argument. This is **De Moivre's theorem**, and the roots, the unit-circle results and the trigonometric identities in the rest of the chapter all follow from it.

FORMULA 17.4 **De Moivre's theorem** FORMULA BOOKLET · 1.14

$$\left(r(\cos\theta + i\sin\theta)\right)^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n \operatorname{cis} n\theta.$$

Here $n \in \mathbb{Z}$, with $r \neq 0$ when $n \leq 0$.

EXAMPLE 17.7

Find $(1+i)^8$.

SOLUTION

In polar form $1 + i = \sqrt{2}\operatorname{cis}\frac{\pi}{4}$. By De Moivre,

$$(1+i)^8 = (\sqrt{2})^8 \operatorname{cis}(8 \cdot \tfrac{\pi}{4}) = 16\operatorname{cis}(2\pi) = \boxed{16}$$

Seven repeated multiplications reduce to one power of the modulus and one multiple of the angle.

Plotting the powers shows the theorem as a picture. Each multiplication by $1 + i$ turns the arrow through $\frac{\pi}{4}$ anticlockwise and stretches it by $\sqrt{2}$, so the powers lie on a spiral that winds outward from the origin:

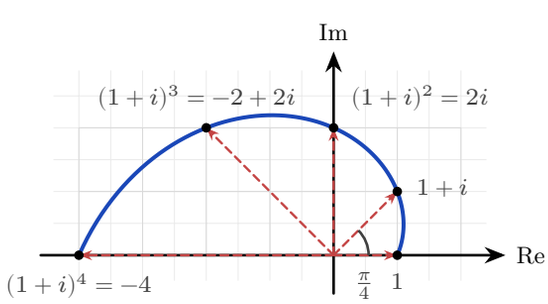


Figure 17.9 The powers $(1+i)^0$ to $(1+i)^4$.

Four more steps continue the spiral through the lower half-plane and back to the positive real axis at $(1+i)^8 = 16$: a full turn of $8 \times \frac{\pi}{4} = 2\pi$, with the modulus grown to $(\sqrt{2})^8 = 16$. The worked example above does the same calculation in algebra.

EXAMPLE 17.8

Prove De Moivre's theorem, $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis} n\theta$, for all $n \in \mathbb{Z}^+$.

SOLUTION

Let P_n be the statement $(r\operatorname{cis}\theta)^n = r^n \operatorname{cis} n\theta$. The four steps are those of §3.4.

Step 1 (P_1): the left side is $r\operatorname{cis}\theta$ and the right side is $r^1 \operatorname{cis}\theta$. They agree, so P_1 is true.

Step 2: assume P_k is true for some $k \geq 1$, that is, $(r\operatorname{cis}\theta)^k = r^k \operatorname{cis} k\theta$.

Step 3: we now show that P_{k+1} is true, that is, P_{k+1}: $(r\operatorname{cis}\theta)^{k+1} = r^{k+1}\operatorname{cis}(k+1)\theta$. Start from the left-hand side and split off one factor. The assumption replaces $(r\operatorname{cis}\theta)^k$ in the second line; then expand with $i^2 = -1$:

$$\begin{aligned}\text{LHS} = (r\operatorname{cis}\theta)^{k+1} &= (r\operatorname{cis}\theta)^k \times r\operatorname{cis}\theta \\ &= r^k \operatorname{cis} k\theta \times r\operatorname{cis}\theta \\ &= r^{k+1}(\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta) \\ &= r^{k+1}\big((\cos k\theta\cos\theta - \sin k\theta\sin\theta) + i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)\big) \\ &= r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta) = \text{RHS}.\end{aligned}$$

The last line uses the compound-angle identities of Ch. 8 for $\cos(k\theta+\theta)$ and $\sin(k\theta+\theta)$. So P_{k+1} is true. So if P_k is true, then P_{k+1} is true.

Step 4: since P_1 is true, and P_k true implies P_{k+1} true, by the principle of mathematical induction P_n is true for all integers $n \geq 1$. ■

End-of-Chapter Problem 29 asks you to write this proof yourself, with the book closed. The example below extends the theorem to the negative integers.

EXAMPLE 17.9

The theorem is claimed for every integer exponent. Taking the case $n \in \mathbb{Z}^+$ as proved, show that it also holds for every negative integer n, for $r > 0$.

SOLUTION

Let n be a negative integer, and write $n = -m$ with $m \in \mathbb{Z}^+$. Applying the positive case to m, and then the reciprocal result above,

$$(r\operatorname{cis}\theta)^n = \frac{1}{(r\operatorname{cis}\theta)^m} = \frac{1}{r^m\operatorname{cis}(m\theta)} = r^{-m}\operatorname{cis}(-m\theta) = r^n\operatorname{cis}(n\theta). \quad \blacksquare$$

The exponent $n = 0$ needs no work either, since $z^0 = 1 = r^0\operatorname{cis}0$. So the theorem holds for every integer n.

17.3.1 Numbers on the unit circle

The case $|z| = 1$ of De Moivre's theorem is used often enough to state separately. Then $z = \operatorname{cis}\theta$, and by the reciprocal result above, $\dfrac{1}{z} = \operatorname{cis}(-\theta) = z^*$. Add the two and the imaginary parts cancel while the real parts double. Subtract them and the real parts cancel while the imaginary parts double. By De Moivre's theorem, $z^n = \operatorname{cis} n\theta$ and $\dfrac{1}{z^n} = \operatorname{cis}(-n\theta)$, so the same addition and subtraction work for every integer n.

KEY 17.5 **Powers of z and $1/z$**

For $z = \operatorname{cis}\theta$ and any $n \in \mathbb{Z}$:

$$z^n + \frac{1}{z^n} = 2\cos n\theta, \qquad z^n - \frac{1}{z^n} = 2i\sin n\theta.$$

These formulas turn powers into multiple angles. §17.5.1 uses them to rewrite a power such as $\cos^6\theta$.

EXAMPLE 17.10

Let $z = \operatorname{cis}\theta$. Expand $\left(z + \dfrac{1}{z}\right)\left(z - \dfrac{1}{z}\right)$ in two ways, and hence express $\sin\theta\cos\theta$ in terms of $\sin 2\theta$.

SOLUTION

Taking $n = 1$ in the two formulas, $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$, so the product is

$$\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right) = (2\cos\theta)(2i\sin\theta) = 4i\sin\theta\cos\theta.$$

Expanding algebraically instead, the two cross terms cancel:

$$\left(z + \frac{1}{z}\right)\left(z - \frac{1}{z}\right) = z^2 - 1 + 1 - \frac{1}{z^2} = z^2 - \frac{1}{z^2} = 2i\sin 2\theta,$$

using the second formula again with $n = 2$. The two expressions are equal, so

$$4i\sin\theta\cos\theta = 2i\sin 2\theta \implies \boxed{\sin\theta\cos\theta = \frac{\sin 2\theta}{2}}$$

This is the double-angle identity for the sine, from Ch. 8, now obtained without using the compound-angle formula directly. Evaluating one product in two ways is the technique of §17.5.

The three forms suit different tasks, and choosing the right one first saves work.

Form	Written	Conversion	Easiest for
Cartesian	$a + bi$	$a = r\cos\theta,\ \ b = r\sin\theta$	adding and subtracting; simple products and quotients
Polar	$r\,\mathrm{cis}\,\theta$	$r = \sqrt{a^2 + b^2}$; θ from $\tan\theta = \frac{b}{a}$ and a sketch	products, quotients, powers and roots
Euler	$re^{i\theta}$	the same r and θ as polar form	the same tasks as polar form, using the laws of exponents

When finding roots (§17.4), list all n of them and check one by substituting it back.

YOUR TURN 17.3 Products, Quotients and De Moivre's Theorem

Products and quotients

15. Evaluate each expression, giving your answer in modulus–argument form.

(a) $\left(2\,\mathrm{cis}\,\frac{\pi}{6}\right)\left(3\,\mathrm{cis}\,\frac{\pi}{3}\right)$ (b) $\dfrac{6\,\mathrm{cis}\,\frac{2\pi}{3}}{2\,\mathrm{cis}\,\frac{\pi}{3}}$

16. Given $|z| = 3$, $|w| = 4$, $\arg z = \frac{\pi}{4}$ and $\arg w = \frac{\pi}{6}$, write down

(a) $|zw|$ (b) $\arg(zw)$

(c) $\left|\frac{z}{w}\right|$ (d) $\arg\left(\frac{z}{w}\right)$

17. Give each answer in modulus–argument form, with the principal argument $-\pi < \theta \leq \pi$.

(a) Find zw, where $z = 2\,\mathrm{cis}\,\frac{5\pi}{6}$ and $w = 3\,\mathrm{cis}\,\frac{\pi}{3}$.

(b) Find $\dfrac{z}{w}$, where $z = 4\,\mathrm{cis}\left(-\frac{2\pi}{3}\right)$ and $w = 2\,\mathrm{cis}\,\frac{\pi}{2}$.

De Moivre's theorem

18. Use De Moivre's theorem for each part.

(a) Write $(\cos\theta + i\sin\theta)^4$ in the form $\cos n\theta + i\sin n\theta$.

(b) Evaluate $\left(\mathrm{cis}\,\frac{\pi}{5}\right)^5$, in modulus–argument form.

