CALCULUS

Chapter 16

Differential Equations

16

SYLLABUS 5.18 5.19 HL only

So far you have solved equations whose unknown is a number: linear, quadratic, exponential and logarithmic equations. When a real situation is modelled, though, the rule you know is often a rule about **change**. A population grows by an amount that depends on its present size. A hot drink cools faster when it is hotter. A planet's velocity changes according to the pull of the Sun. An equation that connects a quantity with its rate of change is a **differential equation**, and its unknown is not a number but a whole function. To **solve** one is to find that function, and with it a rule about change becomes a prediction.

Differential equations are used widely across science:

- **Planets.** Newton's laws of motion are differential equations, and solving them for the planets gives their orbits. In 1846 the difference between the observed and the predicted orbit of Uranus led astronomers to a planet no one had seen: Neptune.
- **The universe.** The equations that describe the expansion of the universe are differential equations. Their solutions give the distance between galaxies at each time.
- **Disease and weather.** Scientists model the spread of a disease, and weather services forecast tomorrow's weather, by solving differential equations step by step on computers.

By the end of this chapter you will be able to classify a first-order differential equation and solve it exactly in three ways (§16.1–§16.3): by separating the variables, by the substitution $y = vx$, or with an integrating factor. You will also approximate a function on an interval around 0 by the first terms of its Maclaurin series (§16.4), approximate a solution step by step with Euler's method (§16.5), and judge how accurate each approximation is.

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16.1 Separable Variables

Equations of this kind model cooling, radioactive decay, population growth and a drug leaving the bloodstream. In each, the rate is known and the quantity is wanted.

16.1.1 What a differential equation is

Many laws in science describe how fast a quantity changes, not the quantity itself: a population grows at a rate set by its size, and a drink cools at a rate set by how much hotter it is than the room. Written as an equation, such a law contains a derivative. Solving the equation turns the rule about change into a formula for the quantity, so it can be used to predict.

A **differential equation** is an equation that connects an unknown function $y(x)$, its derivative $\dfrac{dy}{dx}$ and the variable x. The unknown is a function, not a number. Many equations in this chapter can be written in one general form:

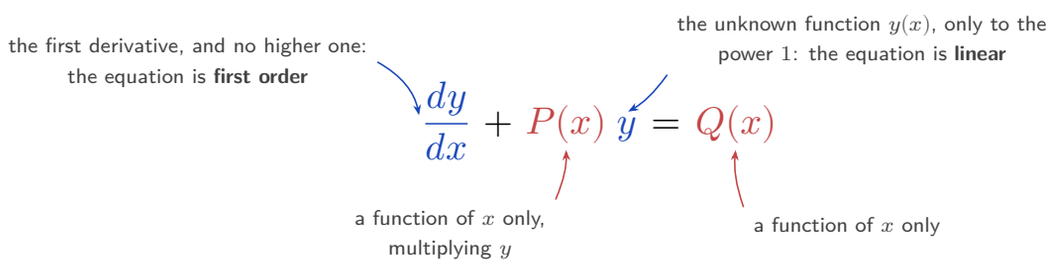


Figure 16.1 The general first-order linear equation. The unknown and its derivative are in cobalt; the given functions of x are in scarlet.

The two words in the labels describe an equation, and both are decided by looking at it.

- It is **first order** because the highest derivative in it is the first, $\dfrac{dy}{dx}$. Every equation in this chapter is first order.
- It is **linear** because y and $\dfrac{dy}{dx}$ appear only to the first power, are never multiplied by one another, and are not inside another function such as $\sin y$ or e^y; each may be multiplied only by a function of x. So $\dfrac{dy}{dx} + xy = 1$ is linear, with $P(x) = x$ and $Q(x) = 1$, and so is $\dfrac{dy}{dx} = ky$, with $P(x) = -k$ and $Q(x) = 0$. The equation $\dfrac{dy}{dx} = 2xy^2$ is not linear, because of the y^2.

A **solution** is a function that makes the equation true. To check one, substitute it. The equation $\dfrac{dy}{dx} = ky$ says that the rate of change of y is proportional to y, the rule behind growth and decay. For $y = Ae^{kx}$, $\dfrac{dy}{dx} = kAe^{kx} = ky$, so $y = Ae^{kx}$ is a solution for every constant A. The family of all solutions, with its constant, is the **general solution**. An

initial condition (a known value, such as $y(0) = 1$) fixes the constant and selects one curve of the family, the **particular solution**.

This chapter solves three types of first-order equation. Each is recognised by its form:

- **separable**, $\dfrac{dy}{dx} = f(x)\,g(y)$, solved in §16.1.2;
- **homogeneous**, $\dfrac{dy}{dx} = F\left(\dfrac{y}{x}\right)$, solved in §16.2;
- **linear**, $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, solved in §16.3.

> EXT The **order** of an equation is the order of the highest derivative in it. Many laws of physics are **second order**. Newton's second law, $F = ma$, says that force fixes the acceleration $a = \frac{d^2x}{dt^2}$. For a mass m on a spring of stiffness k, the force is $-kx$, so
>
> $$m\,\frac{d^2x}{dt^2} = -kx.$$
>
> Its solutions are $x = A\cos\omega t + B\sin\omega t$ with $\omega = \sqrt{k/m}$: the mass oscillates. A second-order equation needs two conditions, such as the starting position and the starting velocity. These equations are beyond the syllabus, which covers first-order equations only.

16.1.2 Separating the variables

The aim: put the y-expressions with dy and the x-expressions with dx, then integrate both sides.

The simplest case can be rearranged so that each variable is on its own side. A first-order equation is **separable** if it can be written as a function of y times a function of x:

$$\frac{dy}{dx} = f(x)\,g(y).$$

Then you separate and integrate both sides. Treating $\dfrac{dy}{dx}$ as a ratio is informal but valid here: gather all the y terms with dy and all the x terms with dx.

Being separable and being linear (§16.1.1) are different properties. An equation can have both, either one, or neither:

Equation	Linear?	Separable?
$\dfrac{dy}{dx} = xy$	yes	yes
$\dfrac{dy}{dx} + xy = 1$	yes	no
$\dfrac{dy}{dx} = 2xy^2$	no	yes
$\dfrac{dy}{dx} = x^2 + y^2$	no	no

KEY 16.1 **Separating the variables**

$$\frac{dy}{dx} = f(x)\,g(y) \quad \Longrightarrow \quad \int \frac{1}{g(y)}\,dy = \int f(x)\,dx$$

Dividing needs $g(y) \neq 0$. If $g(c) = 0$, then $y = c$ is also a solution: check it and keep it.

For example, $y = 0$ solves $\dfrac{dy}{dx} = xy$, but separating divides by y and loses it. It is the required solution only if it fits the initial condition.

The result does not depend on treating $\dfrac{dy}{dx}$ as a ratio. Divide the equation by $g(y)$ and integrate both sides with respect to x:

$$\int \frac{1}{g(y)} \frac{dy}{dx}\,dx = \int f(x)\,dx.$$

By the substitution rule of §15.4, the left side is $\displaystyle\int \frac{1}{g(y)}\,dy$.