(c) Evaluate $\left(2\,\mathrm{cis}\,\frac{\pi}{6}\right)^3$, in modulus–argument form.

(d) Evaluate $(1 + i)^6$.

(e) Evaluate $\left(1 - \sqrt{3}\,i\right)^5$.

Negative powers and reciprocals

19.

(a) Find $\left(2\operatorname{cis}\frac{\pi}{6}\right)^{-2}$ in the form $a+bi$.

(b) Find $(1-i)^{-4}$.

(c) Write $\dfrac{1}{\sqrt{3}+i}$ in modulus–argument form.

Multiplying as a rotation

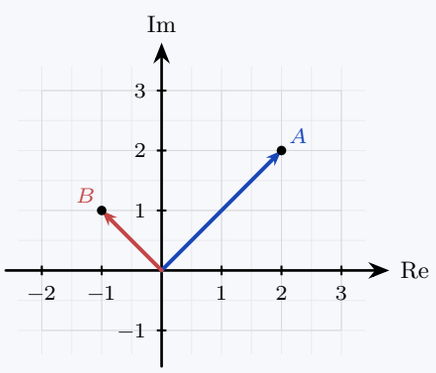
20.

(a) Let $z=3+2i$. Find iz, and describe the transformation of the Argand diagram that maps z to iz.

(b) The point $4+2i$ is rotated anticlockwise about the origin through $\frac{\pi}{3}$. Find its image in the form $a+bi$.

(c) $OABC$ is a square on the Argand diagram, with vertices labelled anticlockwise, O at the origin and A at $2+i$. Find the numbers at C and at B.

(d) The figure shows the points A and B on an Argand diagram. A represents z and B represents w.



Write down z and w in Cartesian form, and find $|z|$ and $\arg z$.

(e) For the points in part (d), find $\dfrac{w}{z}$, and hence describe fully the transformation of the Argand diagram that maps A to B.

(f) Let $z=2+i$. On one Argand diagram, plot and label the points representing z, z^*, iz and z^2.

Numbers on the unit circle

21. Let $z=\operatorname{cis}\theta$. Write each expression in terms of a single cosine or sine.

(a) $z+\dfrac{1}{z}$ (b) $z^3+\dfrac{1}{z^3}$

(c) $z^2-\dfrac{1}{z^2}$

Concept check

22. A student uses De Moivre's theorem and writes
$(1+i)^4=\operatorname{cis}\left(4\times\frac{\pi}{4}\right)=\operatorname{cis}\pi=-1$. Identify the error, and find $(1+i)^4$.

17.4 Roots of Complex Numbers

Solving $z^3 = 125$ over the real numbers gives one answer. Over the complex numbers it gives three, and they are not scattered: they sit at equal spacing on a circle.

EXAMPLE 17.11

Solve $z^3 = 125$.

SOLUTION

Write $z = \rho \operatorname{cis} \phi$. Cubing, by De Moivre's theorem, gives $z^3 = \rho^3 \operatorname{cis} 3\phi$. The modulus must satisfy $\rho^3 = 125$, so $\rho = 5$. The number 125 lies on the positive real axis, so 3ϕ can be 0, 2π, 4π, or any other whole number of turns. Dividing these angles by 3 gives

$$\phi = 0, \qquad \frac{2\pi}{3}, \qquad \frac{4\pi}{3}.$$

The next choice, $3\phi = 6\pi$, gives $\phi = 2\pi$, which repeats the first root. So the three roots are

$$z = 5, \qquad 5 \operatorname{cis} \tfrac{2\pi}{3} = -\frac{5}{2} + \frac{5\sqrt{3}}{2} i, \qquad 5 \operatorname{cis} \tfrac{4\pi}{3} = -\frac{5}{2} - \frac{5\sqrt{3}}{2} i$$

They have the same modulus and are separated by equal angles of $\frac{2\pi}{3}$: an equilateral triangle on the circle of radius 5, one real root and a conjugate pair.

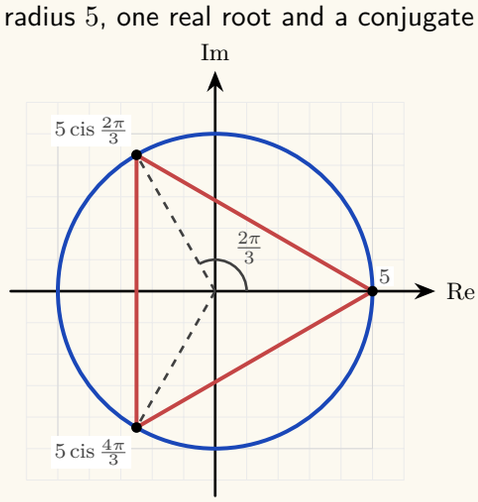


Figure 17.10 The three cube roots of 125, equally spaced on the circle of radius 5.

In general, the roots are found by using De Moivre's theorem in reverse. A non-zero complex number has exactly n distinct nth roots. They all share one modulus, and their arguments are equally spaced. The only nth root of 0 is 0.

KEY 17.6 **nth roots of a complex number**

For $n \in \mathbb{Z}^+$ and $r > 0$, the nth roots of $z = r\,\mathrm{cis}\,\theta$ are $\sqrt[n]{r}\,\mathrm{cis}\left(\dfrac{\theta + 2\pi k}{n}\right)$, for $k = 0, 1, \ldots, n-1$. These arguments may exceed π; subtract 2π when a principal argument is required.

EXAMPLE 17.12

Show that the nth roots of $z = r\,\mathrm{cis}\,\theta$, with $r > 0$, are $\sqrt[n]{r}\,\mathrm{cis}\,\dfrac{\theta + 2\pi k}{n}$ for $k = 0, 1, \ldots, n-1$, and that there are exactly n of them.

SOLUTION

Let $u = \rho\,\mathrm{cis}\,\phi$ be an nth root of z. De Moivre's theorem gives $u^n = \rho^n\,\mathrm{cis}\,n\phi$, and this must equal $r\,\mathrm{cis}\,\theta$. Two non-zero complex numbers are equal exactly when their moduli are equal and their arguments differ by a multiple of 2π, so

$$\rho^n = r \qquad \text{and} \qquad n\phi = \theta + 2\pi k \quad \text{for some integer } k.$$

Hence $\rho = \sqrt[n]{r}$ and $\phi = \frac{\theta + 2\pi k}{n}$. Increasing k by n adds 2π to ϕ and returns the same number. So the values $k = 0, 1, \ldots, n-1$ give every root, and these n roots are distinct because any two of their arguments differ by more than 0 and less than 2π.

The roots share the modulus $\sqrt[n]{r}$, and their arguments differ by equal steps of $\frac{2\pi}{n}$. So for $n \geq 3$ they sit at the vertices of a **regular n-gon** centred at the origin. For $n = 2$ they sit at the two ends of a diameter. So one root is enough to locate all the others. Find any single root, then turn it through $\frac{2\pi}{n}$ at a time to find the rest.

KEY 17.7 **Solving $z^n = w$**

To solve $z^n = w$:

1. Write w in polar form, $w = r\,\mathrm{cis}\,\theta$.
2. The modulus of every root is $\sqrt[n]{r}$.
3. Take $\frac{\theta}{n}$ as the first argument, then add $\frac{2\pi}{n}$ repeatedly until you have n roots.
4. Convert back to Cartesian form only if the question asks for it.

In the example $z^3 = 125$ one step could not be seen: 125 has argument 0, and $0 \div 3 = 0$. When the number has a non-zero argument, dividing the argument by n changes it.

EXAMPLE 17.13

Find the three cube roots of $-27i$, giving each in polar form and in Cartesian form.

SOLUTION

First write $-27i$ in polar form. It lies on the negative imaginary axis, so its modulus is 27 and its argument is $-\frac{\pi}{2}$:

$$-27i = 27\,\mathrm{cis}\left(-\tfrac{\pi}{2}\right).$$

Each root has modulus $\sqrt[3]{27} = 3$. The first argument is the given argument divided by 3:

$$\frac{-\pi/2}{3} = -\frac{\pi}{6}.$$

The others follow by adding $\frac{2\pi}{3}$ each time:

$$-\frac{\pi}{6}, \qquad -\frac{\pi}{6} + \frac{2\pi}{3} = \frac{\pi}{2}, \qquad \frac{\pi}{2} + \frac{2\pi}{3} = \frac{7\pi}{6}, \text{ which is the same direction as } -\frac{5\pi}{6}.$$

The third has been reduced by 2π to bring it back into $-\pi < \theta \le \pi$. In polar form the roots are

$$3\,\mathrm{cis}\left(-\tfrac{\pi}{6}\right), \qquad 3\,\mathrm{cis}\,\tfrac{\pi}{2}, \qquad 3\,\mathrm{cis}\left(-\tfrac{5\pi}{6}\right)$$

and converting each,

$$\frac{3\sqrt{3}}{2} - \frac{3}{2}i, \qquad 3i, \qquad -\frac{3\sqrt{3}}{2} - \frac{3}{2}i$$

Check the easiest one: $(3i)^3 = 27i^3 = -27i$. ✓

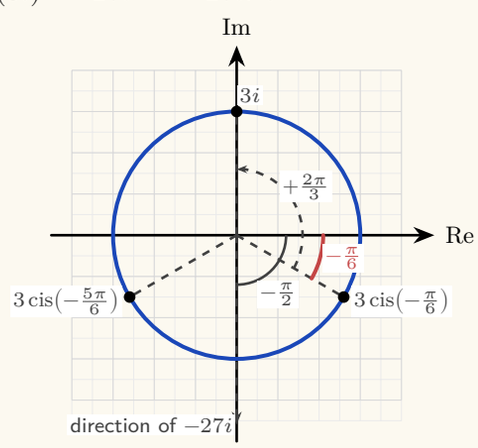


Figure 17.11 The cube roots of $-27i$ on the circle of radius 3. The first root is at $-\frac{\pi}{6}$, one third of the argument $-\frac{\pi}{2}$ of $-27i$; the others follow at steps of $\frac{2\pi}{3}$.

The syllabus uses the theorem for integer n, extends it to *rational* exponents, and asks you to know that it holds for every real n. Read a rational power with care. A power such as $z^{1/n}$ has n values, so $\sqrt[n]{r}\,\mathrm{cis}\,\frac{\theta}{n}$ is one of them, not the whole answer. To check that it is one of them, raise it to the power n: by the integer case this gives $r\,\mathrm{cis}\,\theta$, so it is an nth root of z.

A rational power $z^{p/q}$, with $\frac{p}{q}$ in lowest terms and $q > 0$, has q values: it is **multi-valued** (has more than one value), and a question should state whether every value or only one is required. To find them all, find the q roots of order q first. Then raise each one to the power p: raise the modulus to the power p and multiply the argument by p. Bring each argument back into $-\pi < \theta \leq \pi$ at the end.

EXAMPLE 17.14

Find all values of $(8i)^{2/3}$.

SOLUTION

This question asks for every value, so take the cube roots first and then square them. Since $8i = 8\,\mathrm{cis}\,\frac{\pi}{2}$, its cube roots are $2\,\mathrm{cis}\big(\frac{\pi}{6} + \frac{2k\pi}{3}\big)$ for $k = 0, 1, 2$. Squaring each one squares the modulus and doubles the argument, and subtracting 2π where needed brings each argument into $-\pi < \theta \leq \pi$:

$$4\,\mathrm{cis}\,\tfrac{\pi}{3} = 2 + 2\sqrt{3}\,i, \qquad 4\,\mathrm{cis}\big(-\tfrac{\pi}{3}\big) = 2 - 2\sqrt{3}\,i, \qquad 4\,\mathrm{cis}\,\pi = -4$$

There are three values, one for each cube root. As a check, $(-4)^3 = -64 = (8i)^2$.

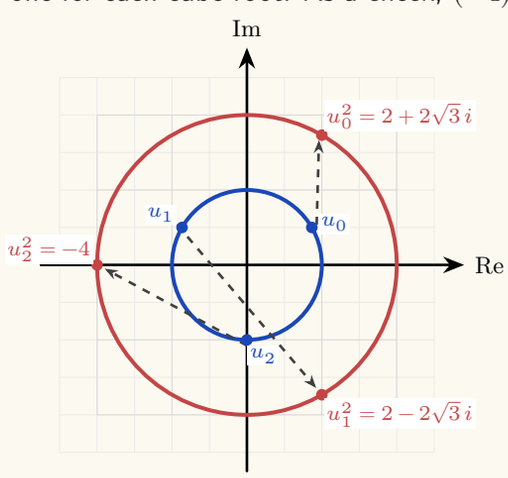


Figure 17.12 The cube roots u_0, u_1, u_2 of $8i$ lie on the circle of radius 2. Squaring each one squares the modulus and doubles the argument, so the three values of $(8i)^{2/3}$ lie on the circle of radius 4.

17.4.1 Roots of Unity

One case is used often enough to study separately: $w = 1$. Solving $z^n = 1$ gives the **roots of unity**. These are n points on the *unit* circle. One of them is 1. For $n \geq 3$ they sit at the vertices of a regular n-gon; for $n = 2$ they are 1 and -1, the two ends of a diameter.

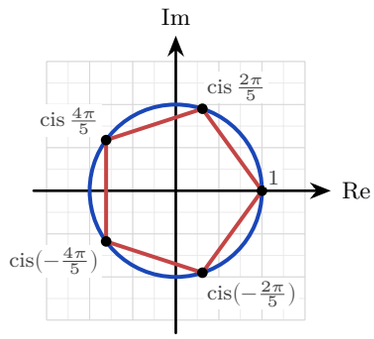


Figure 17.13 The five fifth roots of unity, at the vertices of a regular pentagon inscribed in the unit circle with one vertex at 1.

The first root anticlockwise from 1, $\omega = \operatorname{cis}\dfrac{2\pi}{n}$, gives all the others as its powers. Then $\omega^k = \operatorname{cis}\dfrac{2\pi k}{n}$, which is the kth root round the circle. Every power of ω is again a root, because $\left(\omega^k\right)^n = \left(\omega^n\right)^k = 1^k = 1$. The n roots are therefore

$$1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1},$$

one number and its powers, stepping round the polygon and returning to the start.

The case $n = 3$ is worth memorising. Converting each root to Cartesian form, the cube roots of unity are 1, $\omega = \operatorname{cis}\frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $\omega^2 = \operatorname{cis}\frac{4\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$: a real root and a conjugate pair on the unit circle.

For $n \geq 2$ they also **sum to zero**. There are two ways to see this. Geometrically, the n arrows point in n equally spaced directions. Turning all of them through $\frac{2\pi}{n}$ about 0 gives the same set of arrows, so the turn **does not change their sum**. The only point that a turn about 0 leaves in place is 0 itself, so the sum is 0. Algebraically, the geometric series of Ch. 2 gives the result directly, since $\omega \neq 1$:

$$\begin{aligned} 1 + \omega + \omega^2 + \cdots + \omega^{n-1} &= \frac{\omega^n - 1}{\omega - 1} \\ &= \frac{1 - 1}{\omega - 1} \\ &= 0. \end{aligned}$$

Placing the arrows tip to tail shows the same fact as a picture.

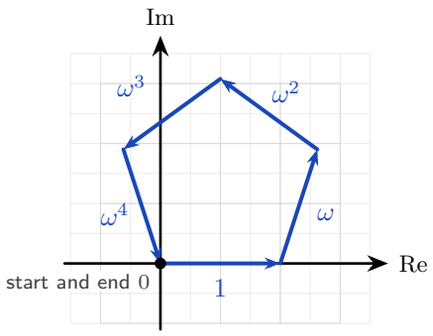


Figure 17.14 The five fifth roots of unity placed tip to tail close into a pentagon and end where they began, so their sum is 0.

The same conclusion follows from the sum of roots in §6.5. For $z^n - 1$ with $n \geq 2$ the coefficient a_{n-1} is zero, so the sum of the roots, $-\frac{a_{n-1}}{a_n}$, is zero. For $n = 3$ this gives the identity $1 + \omega + \omega^2 = 0$, which the next example uses twice.

The roots of unity are the n zeros of the polynomial $z^n - 1$. The factor theorem holds for complex zeros (§6.3), so each $z - \omega^k$ is a factor. The n factors are all different, and their product has degree n and leading coefficient 1, as $z^n - 1$ does. So $z^n - 1 = (z-1)(z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1})$. A product of this kind is written with the sign $\prod$, which is used for a product as $\sum$ is used for a sum: $\prod_{k=0}^{n-1}(z-\omega^k) = z^n - 1$.

EXAMPLE 17.15

Let $\omega = \operatorname{cis}\frac{2\pi}{3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$. Simplify (a) $\omega^{10} + \omega^{20}$, and (b) $(1+\omega)(1+\omega^2)$.

SOLUTION

(a) A power of ω is reduced by dividing the exponent by 3 and keeping the remainder, since $\omega^3 = 1$. Here $10 = 3(3) + 1$ and $20 = 3(6) + 2$, so

$$\omega^{10} + \omega^{20} = \omega + \omega^2 = \boxed{-1}$$

The last step uses $1 + \omega + \omega^2 = 0$.

(b) Expand, and use $\omega^3 = 1$ again:

$$(1+\omega)(1+\omega^2) = 1 + \omega^2 + \omega + \omega^3 = (1 + \omega + \omega^2) + 1 = 0 + 1 = \boxed{1}$$

Both parts used two facts: $\omega^3 = 1$ reduces any power of ω, and $1 + \omega + \omega^2 = 0$ removes what is left. Apply them in that order.

These evenly spaced points are the basis of the *discrete Fourier transform*, which separates a signal, such as a sound, into the pure frequencies it is made of. It does this by multiplying the n samples by powers of these n points on the circle and adding the results. A fast

method of computing it, the fast Fourier transform, uses the symmetry of the polygon to reduce the work. It is one of the methods a phone uses to process sound as it arrives.

17.4.2 Square Roots Without Polar Form

Polar form gives roots of every order, but it needs the argument. When that is not a standard angle (one with exact values of sin and cos, such as $\frac{\pi}{6}$), the argument is an arctan value, and the roots are hard to convert to exact Cartesian form. For *square* roots there is a second route that stays in Cartesian form and gives the answer in surds. Assume the root is $a+bi$, square it, and equate real and imaginary parts as in §17.1.1. A third equation comes from the modulus. By the rule for products, $|(a+bi)^2| = |a+bi|^2 = a^2+b^2$, so a^2+b^2 equals the modulus of the number. Adding and subtracting the equations gives a^2 and b^2. The imaginary part is $2ab$, so its sign tells you whether a and b have the same sign or opposite signs.