Integrating gives one unknown constant, so the result is the general solution, a family of curves; an initial condition determines the constant and selects the particular solution. A particular solution **holds on an interval that contains the starting point**, and on that interval it is differentiable and satisfies the original equation. If the formula, or the equation itself, is undefined at some value of x, the interval stops there.

For a separable equation, work in this order:

1. Check for constant solutions that a division might lose.
2. Separate the variables.
3. Integrate, with one arbitrary constant.
4. Apply the initial condition, if one is given.
5. State the interval on which the solution holds, and check it in the original equation.

Apply the initial condition as early as possible. Carrying C through more steps than necessary makes errors more likely.

When integration gives $\ln|y-a| = F(x) + C$, exponentiate both sides and write $y - a = Ae^{F(x)}$:

$$\ln|y-a| = F(x) + C \implies |y-a| = e^C e^{F(x)} \implies y - a = \pm e^C e^{F(x)} = Ae^{F(x)}.$$

- The constant $A = \pm e^C$ replaces the modulus sign and the old constant together. The constant becomes **a multiplying factor, not an added term**. The value $A = 0$ does not come from exponentiating, so check separately whether $y = a$ is itself a solution.

EXAMPLE 16.1

Solve $\dfrac{dy}{dx} = 2xy^2$ given $y(0) = 1$.

SOLUTION

Separate the variables and integrate:

$$\int \frac{1}{y^2}\,dy = \int 2x\,dx \implies -\frac{1}{y} = x^2 + C.$$

Apply $y(0) = 1$: $-1 = 0 + C \implies C = -1$. So $-\dfrac{1}{y} = x^2 - 1$, giving

$$y = \frac{1}{1-x^2}, \quad -1 < x < 1$$

The solution is valid on $-1 < x < 1$, the interval containing the starting point $x = 0$ on which $1 - x^2 \neq 0$.

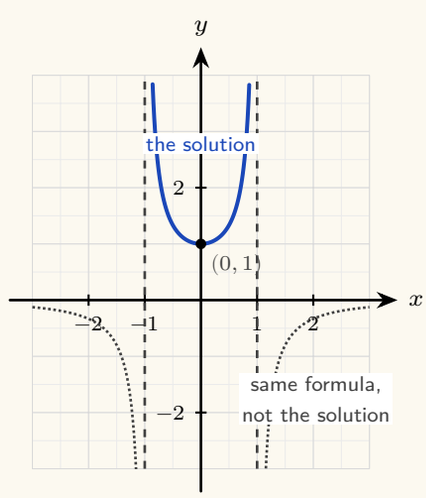


Figure 16.2 $y = \dfrac{1}{1-x^2}$ through $(0, 1)$. The solution is the branch on $-1 < x < 1$, between the asymptotes; the dotted pieces satisfy the same formula but are not joined to the starting point.

A hot drink in a room follows **Newton's law of cooling**: it cools at a rate proportional to the difference between its temperature and the room's. With room temperature $20°$C this is $\dfrac{dT}{dt} = -k(T - 20)$, where $k > 0$, and it is separable.

EXAMPLE 16.2

A cup of coffee obeys $\dfrac{dT}{dt} = -k(T - 20)$, where $k > 0$, with $T(0) = 80$. Solve for $T(t)$.

SOLUTION

Separate and integrate:

$$\int \frac{1}{T-20}\,dT = \int -k\,dt \implies \ln|T-20| = -kt + C.$$

Exponentiating, $T - 20 = Ae^{-kt}$. The condition $T(0) = 80$ gives $A = 60$, so

$$\boxed{T(t) = 20 + 60e^{-kt}}$$

The temperature decays exponentially towards the room's 20 degrees, never quite reaching it. This is the cooling pattern seen in §5.5: the difference from room temperature falls by the same fraction in each minute.

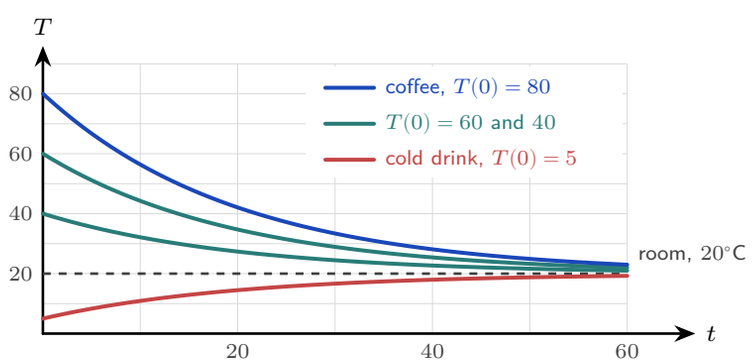


Figure 16.3 Solutions of $\frac{dT}{dt} = -k(T - 20)$ with $k = 0.05$ and t in minutes. Every solution approaches room temperature, from above or below; each starting value gives a different curve.

Separation also solves a standard population model in ecology. In the 1960s, once the cattle disease rinderpest was eliminated around the Serengeti, the wildebeest herd began to grow, and its growth followed the **logistic equation**. In this model the growth rate is proportional both to the size of the herd and to how far the herd is below the **carrying capacity** (the largest population the land can support). In symbols, $\dfrac{dP}{dt} = kP(N - P)$, where N is the carrying capacity and $k > 0$. It is separable, and the partial fractions of Ch. 6 split $\frac{1}{P(N-P)}$ so that each part can be integrated.

EXAMPLE 16.3

The Serengeti wildebeest population P (in millions) satisfies $\dfrac{dP}{dt} = kP(1.4 - P)$, where $k > 0$, with $P(0) = 0.25$ in 1961. Solve for $P(t)$.

SOLUTION

Separate, and split the left side with the partial fractions of Ch. 6:

$$\int \frac{dP}{P(1.4-P)} = \int k\,dt, \qquad \frac{1}{P(1.4-P)} = \frac{1}{1.4}\left(\frac{1}{P} + \frac{1}{1.4-P}\right).$$

Integrating,

$$\frac{1}{1.4}\ln\left|\frac{P}{1.4-P}\right| = kt + C \implies \frac{P}{1.4-P} = Ae^{1.4kt}.$$

Multiply by 1.4 and exponentiate to reach the second form, with $A = \pm e^{1.4C}$. The initial condition gives $A = \dfrac{0.25}{1.15} \approx 0.217$. To solve for P, write $P = Ae^{1.4kt}(1.4-P)$ and collect the P terms:

$$P(1 + Ae^{1.4kt}) = 1.4Ae^{1.4kt} \implies P = \frac{1.4Ae^{1.4kt}}{1 + Ae^{1.4kt}}.$$

Divide the top and the bottom by $Ae^{1.4kt}$. Since $\dfrac{1}{A} = \dfrac{1.15}{0.25} = 4.6$,

$$\boxed{P(t) = \frac{1.4}{1 + 4.6\,e^{-1.4kt}}}$$

This is the logistic **S-curve** (a curve shaped like a stretched letter S): growth that is nearly exponential at first, then approaches the carrying capacity of 1.4 million. The bracket $(1.4-P)$ is what slows it, since it decreases towards zero as P approaches 1.4. The real herd behaved this way, growing to about 1.4 million by 1977 and staying close to that level since. Growth is fastest when $P = 0.7$, half the carrying capacity, because $P(1.4-P)$ is largest there.