EXAMPLE 17.16

Find the square roots of $7+24i$ in the form $a+bi$.

SOLUTION

Let $(a+bi)^2 = 7+24i$. Expanding the left side gives a^2-b^2+2abi, so equating parts,

$$a^2-b^2=7 \qquad \text{and} \qquad 2ab=24.$$

Taking the modulus of both sides gives a third equation, $a^2+b^2=|7+24i|=25$. Add it to $a^2-b^2=7$ to get $a^2=16$, and subtract to get $b^2=9$. Since $2ab=24$ is positive, a and b have the same sign, so the square roots are $4+3i$ and $-4-3i$, that is

$$\pm(4+3i)$$

Alternative: substitution. The equation $2ab=24$ gives $b=\dfrac{12}{a}$. Substituting into the first,

$$a^2-\frac{144}{a^2}=7 \implies a^4-7a^2-144=0 \implies (a^2-16)(a^2+9)=0.$$

Since a is real, $a^2=16$, and the same two roots follow. When the imaginary part is not zero, this route gives a **quadratic in a^2**. It has one positive and one negative solution. The negative one is rejected, because a is real. Either way the two answers differ only by a sign, since square roots lie at opposite ends of a diameter, half a turn apart.

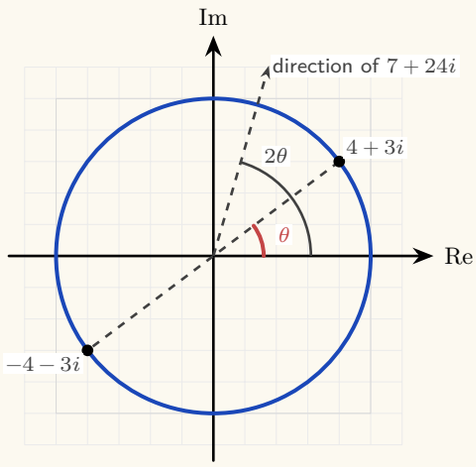


Figure 17.15 The square roots $\pm(4+3i)$ of $7+24i$ lie at opposite ends of a diameter. Each has half the argument of $7+24i$, or half plus π.

CAUTION For cube roots and higher, use polar form. Equating parts would give a cubic equation or one of higher degree, but De Moivre's theorem gives roots of any order directly.

17.4.3 Complex Zeros of Polynomials

Every nth root above was a solution of $z^n = w$, which is a polynomial equation of degree n with exactly n solutions. That is not a coincidence of this one family. The general fact is the **Fundamental Theorem of Algebra**: every polynomial of degree $n \geq 1$ has exactly n zeros in the complex numbers, counted with multiplicity. The proof is beyond this course. The zeros of a polynomial need not lie on the real line, but every one of them is a complex number.

The simplest case has already appeared, in a different form. In Ch. 4, a quadratic equation with negative discriminant had "no real solutions." It always had two roots; they were simply not on the real line.

EXAMPLE 17.17

Solve $z^2 - 4z + 13 = 0$.

SOLUTION

Use the quadratic formula, which works unchanged:

$$\begin{aligned} z &= \frac{4 \pm \sqrt{16 - 52}}{2} \\ &= \frac{4 \pm \sqrt{-36}}{2} \\ &= \frac{4 \pm 6i}{2} \\ &= \boxed{2 \pm 3i} \end{aligned}$$

Here $\sqrt{-36} = 6i$, because $(6i)^2 = -36$; the $\pm$ gives the other square root, $-6i$. The discriminant is negative, so its square root is a multiple of i: the two roots are $2 + 3i$ and $2 - 3i$, a conjugate pair placed symmetrically above and below the real axis.

The symmetry comes from the real coefficients: when a polynomial has *real* coefficients, its non-real zeros always come in conjugate pairs.

KEY 17.8 **Conjugate root theorem**

If a polynomial has real coefficients and $z = a + bi$ is a zero, then $z^* = a - bi$ is also a zero.

EXAMPLE 17.18

Prove the conjugate root theorem: if p is a polynomial with real coefficients and $p(z) = 0$, then $p(z^*) = 0$.

SOLUTION

The proof uses two properties of the conjugate. For any complex numbers z and w,

$$(z + w)^* = z^* + w^*, \qquad (zw)^* = z^* w^*.$$

Check both by writing each number in Cartesian form and expanding the two sides. Applying the second property repeatedly gives $(z^k)^* = (z^*)^k$. A real number is its own conjugate.

Now let $p(z) = a_n z^n + \cdots + a_1 z + a_0$, where every coefficient is real, and suppose $p(z) = 0$. Take the conjugate of $p(z)$. The conjugate of a sum is the sum of the conjugates. Each term $a_k z^k$ becomes $a_k^* (z^*)^k$, and $a_k^* = a_k$ because a_k is real. So

$$p(z^*) = \big(p(z)\big)^* = 0^* = 0.$$

So z^* is also a zero.

This is why a cubic with real coefficients always has a real zero. It has three zeros, counted with multiplicity, and the non-real ones come in pairs. So it has 0 or 2 non-real zeros, and at least one real zero.

The sum and product of roots in §6.5 also hold for complex roots, with each root counted as often as its multiplicity. With z in place of x, the equation $a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 = 0$ has

$$\text{sum of roots} = -\frac{a_{n-1}}{a_n}, \qquad \text{product of roots} = (-1)^n \frac{a_0}{a_n}.$$

For example, the conjugate roots $2 + 3i$ and $2 - 3i$ have sum 4 and product 13. A **monic** quadratic (leading coefficient 1) with these roots therefore has $a_1 = -4$ and $a_0 = 13$, so it is $z^2 - 4z + 13$, the polynomial of the equation solved above. For a cubic

$a_3z^3 + a_2z^2 + a_1z + a_0 = 0$ with roots α, β, γ,

$$\alpha + \beta + \gamma = -\frac{a_2}{a_3}, \qquad \alpha\beta\gamma = -\frac{a_0}{a_3}.$$

§6.5 also gives the relation for the roots taken in pairs, $\alpha\beta + \beta\gamma + \gamma\alpha = \dfrac{a_1}{a_3}$. Use these relations to find a remaining root, or to check roots already calculated.

To use the theorem, turn the pair into a real factor. Here a and b are the real and imaginary parts of the zero, not coefficients. The factor theorem holds for complex zeros (§6.3), so $z - (a + bi)$ and $z - (a - bi)$ are both factors. Their product is the real quadratic factor

$$(z - a)^2 + b^2 = z^2 - 2az + (a^2 + b^2).$$

Divide the polynomial by this factor, or compare coefficients (§6.3), to find the remaining zeros. For a cubic the other factor is linear. For a quartic it is a quadratic, which you solve with the quadratic formula.

EXAMPLE 17.19

Given that $2 + i$ is a root of $z^3 - 7z^2 + 17z - 15 = 0$, solve the equation.

SOLUTION

The coefficients are real, so $2 - i$ is also a root. The pair gives the real quadratic factor

$$(z - (2 + i))(z - (2 - i)) = (z - 2)^2 - i^2 = z^2 - 4z + 5,$$

the form $(z - a)^2 + b^2$ with $a = 2$ and $b = 1$. The other factor is linear, so write

$$z^3 - 7z^2 + 17z - 15 = (z^2 - 4z + 5)(z - c).$$

Compare the constant terms: $-5c = -15$, so $c = 3$. Check with the coefficient of z^2: the right side gives $-c - 4 = -7$. ✓ The three roots are $z = 2 + i,\ 2 - i$ or 3

The sum of the roots gives the same answer faster (§6.5). Here $a_3 = 1$ and $a_2 = -7$, so the three roots add to $-\frac{a_2}{a_3} = 7$. The conjugate pair adds to 4, so the third root is 3.

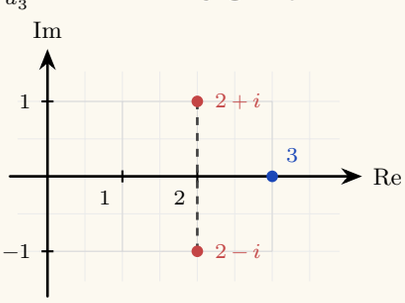


Figure 17.16 The roots of $z^3 - 7z^2 + 17z - 15 = 0$. The conjugate pair $2 \pm i$ are reflections of each other in the real axis, and the third root, 3, lies on it.

YOUR TURN 17.4 Roots of Complex Numbers

How many roots, and where

23.

(a) Write down a square root of -9.

(b) State the modulus of each cube root of 27.

(c) State how many distinct fifth roots a non-zero complex number has.

(d) State the shape formed on the Argand diagram by the nth roots ($n \geq 3$) of a non-zero complex number.

Finding nth roots

24. Find, in modulus–argument form with principal arguments,

(a) the three cube roots of 8;

(b) the four fourth roots of 16;

(c) the three cube roots of i;

(d) all values of $i^{2/3}$, using your answer to (c);

(e) the three cube roots of $-8i$. Plot them on an Argand diagram, on a circle centred at the origin, and show their modulus and their arguments.

Roots of unity

25.

(a) State the three cube roots of unity, in Cartesian form.

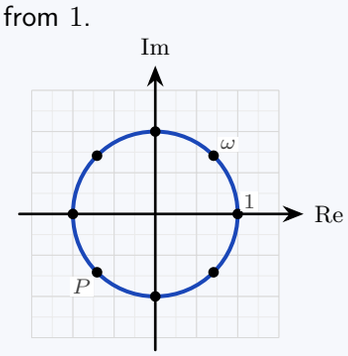
(b) State the four fourth roots of unity.

(c) State the six sixth roots of unity, in modulus–argument form.

(d) Let ω be a cube root of unity with $\omega \neq 1$. Show that $1 + \omega + \omega^2 = 0$.

(e) Use (a) to write $z^3 - 1$ as a product of three linear factors.

(f) The figure shows all the solutions of $z^n = 1$ for one value of n. The solution ω is the first one anticlockwise from 1.



State n and write ω in modulus–argument form. Write the number at P as a power of ω, and in Cartesian form.

26. Let $\omega = \operatorname{cis} \frac{2\pi}{3}$, so that $\omega^3 = 1$ and $1 + \omega + \omega^2 = 0$.

(a) Simplify $(1-\omega)(1-\omega^2)$.

(b) Show that $\left(1+\omega^2\right)^4 = \omega$.

Square roots without polar form

27. Find the square roots of each number in the form $a+bi$ by equating real and imaginary parts.

(a) $5+12i$

(b) $-8+6i$

(c) $-5+12i$

Polynomials with real coefficients

28. Solve each equation.

(a) $z^2+9=0$

(b) $z^2-2z+5=0$

(c) $z^2+2z+2=0$

(d) $2z^2-2z+5=0$

29.

(a) Given that $3+i$ is a zero of a polynomial with real coefficients, state another zero.

(b) Write down a quadratic with real coefficients whose zeros are $2i$ and $-2i$.

(c) Write down a quadratic with real coefficients whose zeros are $-2+5i$ and $-2-5i$.

30.

(a) The equation $z^2+pz+q=0$ has real coefficients and $3-2i$ as a root. Find p and q.

(b) Explain why a cubic with real coefficients must have at least one real zero.

31.

(a) Given that $1-i$ is a root of $z^3-4z^2+6z-4=0$, find the other two roots.

(b) Given that $2i$ is a root of $z^4-2z^3+5z^2-8z+4=0$, solve the equation.

(c) The equation $z^3+pz^2+qz-26=0$, where $p, q \in \mathbb{R}$, has $2-3i$ as a root. Find p and q, and the real root.

Concept check

32. A student solves $z^3=8i$ and writes $z=2\,\mathrm{cis}\left(\frac{\pi}{6}+2k\pi\right)$ for $k=0,1,2$. Explain why this gives only one root, and find all three roots in modulus–argument form, with principal arguments.

33. A student says: "*i is a root of $z^2-3iz-2=0$, so $-i$ is also a root, because complex roots come in conjugate pairs.*" Show that $-i$ is not a root, and state the condition the student has overlooked.

34. A student states that the roots of $z^3-5z^2+11z-15=0$ are 3, $1+2i$ and $1-2i$. Use both the sum and the product of the roots to check this claim.

17.5 Trigonometric Identities from De Moivre

The compound-angle rules of Ch. 8 give $\cos 2\theta$ and $\sin 2\theta$ without much work. For $\cos 3\theta$ they take considerably more. Each multiple angle has to be built from the one before it, so the work grows with every step. Expanding $(\cos\theta + i\sin\theta)^n$ gives any multiple angle at once, and it gives the sine identity and the cosine identity together.

De Moivre's theorem is used here in two opposite directions. Decide first which one a question needs.

Starting expression	Aim	Starting tool
$\cos n\theta$ or $\sin n\theta$	express it using powers of $\sin\theta$ or $\cos\theta$	expand $(\cos\theta + i\sin\theta)^n$
$\cos^n\theta$ or $\sin^n\theta$	express it using multiple angles	use $2\cos\theta = z + z^{-1}$ or $2i\sin\theta = z - z^{-1}$, with $z = \operatorname{cis}\theta$

Direction 1: a multiple angle in terms of powers. The method uses De Moivre's theorem: $(\cos\theta + i\sin\theta)^n$ and $\cos n\theta + i\sin n\theta$ are the same number. So one number has been written in two ways. Expand the first way with the binomial theorem of Ch. 2. The two ways must then match in both parts: the real part of one equals the real part of the other, and the imaginary parts match in the same way.

KEY 17.9 **Deriving multiple-angle identities**

To find an identity for $\cos n\theta$ or $\sin n\theta$:

1. Expand $(\cos\theta + i\sin\theta)^n$ by the binomial theorem.
2. Collect the terms into a real part and an imaginary part, using $i^2 = -1$.
3. Equate the real part to $\cos n\theta$ and the imaginary part to $\sin n\theta$.
4. Replace $\sin^2\theta$ by $1 - \cos^2\theta$, or the reverse, to leave powers of one function. For $\sin n\theta$ with n even, one factor of $\cos\theta$ always remains, as in $\sin 4\theta = \cos\theta\,(4\sin\theta - 8\sin^3\theta)$.

Divide one identity by the other and you have a third. The imaginary part over the real part is $\tan n\theta$, and the real part over the imaginary part is $\cot n\theta$. Use the two parts before rewriting them in one function. Every term then has total power n in $\cos\theta$ and $\sin\theta$, so dividing every term by $\cos^n\theta$ gives powers of $\tan\theta$, and dividing by $\sin^n\theta$ gives powers of $\cot\theta$.

EXAMPLE 17.20

Show that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, and find a matching identity for $\sin 3\theta$ in terms of $\sin\theta$.

SOLUTION

Write $c = \cos\theta$, $s = \sin\theta$, and expand by the binomial theorem:

$$(c+is)^3 = c^3 + 3c^2(is) + 3c(is)^2 + (is)^3 = (c^3 - 3cs^2) + i\,(3c^2 s - s^3).$$

The real terms, in blue, give $\cos 3\theta$; the imaginary terms, in red, give $\sin 3\theta$. By De Moivre this equals $\cos 3\theta + i\sin 3\theta$, so the parts must match separately. Real parts, using $s^2 = 1 - c^2$:

$$\cos 3\theta = c^3 - 3c(1-c^2) = 4\cos^3\theta - 3\cos\theta.$$

Imaginary parts, using $c^2 = 1 - s^2$:

$$\sin 3\theta = 3(1-s^2)s - s^3 \implies \boxed{\sin 3\theta = 3\sin\theta - 4\sin^3\theta}$$

EXAMPLE 17.21

Use the expansion above to show that $\cot 3\theta = \dfrac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}$, for all θ where both sides are defined.

SOLUTION

Take the two parts of the expansion *before* either was rewritten in a single function:

$$\cos 3\theta = c^3 - 3cs^2, \qquad \sin 3\theta = 3c^2 s - s^3.$$

Since $\cot 3\theta$ is the first divided by the second,

$$\cot 3\theta = \frac{c^3 - 3cs^2}{3c^2 s - s^3}.$$

Now divide every term, top and bottom, by s^3. Each division turns a c over an s into a $\cot\theta$:

$$\cot 3\theta = \frac{\dfrac{c^3}{s^3} - \dfrac{3cs^2}{s^3}}{\dfrac{3c^2 s}{s^3} - \dfrac{s^3}{s^3}} = \frac{\cot^3\theta - 3\cot\theta}{3\cot^2\theta - 1}. \qquad \blacksquare$$

Both choices from the method above keep the division simple: the mixed form of the parts, and division by the highest power, s^3.

17.5.1 Powers of Cosine and Sine

Direction 2: a power in terms of multiple angles. Direction 1 writes a multiple angle in powers of $\cos\theta$. The same idea works in the other direction. A power such as $\cos^6\theta$ becomes a sum of multiple angles, and a sum of multiple angles can be integrated directly with the methods of Ch. 15.

The tool is the pair of results from §17.3.1. For $z = \operatorname{cis}\theta$,

$$z + \frac{1}{z} = 2\cos\theta, \qquad z^n + \frac{1}{z^n} = 2\cos n\theta,$$

and the same expressions with a minus sign give $2i\sin\theta$ and $2i\sin n\theta$. So raise $z + \frac{1}{z}$ to a power and expand it. Each pair of terms then combines into a cosine of a multiple angle. The pairing works because binomial coefficients are symmetric. The term in z^m and the term in $\frac{1}{z^m}$ have the same coefficient, so together they give that coefficient times $2\cos m\theta$. When the power is even, the middle term has no partner and is a constant.

EXAMPLE 17.22

Express $\cos^6\theta$ in terms of $\cos 2\theta$, $\cos 4\theta$ and $\cos 6\theta$.