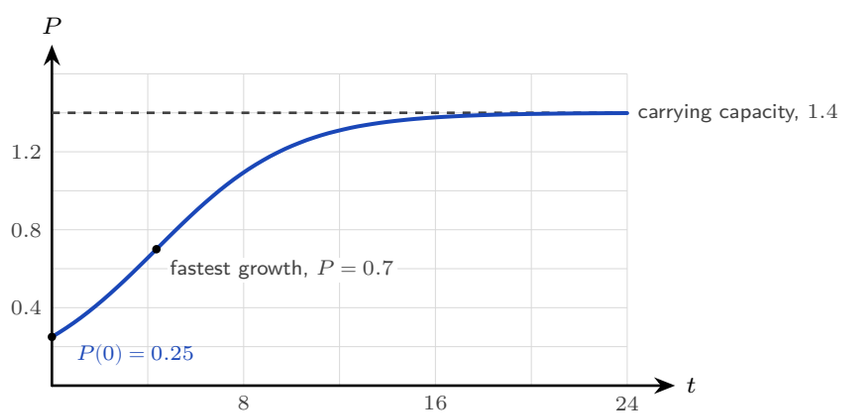


Figure 16.4 The wildebeest solution $P(t)$ with $k = 0.25$, a value chosen for illustration, and t in years after 1961.

YOUR TURN 16.1 Separable Variables

Recognising the type of equation

1. Each equation is first order. State whether it is linear and whether it is separable, giving a reason for each.

(a) $\dfrac{dy}{dx} = x^2 y$

(b) $\dfrac{dy}{dx} = y^2 + x$

(c) $\dfrac{dy}{dx} + y\cos x = \sin x$

(d) $y\dfrac{dy}{dx} = x$

General solutions

2. Find the general solution of each equation.

(a) $\dfrac{dy}{dx} = \dfrac{x}{y}$

(b) $\dfrac{dy}{dx} = y$

(c) $\dfrac{dy}{dx} = \dfrac{1+y}{1+x}$

(d) $\dfrac{dy}{dx} = e^{x-y}$

(e) $\dfrac{dy}{dx} = x(y-2)$. Show also that $y = 2$ is a solution, and explain why dividing by $y - 2$ loses it.

Particular solutions and where they are valid

3. Find the particular solution of each equation.

(a) $\dfrac{dy}{dx} = xy$ with $y(0) = 2$.

(b) $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 1$. State the interval containing $x = 0$ on which the solution is valid.

(c) $\dfrac{dy}{dx} = y\tan x$ with $y(0) = 4$, for $0 \le x < \frac{\pi}{2}$.

(d) $\dfrac{dy}{dx} = y^2\cos x$ with $y(0) = 1$. State the interval containing $x = 0$ on which the solution is valid.

(e) Explain why a general solution contains one arbitrary constant, and what the initial condition does to it.

Models

4. A metal bar is taken from a furnace at $300°$C into a room at $25°$C. Its temperature $T\,°$C after t minutes satisfies $\dfrac{dT}{dt} = -0.05(T - 25)$.

(a) Solve the equation to find T in terms of t.

(b) Find the time the bar takes to cool to 100°C.

5. The number N of bacteria in a culture grows at a rate proportional to N. Initially $N = 500$, and after 2 hours $N = 800$.

(a) Write down a differential equation for N, and solve it to find N in terms of t, the time in hours.

(b) Find N after 5 hours.

6. A population of P thousand animals satisfies $\dfrac{dP}{dt} = 0.3P(2 - P)$, where t is the time in years, and $P(0) = 0.5$.

(a) Use partial fractions to show that $P = \dfrac{2}{1 + 3e^{-0.6t}}$.

(b) Find the time at which $P = 1.5$, and state the value that P approaches as t increases.

Concept check

7. A student solves $\dfrac{dy}{dx} = xy$, for $y > 0$, and writes

$$\ln y = \tfrac{1}{2}x^2 + C \quad \Longrightarrow \quad y = e^{x^2/2} + C.$$

Identify the error, and give the correct general solution.

16.2 Homogeneous Differential Equations

Equations of this kind appear when only the ratio $\dfrac{y}{x}$ matters, as in **pursuit curves** (the path of something that always aims at a target), such as the ferry of the challenge problems, and in the shapes of reflectors.

Some equations cannot be separated as they stand, but a substitution makes them separable.

The aim: replace the repeated ratio $\dfrac{y}{x}$ by a single variable v, so that the equation separates.

A first-order equation is **homogeneous** if the right-hand side depends only on the ratio $\dfrac{y}{x}$, and not on x and y separately.

The substitution $y = vx$ turns such an equation into a separable one. Here v is a function of x, not a constant, so differentiating $y = vx$ needs the product rule: $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

KEY 16.2 **Homogeneous equations**

For $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, substitute $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$.

After the substitution the equation becomes $v + x\frac{dv}{dx} = f(v)$, that is $x\frac{dv}{dx} = f(v) - v$. The variables v and x can now be separated. Separating divides by $f(v) - v$, so first check whether $f(v) = v$ for some value $v = m$. Each such value gives a straight-line solution $y = mx$, which the division would lose. The method has four steps.

1. Check that the right side depends only on $\frac{y}{x}$: divide its top and bottom by x^n, where n is the degree of every term (for example x^2 for $\frac{2x^2 + y^2}{xy}$).
2. Substitute $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$.
3. Check for values $v = m$ with $f(m) = m$ (each gives a solution $y = mx$), then separate the variables and integrate.
4. Replace v by $\frac{y}{x}$. The substitution is only a step: the answer must be in x and y.

EXAMPLE 16.4

Solve $\frac{dy}{dx} = \frac{2x^2 + y^2}{xy}$ for $x > 0$.

SOLUTION

Divide the top and bottom by x^2 to show that the right side depends only on the ratio:

$$\frac{2x^2 + y^2}{xy} = \frac{2 + (y/x)^2}{y/x}.$$

Substitute $y = vx$, so $\frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{2 + v^2}{v} \implies x\frac{dv}{dx} = \frac{2 + v^2}{v} - v = \frac{2}{v}.$$

The terms in v cancel, leaving a separable equation:

$$\int v\,dv = \int \frac{2}{x}\,dx \implies \frac{v^2}{2} = 2\ln x + C.$$

Replace $v = \frac{y}{x}$ and clear the fraction:

$$\boxed{y^2 = 2x^2(2\ln x + C)}$$

In the example above the terms in v cancelled, leaving $\frac{2}{v}$, which is easy to integrate. When v does not cancel, the integral in v needs one of the methods of Ch. 15, as in the next example.