SOLUTION

Start from $2\cos\theta = z + \dfrac{1}{z}$ and raise both sides to the sixth power, with binomial coefficients $1, 6, 15, 20, 15, 6, 1$:

$$64\cos^6\theta = \left(z + \frac{1}{z}\right)^6 = z^6 + 6z^4 + 15z^2 + 20 + \frac{15}{z^2} + \frac{6}{z^4} + \frac{1}{z^6}.$$

Now pair the terms that are reciprocals of each other, shown above in the same colour:

$$\begin{aligned} 64\cos^6\theta &= \left(z^6 + \frac{1}{z^6}\right) + 6\left(z^4 + \frac{1}{z^4}\right) + 15\left(z^2 + \frac{1}{z^2}\right) + 20 \\ &= 2\cos 6\theta + 12\cos 4\theta + 30\cos 2\theta + 20. \end{aligned}$$

Divide by 64:

$$\boxed{\cos^6\theta = \tfrac{1}{32}\cos 6\theta + \tfrac{3}{16}\cos 4\theta + \tfrac{15}{32}\cos 2\theta + \tfrac{5}{16}}$$

Here the pairs have coefficients 1, 6 and 15, and the middle term, 20, is the constant.

TIP A power of $\sin\theta$ is rewritten in the same way, starting from $2i\sin\theta = z - \frac{1}{z}$. The factor $(2i)^n$ on the left must be divided out at the end, and an odd power gives sines rather than cosines. Check your answer by putting $\theta = \frac{\pi}{2}$, where $\sin\theta = 1$.

TIP Complex numbers offer several good IA topics.

- Iterate $z \mapsto z^2 + c$ from $z = 0$ and investigate which values of c make the sequence escape to infinity. This is how the Mandelbrot set is defined. Once $|z| > 2$ the sequence is certain to escape, but a finite run that has not escaped does not prove that c is in the set.

- Investigate which regular polygons can be constructed with ruler and compass. The Greeks asked this question. Gauss made progress on it at eighteen, using the roots of unity to construct the regular 17-gon, and Wantzel completed it in 1837.
- Measure the phase shift in a simple AC circuit and model it with rotating complex numbers.

YOUR TURN 17.5 Trigonometric Identities from De Moivre

Multiple angles as powers

35. Use De Moivre's theorem and the binomial theorem for each part.

(a) Show that $\cos 2\theta = 2\cos^2\theta - 1$.

(b) Find a matching identity for $\sin 2\theta$.

(c) Show that $\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1$.

(d) Hence, or otherwise, express $\tan 2\theta$ in terms of $\tan\theta$.

Powers as multiple angles

36. Let $z = \operatorname{cis}\theta$. Use $z \pm \dfrac{1}{z}$, where $z + \dfrac{1}{z} = 2\cos\theta$ and $z - \dfrac{1}{z} = 2i\sin\theta$.

(a) Express $\cos^2\theta$ in terms of $\cos 2\theta$.

(b) Express $\sin^2\theta$ in terms of $\cos 2\theta$.

(c) Express $\cos^3\theta$ in terms of $\cos 3\theta$ and $\cos\theta$.

37.

(a) Show that $\sin^3\theta = \frac{1}{4}(3\sin\theta - \sin 3\theta)$.

(b) Express $\cos^5\theta$ in terms of cosines of multiple angles.

Integrating a power of sine or cosine

38.

(a) Use your answer to Question 36(c) to evaluate $\displaystyle\int_0^{\pi/2} \cos^3\theta\, d\theta$.

(b) Use the result of Question 37(a) to find $\displaystyle\int \sin^3\theta\, d\theta$.

Concept check

39. A student uses $z = \operatorname{cis}\theta$ and writes

$$\left(z - \frac{1}{z}\right)^2 = (2i\sin\theta)^2 = 4\sin^2\theta \quad \text{and} \quad \left(z - \frac{1}{z}\right)^2 = z^2 - 2 + \frac{1}{z^2} = 2\cos 2\theta - 2,$$

and concludes that $\sin^2\theta = \frac{1}{2}(\cos 2\theta - 1)$. Show that this result is false by taking $\theta = \frac{\pi}{2}$, identify the error, and correct the result.

Chapter 17 · Complex Numbers

Reflections

TOK Euler's identity $e^{i\pi} + 1 = 0$ has often been called the most beautiful equation in mathematics. It links five fundamental numbers, 0, 1, π, e and i, in one short line. Mathematicians who call a result beautiful often mean qualities like these: shortness, surprise and an unexpected connection. What is being judged when a mathematician calls a result beautiful, and is that judgement evidence of anything?

IM The name "imaginary" was not a compliment. Descartes used it for roots he regarded as not real. Acceptance came slowly, helped by the picture of the plane drawn independently by Wessel and Argand. Gauss remarked that these numbers would have met less resistance if they had been called "lateral". Does the name of an idea change how readily it is believed?

RW Complex numbers are a standard tool for describing oscillations. An alternating current, a radio wave or a vibration is modelled as a complex number rotating about the origin. Its real part is the signal you measure: $A\cos(\omega t + \phi) = \mathrm{Re}\left[(Ae^{i\phi})\,e^{i\omega t}\right]$. Here ω is the angular frequency $(\text{the rate of turning, in radians per second})$, not a root of unity. The fixed number $Ae^{i\phi}$ records the amplitude and the phase, and the factor $e^{i\omega t}$ is the rotation. Engineers add and scale these fixed numbers, called **phasors**, directly, instead of working through long trigonometric calculations.

EXT Euler's formula connects the exponential and trigonometric functions, and De Moivre's theorem describes how they behave under powers. The identity $e^{i\theta} = \cos\theta + i\sin\theta$ shows that the exponential and trigonometric functions, which look nothing alike on the real line, are closely related once complex inputs are allowed: $\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and $\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}$. Exponential growth and oscillation both come from the one function e^z. Push this further and you reach complex analysis, the study of functions of a complex variable. Functions that can be differentiated throughout a region of the plane follow much stricter rules than real functions. Some problems about ordinary integers, such as how the prime numbers are distributed, have been partly answered by studying such functions. Physics uses complex numbers at a deeper level too: in quantum mechanics a particle's state is a complex-valued wave, and i appears explicitly in the Schrödinger equation.

End-of-Chapter Problems

1. Given $z = 3 + 4i$ and $w = 1 - 2i$:

(a) find $z + w$; [1]
(b) find zw; [2]
(c) find $\frac{z}{w}$; [2]
(d) find $|zw|$. [1]

2. Express $\frac{2+i}{3-i}$ in the form $a + bi$. [4]

3. Simplify

(a) $i^{2027} + i^{2028}$; [2]
(b) i^{-7}. [2]

4. The points representing 1, i, -1 and $-i$ are plotted on an Argand diagram. State the shape they form and find its area. [3]

5. Find real numbers x and y such that $(x + yi)(2 - i) = 3 + i$. [4]

6. Find the complex number z satisfying $3z + iz^* = 5 + 7i$. [4]

7. Find the modulus and the principal argument of

(a) $z = -1 + i$; [2]
(b) $z = \sqrt{3} - i$. [2]

8. Write $z = -\sqrt{3} + 3i$ in modulus–argument form and in Euler form. [4]

9. Given $z = 2 + 2i$, find $|z|$ and $\arg z$, and hence write z in Euler form. [4]

10.

(a) Express -8 in the form $re^{i\theta}$, with $-\pi < \theta \leq \pi$. [3]
(b) The complex number z has $|z| = 6$ and $\arg z = \frac{3\pi}{4}$. Write z in Cartesian form. [3]

11. Use De Moivre's theorem to evaluate

(a) $(-\sqrt{3} + i)^5$, giving your answer in the form $a + bi$; [3]
(b) $(1 + i)^{10}$; [3]

(c) $\left(\dfrac{1+i}{\sqrt{2}}\right)^{12}$. [3]

12. Given $z = 3\,\mathrm{cis}\,\frac{3\pi}{4}$, giving each answer in modulus–argument form with $-\pi < \theta \leq \pi$:

(a) find z^2; [2]

(b) find z^3. [2]

13. Let $z_1 = 4\,\mathrm{cis}\,\frac{5\pi}{6}$ and $z_2 = 2\,\mathrm{cis}\,\frac{\pi}{4}$.

(a) Find $z_1 z_2$ and $\dfrac{z_1}{z_2}$, giving each in modulus–argument form with $-\pi < \theta \leq \pi$. [4]

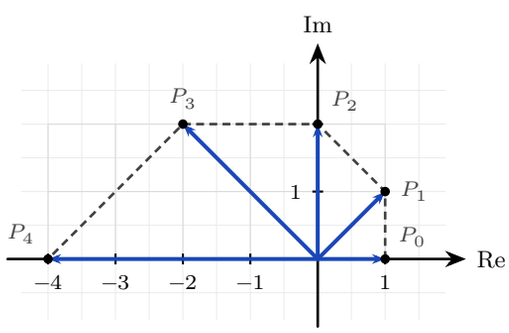
(b) Describe fully the single transformation of the Argand diagram that maps the point representing z to the point representing zz_2. [2]

(c) The point A represents $w = 3 + i$. Point B is the image of A under an anticlockwise rotation of $\frac{2\pi}{3}$ about the origin O. Find the complex number that B represents, in exact Cartesian form. [3]

(d) Find the area of triangle OAB. [2]

14. Given $z = \cos\theta + i\sin\theta$ with $\cos\theta \neq 0$, show that $\dfrac{z^2 - 1}{z^2 + 1} = i\tan\theta$. [4]

15. The figure shows the points P_0, P_1, P_2, P_3 and P_4 on an Argand diagram. They represent 1, z, z^2, z^3 and z^4 for a complex number z.



(a) Write down z, and find $|z|$ and $\arg z$. [3]

(b) Use De Moivre's theorem to write z^n in modulus–argument form. Hence find the least positive integer n for which z^n is a positive real number, and state z^n. [3]

(c) Show that the area of triangle OP_kP_{k+1} is 2^{k-1} for every integer $k \geq 0$. [3]

(d) Check the result of part (c) for $k = 3$, using the coordinates of P_3 and P_4. [2]

16. Solve $2z^2 + 2z + 5 = 0$. [4]

17. By writing $-2i$ in modulus–argument form, solve $z^2 = -2i$, giving the roots in the form $a + bi$. [4]

18. Find the square roots of each number in the form $a + bi$.

(a) $3 + 4i$ [4]
(b) $-21 - 20i$ [4]

19. The equation $z^2 + pz + q = 0$, where $p, q \in \mathbb{R}$, has $\dfrac{5}{2 - i}$ as a root. Find p and q. [5]

20. Find the three cube roots of $8i$, giving each

(a) in modulus–argument form; [3]
(b) in exact Cartesian form. [3]

21. Find all solutions of $z^3 = -4\sqrt{3} + 4i$, giving each in modulus–argument form with $-\pi < \theta \leq \pi$. [4]

22.

(a) Find the four fourth roots of -16, giving each in the form $a + bi$. [5]
(b) Find all solutions of $z^6 = 64$, giving each in the form $a + bi$, and show that they sum to zero. [4]

23. GDC Let $w = 3 + 4i$.

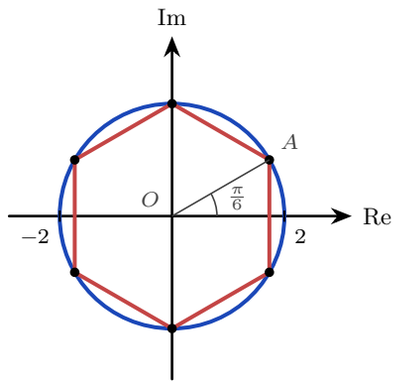
(a) Write w in the form $re^{i\theta}$, giving θ correct to 3 s.f. [2]
(b) Find the five solutions of $z^5 = w$ in the form $re^{i\theta}$, with $-\pi < \theta \leq \pi$, giving r and θ correct to 3 s.f. [4]
(c) Find the solution that lies in the third quadrant, in the form $a + bi$ with a and b correct to 3 s.f. [2]
(d) The five solutions are the vertices of a regular pentagon. Find its area. [2]

24. One cube root of a complex number w is $1 + \sqrt{3}\,i$. Find w and the other two cube roots. [6]

25. The polynomial $P(z) = z^4 - 2z^3 + 14z^2 - 18z + 45$ has real coefficients, and $1 + 2i$ is a zero.

(a) Write down another zero of $P(z)$, and give a reason. [2]
(b) Find a real quadratic factor of $P(z)$. [3]
(c) Find the other quadratic factor. [3]
(d) Hence write down all four zeros of $P(z)$. [3]
(e) The four zeros are plotted on an Argand diagram. Describe the symmetry of the picture, and state the feature of $P(z)$ that guarantees it. [3]

26. The figure shows a regular hexagon on an Argand diagram, centred at the origin O, with its vertices on a circle. One vertex is A.



(a) Write down the number a represented by A in the form $re^{i\theta}$, and in Cartesian form. [2]
(b) The six vertices represent the solutions of $z^6 = w$. Find w. [2]
(c) Write down the other five solutions in the form $re^{i\theta}$, with $-\pi < \theta \leq \pi$. [2]
(d) Find the exact area of the hexagon. [2]
(e) Find the product of the six solutions. [2]

27. Use De Moivre's theorem to show that $\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$. [6]

28. Let $z = \operatorname{cis}\theta$.

(a) Show that $z^n + \dfrac{1}{z^n} = 2\cos n\theta$ for any $n \in \mathbb{Z}^+$. [3]
(b) Expand $\left(z + \dfrac{1}{z}\right)^4$ using the binomial theorem, and hence show that $16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$. [5]
(c) Deduce that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$. [2]
(d) Hence evaluate $\displaystyle\int_0^{\pi/2} \cos^4\theta\, d\theta$ exactly. [4]
(e) Explain what this method achieved that integrating $\cos^4\theta$ directly would not. [2]

29. Prove De Moivre's theorem, $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, for all $n \in \mathbb{Z}^+$ by mathematical induction. [7]

30. Use De Moivre's theorem to show that $\tan 4\theta = \dfrac{4t - 4t^3}{1 - 6t^2 + t^4}$, where $t = \tan\theta$, for values of θ at which both sides are defined. [7]

31. In this question you will investigate the fifth roots of unity, the solutions of $z^5 = 1$.

(a) Write down the five roots in the form $\operatorname{cis}\theta$. [3]
(b) Show that the five roots sum to zero, using the sum of a geometric series. [3]
(c) Let $\omega = \operatorname{cis}\frac{2\pi}{5}$. Show that every fifth root of unity is a power of ω. [2]
(d) Deduce that $\omega + \omega^2 + \omega^3 + \omega^4 = -1$. [2]

(e) The five roots are plotted on an Argand diagram and joined to form a regular pentagon. Show that its area is $\frac{5}{2}\sin\frac{2\pi}{5}$. Given that $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$, find the exact area in the form $p\sqrt{10+2\sqrt{5}}$, where $p \in \mathbb{Q}$. [5]

32.

(a) Find all values of $(-8)^{2/3}$, giving each in Cartesian form. [4]

(b) A student writes $(-8)^{2/3} = \left((-8)^{1/3}\right)^2 = (-2)^2 = 4$. Show that the values in (a) are exactly the three cube roots of 64, and explain what the student has missed. [2]

33. Given that $1+i$ is a root of $z^2 - (3+2i)z + 1 + 3i = 0$, find the other root, and explain why it is not $1-i$. [4]

34. Let $z = \operatorname{cis}\theta$.

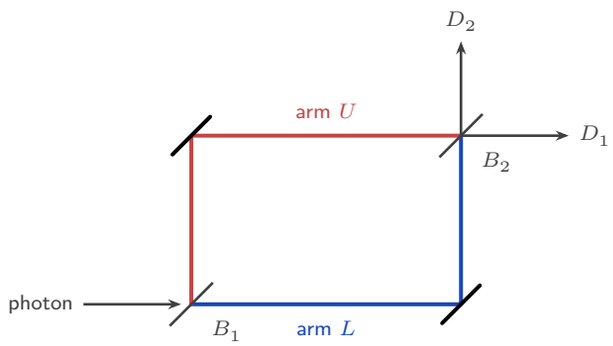
(a) By expanding $\left(z - \frac{1}{z}\right)^4$, express $\sin^4\theta$ in the form $p\cos 4\theta + q\cos 2\theta + r$, where $p, q, r \in \mathbb{Q}$. Check your answer at $\theta = \frac{\pi}{2}$. [4]

(b) Hence find the exact value of $\displaystyle\int_0^{\pi/2} \sin^4\theta\, d\theta$. [2]

Challenge Questions

C1. One photon, two paths. In quantum physics each possible outcome of an experiment has a complex number called its **amplitude**, and the probability of the outcome is the square of its modulus. When an outcome can happen in more than one way, the amplitudes of the ways are *added* first, and only then is the modulus squared. Along a single route, the amplitudes of the steps are multiplied.

In a Mach–Zehnder interferometer, one photon at a time meets a half-silvered mirror B_1, travels along the lower arm L or the upper arm U, and meets a second half-silvered mirror B_2 before reaching a detector D_1 or D_2. At each half-silvered mirror, passing straight through multiplies the amplitude by $\frac{1}{\sqrt{2}}$, and reflecting multiplies it by $\frac{i}{\sqrt{2}}$. The two ordinary mirrors affect both arms in the same way, so leave them out.



At B_1, passing through leads to arm L and reflecting leads to arm U. At B_2, a photon from arm L reaches D_2 by passing through and D_1 by reflecting; a photon from arm U reaches D_1 by passing through and D_2 by reflecting.