EXAMPLE 16.5

Solve $\dfrac{dy}{dx} = \dfrac{y^2 - x^2}{2xy}$ for $x > 0$, given $y(1) = 2$.

SOLUTION

Divide the top and bottom by x^2 and substitute $y = vx$:

$$v + x\frac{dv}{dx} = \frac{v^2 - 1}{2v} \implies x\frac{dv}{dx} = \frac{v^2 - 1}{2v} - v = \frac{v^2 - 1 - 2v^2}{2v} = -\frac{1 + v^2}{2v}.$$

This time v does not cancel. Separate the variables, with every v on the left:

$$\int \frac{2v}{1 + v^2}\, dv = -\int \frac{1}{x}\, dx \implies \ln(1 + v^2) = -\ln x + C.$$

The left side is the form $\dfrac{f'(v)}{f(v)}$ of §15.4. Exponentiate, writing $A = e^C$, then replace v by $\dfrac{y}{x}$:

$$1 + v^2 = \frac{A}{x} \implies 1 + \frac{y^2}{x^2} = \frac{A}{x} \implies x^2 + y^2 = Ax.$$

The condition $y(1) = 2$ gives $1 + 4 = A$, so the particular solution is $x^2 + y^2 = 5x$. Since $y(1) = 2$ is positive, the solution is the upper branch

$$y = \sqrt{5x - x^2}, \quad 0 < x < 5$$

At both ends of the interval $y = 0$, where the equation is undefined. You can also apply the condition before replacing v: at $x = 1$, $v = \frac{y}{x} = 2$, and $1 + v^2 = \frac{A}{x}$ again gives $A = 5$.

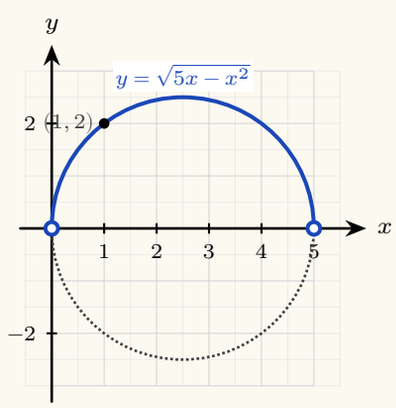


Figure 16.5 The particular solution lies on the circle $x^2 + y^2 = 5x$. The condition $y(1) = 2$ selects the upper half; it ends at $x = 0$ and $x = 5$, where $y = 0$ and the equation is undefined.

YOUR TURN 16.2 Homogeneous Differential Equations

Recognising a homogeneous equation

8. State whether each equation is homogeneous, giving a reason.

(a) $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$

(b) $\dfrac{dy}{dx} = \dfrac{x^2+y}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{y^2+xy}{x^2}$

General solutions

9. Use the substitution $y = vx$ to solve each equation, for $x > 0$.

(a) $\dfrac{dy}{dx} = \dfrac{y}{x}$

(b) $\dfrac{dy}{dx} = \dfrac{y-x}{x}$

(c) $\dfrac{dy}{dx} = \dfrac{x-y}{x}$

(d) $\dfrac{dy}{dx} = \dfrac{x+2y}{x}$

10.

(a) Show that $\dfrac{dy}{dx} = \dfrac{x^2+3y^2}{2xy}$ is homogeneous, by writing the right-hand side in terms of $\dfrac{y}{x}$.

(b) Hence find its general solution, for $x > 0$.

Particular solutions

11. Find the particular solution of each equation, for $x > 0$.

(a) $\dfrac{dy}{dx} = \dfrac{2y-x}{x}$ with $y(1) = 3$.

(b) $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}$ with $y(1) = 2$.

(c) State the largest interval containing $x = 1$ on which the solution in part (b) is valid.

Concept check

12. A student solves $\dfrac{dy}{dx} = \dfrac{x+y}{x}$, for $x > 0$, with the substitution $y = vx$, and writes $\dfrac{dy}{dx} = \dfrac{dv}{dx}$, so that $\dfrac{dv}{dx} = 1 + v$. Identify the error, and find the correct general solution.

16.3 The Integrating Factor

A linear equation appears when a quantity is added and removed at the same time. Examples are a mixing tank, a drug drip, a circuit with a resistor and a capacitor, and savings with steady deposits. In a mixing tank whose volume changes, the result is often a linear equation that needs an integrating factor.

The aim: multiply by a function that turns the left side into the derivative of a single product.

A linear first-order differential equation, as defined in §16.1, has the form

$$\frac{dy}{dx} + P(x)\,y = Q(x).$$

It is usually not separable. The method is to multiply through by a chosen function, the **integrating factor**. The coefficient of $\frac{dy}{dx}$ **must be** 1 **first**: if the equation starts as $x\frac{dy}{dx} + 2y = x^3$, divide by x before reading off $P(x)$.

Why does multiplying help? Multiply the equation by a function $I(x)$, and write I' for $\frac{dI}{dx}$ and y' for $\frac{dy}{dx}$. The left side becomes $Iy' + IPy$, and the product rule gives $\frac{d}{dx}(Iy) = Iy' + I'y$. Set one above the other:

$$\frac{d}{dx}(Iy) = Iy' + I'\,y,$$
$$I(y' + Py) = Iy' + IP\,y.$$

The two are equal if the coloured terms are equal, that is if $I' = PI$, and $I = e^{\int P\,dx}$ has exactly that derivative.

FORMULA 16.3 **Integrating factor** FORMULA BOOKLET · 5.18

For $y' + P(x)y = Q(x)$, the integrating factor is

$$e^{\int P(x)\,dx}.$$

So, with this I, the left side is a single derivative, $\frac{d}{dx}(Iy)$. The method has six steps.

1. Write the equation as $y' + P(x)y = Q(x)$, with the coefficient of y' equal to 1.
2. Find $I = e^{\int P(x)\,dx}$.
3. Multiply **every** term by I.
4. Write the left side as $\frac{d}{dx}(Iy)$.
5. Integrate both sides, then divide by I.
6. Apply the initial condition, if given, and check the solution.

No constant is needed when you find I. A constant would only multiply every term of the equation by the same number, which does not change the solution. The constant C enters at step 5.

If $\int P\,dx$ gives $\ln|g(x)|$, then $I = |g(x)|$. On an interval where $g(x) > 0$, simply use $I = g(x)$. For example, $P(x) = \tan x$ gives $\int \tan x\,dx = -\ln|\cos x|$ §15.4, so $I = \dfrac{1}{\cos x} = \sec x$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

EXAMPLE 16.6

Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$ for $x > 0$.