(a) Show that a single half-silvered mirror sends the photon through or reflects it with probability $\frac{1}{2}$ each. [2]

(b) Find the amplitude of each of the four routes: L to D_1, L to D_2, U to D_1 and U to D_2. [3]

(c) Hence show that the photon reaches D_1 with probability 1 and D_2 with probability 0. [3]

(d) A student argues: "Each of the four routes has probability $\frac{1}{4}$, so each detector is reached with probability $\frac{1}{2}$." Explain how the student's method differs from the quantum rule, and why the factor i makes the result at D_2 zero. [2]

(e) A thin glass plate is placed in arm U. It multiplies the amplitude of any photon passing through it by $e^{i\varphi}$, where φ is real. Show that the amplitude at D_2 is now $\frac{1}{2}\left(1 - e^{i\varphi}\right)$ and that the probability of reaching D_2 is $\sin^2\dfrac{\varphi}{2}$. Find the probability of reaching D_1, and check that the two probabilities add to 1. *(Hint: $1 - \operatorname{cis}\theta = -2i\sin\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}$ and $1 + \operatorname{cis}\theta = 2\cos\frac{\theta}{2}\operatorname{cis}\frac{\theta}{2}$)* [5]

(f) Remove the plate and block arm U, so that any photon entering it is absorbed. Find the probability that the photon reaches D_1, reaches D_2, or is absorbed. Compare your answer for D_2 with part (c), and comment. [3]

(g) Between the mirrors, the amplitude $a(t)$ of a photon of fixed energy satisfies $\dfrac{da}{dt} = -i\omega a$, where $\omega > 0$ is a constant. By writing $e^{-i\omega t} = \cos\omega t - i\sin\omega t$ and differentiating its real and imaginary parts, show that $a(t) = a(0)\,e^{-i\omega t}$ satisfies this equation. Show also that $|a(t)| = |a(0)|$ for all t, and explain why the equation $\dfrac{da}{dt} = -\omega a$, without the i, could not describe a photon that is certain to arrive somewhere. [3]

C2. The escape radius of the Mandelbrot set. Fix a complex number c, start at $z_0 = 0$, and iterate

$$z_{n+1} = z_n^2 + c.$$

For some c the sequence stays bounded forever; for others it grows without bound. The set of c for which it stays bounded is the **Mandelbrot set**, and a computer draws it by testing

one c per pixel. This question finds the circle outside which no point of the set can lie, and the test a computer uses to stop.

(a) Take $c = i$. Compute z_1, z_2, z_3 and z_4, and show that the sequence then repeats with period 2. Is i in the set? [4]

(b) Take $c = 1$. Compute z_1, z_2, z_3, z_4 and say what is happening. [2]

(c) Show that if $|z| > 2$ and $|z| \geq |c|$ then $\left|z^2 + c\right| \geq |z|(|z| - 1) > |z|$. *(Hint: $|z^2 + c| \geq |z|^2 - |c|$)* [4]

(d) Suppose $|c| > 2$. Show that z_1 satisfies both conditions of (c), and prove by induction that $|z_n| \geq |c|(|c| - 1)^{n-1}$ for $n \geq 1$. Deduce that every c in the Mandelbrot set satisfies $|c| \leq 2$. [4]

(e) Now suppose $|c| \leq 2$, and that $|z_m| > 2$ for some m. Use (c) to show that the sequence then grows without bound. Combining this with (d), explain why a computer can stop testing a pixel, whatever c is, as soon as $|z_n| > 2$. [3]

(f) Investigate which real numbers c lie in the Mandelbrot set. Use the fixed points of $z \mapsto z^2 + c$ and a few iterations to suggest the interval, and justify at least one of its endpoints. [3]

C3. Adding waves. Two loudspeakers play the same pure note. At a listening point the two arrivals are $A\cos\omega t$ and $B\cos(\omega t + \phi)$, where $A > 0$, $B > 0$, and ϕ is the phase difference caused by the extra distance one sound has travelled. Adding them with trigonometry is lengthy; with complex numbers each wave becomes a vector in the Argand diagram, and the waves add as vectors do.

(a) Show that $A\cos\omega t + B\cos(\omega t + \phi) = \operatorname{Re}\left[(A + B\operatorname{cis}\phi)\operatorname{cis}\omega t\right]$. [3]

(b) Deduce that the combined sound is a wave of the same frequency, with amplitude $|A + B\operatorname{cis}\phi|$ and, when $A + B\operatorname{cis}\phi \neq 0$, phase $\arg(A + B\operatorname{cis}\phi)$. [2]

(c) Two speakers each have amplitude 3 and the phase difference is $\frac{\pi}{3}$. Find the amplitude and phase of the combined sound, in exact form. [4]

(d) Now let both amplitudes equal A, and take $-\pi \leq \phi \leq \pi$. Show that $A + A\operatorname{cis}\phi = 2A\cos\dfrac{\phi}{2}\operatorname{cis}\dfrac{\phi}{2}$. Deduce that the combined amplitude is $2A\cos\dfrac{\phi}{2}$, that the phase is $\dfrac{\phi}{2}$ when $-\pi < \phi < \pi$, and state what happens when $\phi = \pm\pi$. [4]

(e) Use (d) to explain what a listener hears when $\phi = 0$ and when $\phi = \pi$, and why noise-cancelling headphones work by generating a wave with $\phi = \pi$. [3]

(f) Now n speakers in a row each have amplitude A, and each arrival is a phase ϕ ahead of the one before, so the arrivals are $A\cos(\omega t + k\phi)$ for $k = 0, 1, \ldots, n-1$, with $0 < \phi < 2\pi$. Use the sum of a geometric series in $\operatorname{cis}\phi$ to show that the combined amplitude is

$$A\left|\frac{\sin(n\phi/2)}{\sin(\phi/2)}\right|.$$

(Hint: $\operatorname{cis}\alpha - 1 = 2i\sin\frac{\alpha}{2}\operatorname{cis}\frac{\alpha}{2}$, found by taking out the factor $\operatorname{cis}\frac{\alpha}{2}$ as in (d)) [4]

(g) For $n = 4$, find every ϕ in $0 < \phi < 2\pi$ that gives silence. Then investigate how the loud region around $\phi = 0$ changes as n increases, and suggest why an array of many speakers can aim sound in one direction. [3]

C4. From a circle to an aeroplane wing. The map $w = z + \dfrac{1}{z}$ is called the **Joukowski transformation**. Early aerodynamicists used it because it turns circles, whose airflow is easy to calculate, into wing-shaped curves, whose airflow is not.

(a) Let $z = \operatorname{cis}\theta$, so $|z| = 1$. Show that $w = 2\cos\theta$, and hence that the unit circle maps onto the line segment from -2 to 2. [4]

(b) Now let $z = r\operatorname{cis}\theta$ with $r > 1$. Writing $w = x + iy$, show that

$$x = \left(r + \tfrac{1}{r}\right)\cos\theta, \qquad y = \left(r - \tfrac{1}{r}\right)\sin\theta,$$

and deduce that $\dfrac{x^2}{\left(r + \frac{1}{r}\right)^2} + \dfrac{y^2}{\left(r - \frac{1}{r}\right)^2} = 1$. [5]

(c) That equation describes an ellipse with semi-axes $r + \frac{1}{r}$ and $r - \frac{1}{r}$. Find both for $r = 2$. [2]

(d) Describe what happens to the ellipse as $r \to 1^+$, and reconcile your answer with part (a). [3]

(e) A wing needs a rounded front and a sharp trailing edge. Let $0 < a < 1$ and let Γ be the circle with centre $-a$ and radius $1 + a$. Show that Γ passes through $z = 1$, and that $z = -1$ lies inside Γ. [2]

(f) The derivative $\dfrac{dw}{dz} = 1 - \dfrac{1}{z^2}$ is zero only at $z = \pm 1$, and at a point where it is zero the map doubles angles instead of preserving them. Use this and (e) to explain why the image of Γ has a sharp point at $w = 2$ and is smooth everywhere else. [3]

(g) Real wings are curved (cambered) as well as thick. Investigate the image of the circle through $z = 1$ whose centre is $-a + bi$, with $a > 0$ and $b > 0$: state its radius, and describe how a and b separately change the shape of the image. [2]

C5. How a phone hears a chord. A microphone records a sound as a list of numbers, one per sampling instant. To find which frequencies are present, the recording is compared against the roots of unity. For four samples x_0, x_1, x_2, x_3, the **discrete Fourier transform** is

$$X_k = \sum_{n=0}^{3} x_n \, i^{-nk}, \qquad k = 0, 1, 2, 3,$$

where $i = \operatorname{cis}\frac{2\pi}{4}$ is the fourth root of unity next to 1. Each X_k measures how strongly the frequency k is present, meaning k complete cycles across the four samples. For real samples X_3 is the conjugate of X_1, and the two together describe one oscillation of frequency 1.

(a) Write down the four fourth roots of unity and verify that they sum to zero. [2]

(b) A constant signal gives samples $(1, 1, 1, 1)$. Compute X_0, X_1, X_2, X_3. [4]

(c) Explain, using part (a), why every X_k with $k \neq 0$ had to vanish in (b) (for $k = 2$, note that $i^{-2} = -1$), and say what X_0 measures. [3]

(d) A signal that rises and falls once across the four samples gives $(1, 0, -1, 0)$. Compute X_0, X_1, X_2, X_3. [4]

(e) Interpret your answer to (d): which frequency is present, and what does $X_0 = 0$ tell you about the signal? [3]

(f) A chord is two notes at once. The samples $(2, -1, 0, -1)$ are the sum of the signal in (d) and the signal $(1, -1, 1, -1)$. Compute the transform of $(2, -1, 0, -1)$, and explain how it shows both notes. [3]

(g) Prove that the samples can be recovered from the transform by

$$x_n = \frac{1}{4} \sum_{k=0}^{3} X_k \, i^{nk}, \qquad n = 0, 1, 2, 3,$$

and check this for x_0 in (f). *(Hint: substitute the definition of X_k and change the order of the two sums; the inner sum of powers of i is 4 when the two indices agree and 0 otherwise, by part (a))* [4]

(h) Suggest how the transform and the formula in (g) extend to N samples. Then investigate what the transform gives for a tone that does not complete a whole number of cycles across the samples. [2]