SOLUTION

Here $P(x) = \frac{1}{x}$, so the integrating factor is

$$I = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = x.$$

Multiply through by x:

$$x\frac{dy}{dx} + y = x^3 \implies \frac{d}{dx}(xy) = x^3.$$

The left side is now a single derivative. Integrate both sides:

$$xy = \frac{x^4}{4} + C \implies \boxed{y = \frac{x^3}{4} + \frac{C}{x}}$$

The last example of this section is a mixing tank, the first application in the list above. To model a well-stirred mixing tank, write $\dfrac{dS}{dt} = (\text{rate in}) - (\text{rate out})$, where S is the amount of dissolved substance. It enters at the inflow concentration times the inflow rate, and leaves at the current concentration $\dfrac{S}{V}$ times the outflow rate. When the two flow rates differ, the volume changes too: $V(t) = V_0 + (\text{inflow rate} - \text{outflow rate})\,t$.

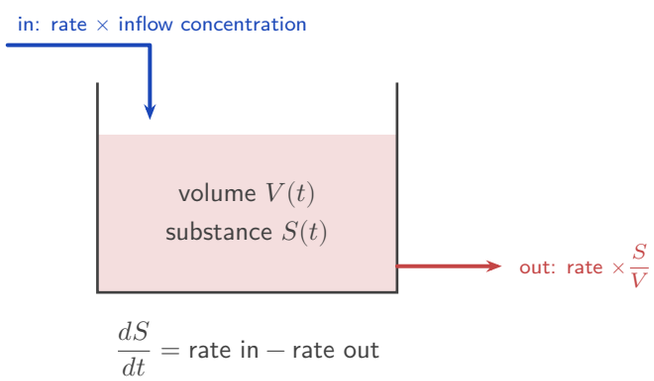


Figure 16.6 A well-stirred tank. Substance enters at the inflow concentration and leaves at the tank's own concentration $\frac{S}{V}$.

EXAMPLE 16.7

A vat in a brewery contains 100 litres of liquid holding 5 kg of dissolved sugar. Syrup carrying 0.2 kg of sugar per litre flows in at 1 litre per minute, the vat is kept well stirred, and the mixture drains at 2 litres per minute. Find the sugar $S(t)$ in the vat at time t minutes, for $0 \le t < 100$.

SOLUTION

The volume shrinks: $V(t) = 100 - t$. Sugar arrives at $0.2 \times 1 = 0.2$ kg min^{-1} and leaves at concentration times outflow, $\dfrac{S}{100-t} \times 2$. So

$$\frac{dS}{dt} = 0.2 - \frac{2S}{100-t} \implies \frac{dS}{dt} + \frac{2}{100-t}\,S = 0.2.$$

The integrating factor is

$$I = e^{\int \frac{2}{100-t}\,dt} = e^{-2\ln(100-t)} = (100-t)^{-2}.$$

The minus sign appears because $\dfrac{d}{dt}\ln(100-t) = -\dfrac{1}{100-t}$. Multiplying through turns the left side into a single derivative:

$$\frac{d}{dt}\Big(S\,(100-t)^{-2}\Big) = 0.2\,(100-t)^{-2} \implies S\,(100-t)^{-2} = \frac{0.2}{100-t} + C.$$

So $S = 0.2(100-t) + C(100-t)^2$, and $S(0) = 5$ gives $5 = 20 + 10\,000\,C$, so $C = -0.0015$:

$$\boxed{S(t) = 0.2(100-t) - 0.0015(100-t)^2}$$

Check the ends: $S(0) = 5$, and $S(t) \to 0$ as $t \to 100^-$, as the vat empties. The equation itself is undefined at $t = 100$, where the volume is zero, so the solution holds for $0 \le t < 100$. In between, the sugar first *rises*, to about 6.7 kg near $t = 33$, and then falls. That rise is not obvious from the description. You find it by solving the equation, not by intuition.

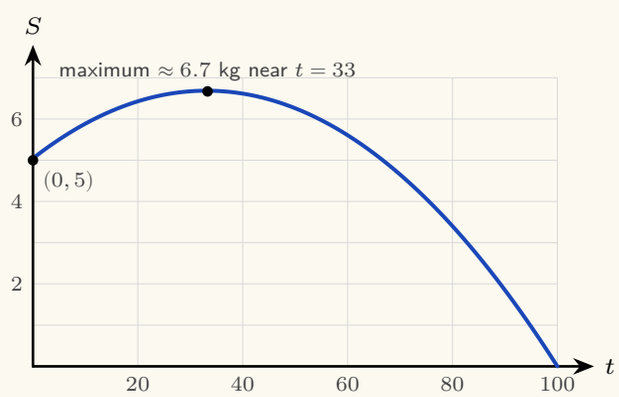


Figure 16.7 The sugar $S(t)$ in the vat. It rises from 5 kg to about 6.7 kg, because at first the syrup brings sugar in faster than the outflow removes it. Then it falls to 0 as the vat empties at $t = 100$.

16.3.1 Choosing a method

Three exact methods are now available, and a question may not say which to use. You choose the method from the form of the equation, and the test is quick. The table sets the three exact methods side by side, with the routes for an equation that fits none of them.

A sensible order is the order of the table below: when an equation separates, separating is usually the shortest route. The methods can overlap. An equation can be both separable and homogeneous; then separating directly is shorter. A linear equation can also be separable. When Q/P is a constant, $\frac{dy}{dx} + P(x)y = Q(x)$ can be written $\frac{dy}{dx} = P(x)\left(\frac{Q}{P} - y\right)$, which separates. This is why $\frac{dy}{dx} + y = 4$ can be separated, and why the cooling equation of §16.1 could be. In general Q/P is not constant, the equation does not separate, and you need the integrating factor.

	Separable	**Homogeneous**	**Integrating factor**	**None of these**
Recognise	$\frac{dy}{dx} = f(x)\,g(y)$	right side depends only on $\frac{y}{x}$	$y' + P(x)y = Q(x)$	fits none of the three
Key step	divide by $g(y)$, after checking where $g(y) = 0$	substitute $y = vx$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$	multiply by $I = e^{\int P\,dx}$	a series (§16.4) or Euler's method (§16.5)
Then	integrate both sides	separate v and x, then integrate	write the left side as $\frac{d}{dx}(Iy)$ and integrate	find derivatives at 0, or step from the start
Finish	apply the condition; state the interval	replace v by $\frac{y}{x}$; apply the condition	divide by I; apply the condition	judge the accuracy
Example	$\frac{dy}{dx} = 2xy^2$	$\frac{dy}{dx} = \frac{x+y}{x}$	$\frac{dy}{dx} + \frac{1}{x}y = x^2$	$\frac{dy}{dx} = x^2 + y^2$

Whichever method you use, finish by applying any initial condition and checking the solution in the original equation.

YOUR TURN 16.3 The Integrating Factor

Solving a linear equation

13. Solve each linear equation.

(a) $\dfrac{dy}{dx} + y = 4$

(b) $\dfrac{dy}{dx} + 2y = 6$

(c) $\dfrac{dy}{dx} + 2y = 4x$

(d) $\dfrac{dy}{dx} - y = e^{2x}$

14.

(a) Write down the integrating factor for $\dfrac{dy}{dx} + \dfrac{3}{x}y = x^2$, for $x > 0$.

(b) Solve $\dfrac{dy}{dx} + \dfrac{2}{x}y = x$ for $x > 0$.

(c) Solve $\dfrac{dy}{dx} - \dfrac{1}{x}y = x^2$ with $y(1) = 1$, for $x > 0$.

(d) Explain what the integrating factor is chosen to achieve on the left-hand side.

Making the coefficient of $\frac{dy}{dx}$ equal to 1 first

15. Divide by the coefficient of $\dfrac{dy}{dx}$ first, then solve each equation.

(a) $x\dfrac{dy}{dx} + 3y = x^2$, for $x > 0$.

(b) $(x^2 + 1)\dfrac{dy}{dx} + 2xy = 4x$

(c) $\dfrac{dy}{dx} + y\tan x = \cos x$, for $0 \le x < \frac{\pi}{2}$.

(d) $\cos x\dfrac{dy}{dx} + y\sin x = 1$, for $0 \le x < \frac{\pi}{2}$.

A mixing tank

16. A tank holds 50 litres of water in which 2 kg of salt is dissolved. Brine containing 0.1 kg of salt per litre flows in at 3 litres per minute, the tank is kept well stirred, and the mixture drains at 2 litres per minute. Let S kg be the salt in the tank after t minutes.

(a) Show that $\dfrac{dS}{dt} + \dfrac{2S}{50 + t} = 0.3$.

(b) Solve the equation to find S in terms of t.

(c) Find the salt in the tank after 10 minutes.

Choosing a method

17. For each equation, state which method applies, and find the general solution.

(a) $\dfrac{dy}{dx} = xe^{-y}$

(b) $\dfrac{dy}{dx} - \dfrac{2y}{x} = x^3$, for $x > 0$.

(c) $\dfrac{dy}{dx} = \dfrac{y^2 + 2xy}{x^2}$, for $x > 0$.

(d) Dividing by an expression in v in part (c) can lose a straight-line solution $y = mx$. Find the straight-line solution, other than $y = 0$, that the method in part (c) does not give.

Concept check

18. A student solves $\dfrac{dy}{dx} + \dfrac{y}{x} = 1$, for $x > 0$, with the integrating factor x, and writes

$$xy = \tfrac{1}{2}x^2 + C \quad \Longrightarrow \quad y = \tfrac{1}{2}x + C.$$

Identify the error, and show that the student's answer does not satisfy the equation unless $C = 0$.

16.4 Maclaurin Series

Many functions can be written, at least for some values of x, as an endless sum of powers of x. This idea is different from the three methods above. It has two uses: it approximates a function, and it gives a fourth way to approach a differential equation.

A polynomial is wanted because it needs only addition and multiplication, which are fast to compute. A function such as $\sin x$ has no such direct method of calculation, so software often evaluates it from a short polynomial.

The aim: approximate a function near $x = 0$ by a polynomial. Start with the tangent line of Ch. 13,

$$f(x) \approx f(0) + f'(0)\,x,$$

which matches the function's value and gradient at 0. Adding a term in x^2 makes the polynomial match the second derivative as well, so it has the same concavity (direction of bending) at 0:

$$f(x) \approx f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2.$$

Matching more derivatives adds more terms. The more derivatives agree, the closer the polynomial stays to the curve near 0. For the standard functions of this chapter this is true only inside an interval, the interval on which the series **converges** (its sum approaches a finite value as more terms are added). The result is the **Maclaurin series**.

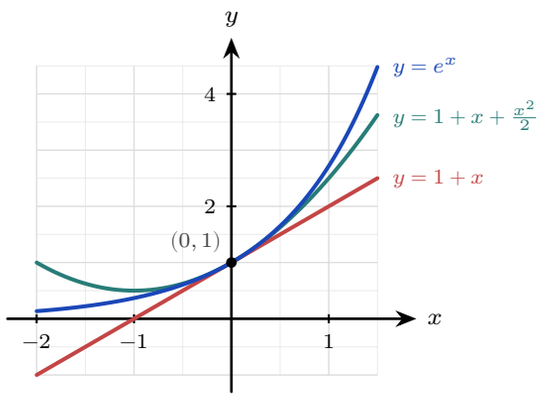


Figure 16.8 Near $x = 0$, $1 + x$ matches the value and gradient of e^x; adding $\frac{x^2}{2}$ also matches the concavity, and the curve and the polynomial stay close over a wider interval.

The coefficients are completely determined by the function. Suppose the series exists and write it as

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

Putting $x = 0$ makes every term except the first equal to zero, so $a_0 = f(0)$. Differentiating gives $f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$, and $x = 0$ now leaves $a_1 = f'(0)$. Differentiating again gives $f''(0) = 2a_2$, and once more $f'''(0) = 6a_3 = 3!\, a_3$. Each differentiation removes the constant term and multiplies each remaining coefficient by its power, so in general $f^{(n)}(0) = n!\, a_n$.

FORMULA 16.4 **Maclaurin series** FORMULA BOOKLET · 5.19

$$f(x) = f(0) + x\, f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

EXAMPLE 16.8

Find the Maclaurin series for $f(x) = e^x$.

SOLUTION

Every derivative of e^x is e^x, and $e^0 = 1$, so $f^{(n)}(0) = 1$ for all n:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

This series can be taken as the definition of e^x, and enough of its terms give the value of e^x to any accuracy you want. It also shows why e^x is its own derivative: differentiate the series term by term and you get the same series back.

As a first use, keep the terms up to x^2 at $x = 0.1$: $e^{0.1} \approx 1 + 0.1 + \frac{0.1^2}{2} = 1.105$, against the true value $1.10517\ldots$. Keep the two signs apart: the series is an equality,

$e^x = 1 + x + \frac{x^2}{2!} + \cdots$, while a **truncation** (the series cut off after a number of terms) is an approximation, $e^x \approx 1 + x + \frac{x^2}{2}$.

The series below are provided in the formula booklet. Each one equals its function exactly on an interval around 0, given in the note under the card. Truncating a series gives a polynomial approximation that is very accurate near $x = 0$. When x is small, each term is much smaller than the one before, so the smaller x is, the fewer terms you need.

FORMULA 16.5 **Standard Maclaurin series** FORMULA BOOKLET · 5.19

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad \ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \qquad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

$$\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

e^x, $\sin x$ and $\cos x$: valid for every x. $\ln(1+x)$: only for $-1 < x \leq 1$. $\arctan x$: only for $|x| \leq 1$.

Each of these comes from the Maclaurin series formula or from a series already found. For $\sin x$ the derivatives repeat in a cycle of four:

n	0	1	2	3	4
$f^{(n)}(x)$	$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$
$f^{(n)}(0)$	0	1	0	-1	0

So only the odd powers remain, and their signs alternate. For $\ln(1+x)$, start from the geometric series of Ch. 2: $\frac{1}{1+x} = 1 - x + x^2 - \cdots$ for $|x| < 1$. Integrate term by term, one of the operations in the next subsection. This gives $x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$. The constant of integration is 0, because $\ln 1 = 0$. The series for $\cos x$ and $\arctan x$ are built later in this section. The interval on which each series equals its function is stated without proof, because the proof that a series converges is beyond this course.

The binomial series $(1+x)^p$ is named by the syllabus but printed elsewhere in the booklet, so it is easy to miss. It is a Maclaurin series like any other, and it is the extended binomial theorem of Ch. 2 (booklet 1.10), now obtained as a Maclaurin series.

KEY 16.6 **Binomial series**

For $p \in \mathbb{Q}$,

$$(1+x)^p = 1 + px + \frac{p(p-1)}{2!}x^2 + \frac{p(p-1)(p-2)}{3!}x^3 + \cdots.$$

Like $\ln(1+x)$ and $\arctan x$, it needs a condition: unless p is a non-negative integer, use it only for $|x| < 1$.

The coefficients come from the Maclaurin series formula. Differentiating $f(x) = (1+x)^p$ a total of n times gives $f^{(n)}(x) = p(p-1)\cdots(p-n+1)(1+x)^{p-n}$, so $f^{(n)}(0) = p(p-1)\cdots(p-n+1)$. Dividing by $n!$ gives the coefficient of x^n.

EXAMPLE 16.9

Use a Maclaurin series to approximate $\sqrt{1.1}$ to four decimal places.

SOLUTION

Write $\sqrt{1.1} = (1+x)^{1/2}$ with $x = 0.1$, which satisfies $|x| < 1$. With $p = \frac{1}{2}$,

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots.$$

Substituting $x = 0.1$ gives

$$1 + 0.05 - 0.00125 + 0.0000625 = 1.0488125.$$

The true value is $\sqrt{1.1} = 1.0488088\ldots$, so four terms agree with it to four decimal places: $\sqrt{1.1} \approx \boxed{1.0488}$

The figure below shows the first three odd-degree truncations of the sine series: the line $y = x$, the cubic, and the quintic. Each new term extends the interval over which the polynomial and the curve agree.

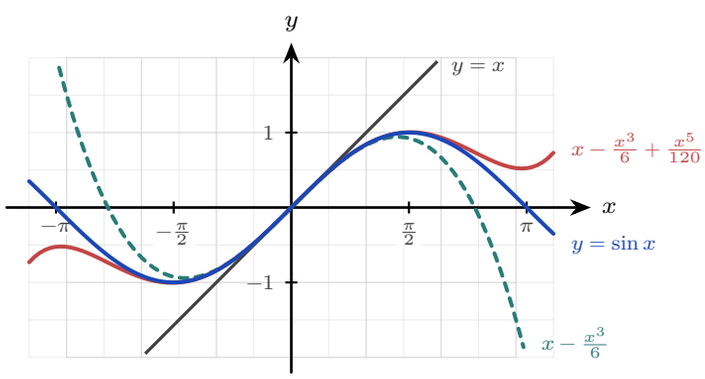


Figure 16.9 The first three odd-degree truncations of the sine series against $y = \sin x$. Each truncation moves away from the curve as $|x|$ grows: the cubic near $x = 2$, the quintic before $x = \pi$.

The same convergence can be watched numerically. Below are the successive truncations of the sine series at $x = 1$, against the true value $\sin 1 = 0.841471$.

terms up to	x	x^3	x^5	x^7
estimate	1	0.833333	0.841667	0.841468
error	0.159	0.0081	0.0002	0.0000027

Each new term divides the error by a factor of about 20 or more. The ratios along this row are about 20, 40 and 70, so the factor grows as more terms are taken. So a short polynomial is easy to compute, and each extra term gives more accuracy for a little more work.

16.4.1 Building new series

New series are usually not built from the definition. They are made from the standard five by four operations, all of which the syllabus names. Each operation is valid for values of x inside the interval on which the series converge. The proof of that is beyond this course.

- **Substitute.** Replacing x with $-x^2$ in the series for e^x gives

$$e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

- **Differentiate.** Differentiating the sine series term by term gives the cosine series:

$$\frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

  This is the identity $\frac{d}{dx}\sin x = \cos x$ in polynomial form.
- **Integrate.** The geometric series of Ch. 2 gives $\dfrac{1}{1+x^2} = 1 - x^2 + x^4 - \cdots$ for $|x| < 1$, and integrating term by term yields $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$. The constant of integration is 0 because $\arctan 0 = 0$.
- **Multiply.** For a product, decide the highest power you need first. Cut each factor's series off at that power, and when you multiply, skip every product of higher degree.

Before using a series, check four things:

1. its interval of validity;
2. the highest power, or the number of non-zero terms, that the question asks for;
3. brackets when you substitute: replacing x by $2x$ turns x^n into $(2x)^n$;
4. enough terms kept for the calculation, before you simplify.

EXAMPLE 16.10

Find the Maclaurin series of $e^x \sin x$ up to the term in x^3.

SOLUTION

Multiply the two standard series, keeping only terms of degree ≤ 3:

$$\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}\right)\left(x-\frac{x^3}{6}\right)=x+x^2+\left(\frac{1}{2}-\frac{1}{6}\right)x^3+\cdots=\boxed{x+x^2+\frac{x^3}{3}+\cdots}$$

16.4.2 Small angles, limits, and series from differential equations

Truncating at the first term or two gives the small-angle approximations Ch. 8, which are used widely in physics. For θ near 0, in radians (see the line $y=x$ in the sine figure above),

$$\sin\theta\approx\theta,\qquad\cos\theta\approx1-\frac{\theta^2}{2},\qquad\tan\theta\approx\theta.$$

The third follows from the first two: $\tan\theta=\dfrac{\sin\theta}{\cos\theta}\approx\dfrac{\theta}{1}$ when only the first term of each series is kept. The usual formula for the period of a pendulum, $T=2\pi\sqrt{\frac{l}{g}}$, depends on the first of them. The force that pulls a pendulum back towards the vertical depends on $\sin\theta$. For small swings, replacing $\sin\theta$ by θ turns the pendulum's equation into one that can be solved exactly; physics calls the result simple harmonic motion.

The series also gives the size of the error. The first term discarded is $\frac{\theta^3}{6}$, so that is roughly the error. Dividing by θ gives the **relative error** (the error divided by the true value), about $\frac{\theta^2}{6}$, which stays under 1% for swings up to about 14° (0.245 radians). For larger swings the approximation is less accurate, and the real period of a pendulum differs noticeably from the formula.

Truncated series can also be used to find an **indeterminate** limit (one where substitution gives a form such as $\frac{0}{0}$): replace each function by the first terms of its series, and read off the limit.

EXAMPLE 16.11

Use Maclaurin series to evaluate $\displaystyle\lim_{x\to0}\frac{e^x-1-x}{x^2}$.

SOLUTION

Expand the numerator with the series for e^x:

$$e^x-1-x=\left(1+x+\frac{x^2}{2}+\frac{x^3}{6}+\cdots\right)-1-x=\frac{x^2}{2}+\frac{x^3}{6}+\cdots$$

Divide by x^2:

$$\frac{e^x-1-x}{x^2}=\frac{1}{2}+\frac{x}{6}+\cdots\xrightarrow{x\to0}\boxed{\frac{1}{2}}$$

This method is as direct as l'Hôpital's rule, and you can also see *why* the answer is $\frac{1}{2}$: it is the leading coefficient.

A Maclaurin series can also be built directly *from a differential equation*, by differentiating the equation itself to obtain the derivatives at 0.

EXAMPLE 16.12

The function y satisfies $\dfrac{dy}{dx} = 1 + y^2$ with $y(0) = 0$. Find its Maclaurin series up to the term in x^3.

SOLUTION

Take the derivatives at 0 from the equation itself. First, $y'(0) = 1 + 0 = 1$. Differentiating the equation,

$$y'' = 2y\,y' \implies y''(0) = 0, \qquad y''' = 2(y')^2 + 2y\,y'' \implies y'''(0) = 2.$$

Substitute these into the Maclaurin series formula:

$$y = 0 + x + \frac{0}{2!}x^2 + \frac{2}{3!}x^3 + \cdots = \boxed{x + \frac{x^3}{3} + \cdots}$$

You may recognise the answer: $y = \tan x$ solves this equation, since $\frac{d}{dx}\tan x = \sec^2 x = 1 + \tan^2 x$, and these are indeed the first terms of its series. You found the series without solving the equation.

TIP Maclaurin series suggest several good IA topics. One is Madhava's series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$, which comes from setting $x = 1$ in the arctangent series. It converges slowly, and a better choice such as $\arctan \frac{1}{\sqrt{3}}$ reaches the same accuracy in a fraction of the terms. Another is how software evaluates $\sin x$ quickly: it first reduces the angle to a small interval, then uses a short polynomial.

YOUR TURN 16.4 Maclaurin Series

Using the standard series

19. Find the first three non-zero terms of the Maclaurin series for each function.

(a) e^{2x} (b) $\cos 3x$

(c) $\ln(1 + 2x)$ (d) e^{-x^2}

20. Find the first two non-zero terms of the Maclaurin series for each function.

(a) $\sin 2x$ (b) $\sin(x^2)$

(c) $\arctan 2x$ (d) $\sqrt{1 + x}$

Series from the definition

21. Use the Maclaurin series formula $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$, differentiating each function yourself, to find its series up to the term in x^4.

(a) $f(x) = \cos 2x$

(b) $f(x) = \ln(\cos x)$

Products of series

22. Find the Maclaurin series of each product up to the term in x^3.

(a) $e^x \cos x$

(b) $e^x \ln(1 + x)$

Differentiating and integrating a series

23. GDC

(a) Differentiate the series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$ term by term, and verify that the result is the expansion of $\frac{1}{1+x^2}$.

(b) Differentiate the series for $\ln(1 + x)$ term by term, and verify that the result is the expansion of $\frac{1}{1+x}$.

(c) Integrate the geometric series $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ term by term to find the first four terms of the series for $-\ln(1 - x)$, for $|x| < 1$.

(d) Use the first three non-zero terms of the series for e^{-x^2} to estimate $\int_0^{0.5} e^{-x^2}\,dx$, giving your estimate to four decimal places.

Approximating values

24. GDC

(a) Use the first three terms of the series for e^x to estimate $e^{0.1}$.

(b) Use the first three terms of the cosine series to estimate $\cos 0.2$ to four decimal places.

(c) Compare each estimate with the true value, and state which is the more accurate. Explain why.

25. The small-angle approximations are $\sin\theta \approx \theta$, $\cos\theta \approx 1 - \frac{\theta^2}{2}$ and $\tan\theta \approx \theta$, for θ in radians close to 0.

(a) Find the value that $\frac{\sin 3\theta \tan\theta}{1 - \cos 2\theta}$ is close to when θ is small.

(b) Show that $\cos\theta + \theta\sin\theta \approx 1 + \frac{\theta^2}{2}$ when θ is small.

(c) Show that the relative error in $\sin\theta \approx \theta$ is about $\dfrac{\theta^2}{6}$, and find the largest θ for which this estimate of the relative error is below 1%.

Limits by series

26. Use Maclaurin series to evaluate each limit.

(a) $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x^2}$

(b) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)-x}{x^2}$

(c) $\lim\limits_{x\to 0} \dfrac{x-\arctan x}{x^3}$

Series from a differential equation

27. GDC A function satisfies $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$.

(a) Write down $y'(0)$.

(b) By differentiating the equation, find $y''(0)$ and $y'''(0)$.

(c) Hence find the Maclaurin series for y up to the term in x^3.

(d) The exact solution is $y = 2e^x - x - 1$. Use the series to estimate $y(0.3)$, and compare it with the exact value.

28. A function satisfies $\dfrac{dy}{dx} = x - y^2$ with $y(0) = 1$.

(a) Find $y'(0)$, $y''(0)$ and $y'''(0)$.

(b) Hence find the Maclaurin series for y up to the term in x^3.

Concept check

29. A student replaces x with $3x$ in the cosine series and writes $\cos 3x = 1 - \dfrac{3x^2}{2} + \dfrac{3x^4}{24} - \cdots$. Identify the error, and give the correct first three terms.

30. GDC A student estimates $\ln 3$ by putting $x = 2$ into $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$, and gets $2 - 2 + \frac{8}{3} \approx 2.67$. Explain why this estimate is not valid, and why taking more terms does not help.

16.5 Euler's Method

Not every differential equation can be solved exactly, and many of the equations that describe the real world cannot. When a solution cannot be written with the functions available, it can still be approximated numerically. **Euler's method** does this in short straight steps that stay close to the curve, taking the gradient of each step from the differential equation itself.

The aim: use the current gradient to estimate the next value, since the change in y is about the gradient times the step length. The idea is the tangent line of Ch. 13. Near a known point the curve is close to its tangent, so moving a small distance h along the tangent ends close to the curve:

$$y \approx y_0 + \underbrace{f(x_0, y_0)}_{\text{gradient}} \times \underbrace{h}_{\text{step}} .$$

At each new point you find the gradient from the differential equation, so you can repeat the same move from that point, and again from the one after it.

FORMULA 16.7 **Euler's method** FORMULA BOOKLET · 5.18

$$y_{n+1} = y_n + h \times f(x_n, y_n), \qquad x_{n+1} = x_n + h,$$

where h is a constant (step length).

The **step length** h stays the same at every step. Each step has the correct gradient only at its starting point, and it moves away from the curve after that. Shorter steps stay closer to the curve, but more steps are then needed.

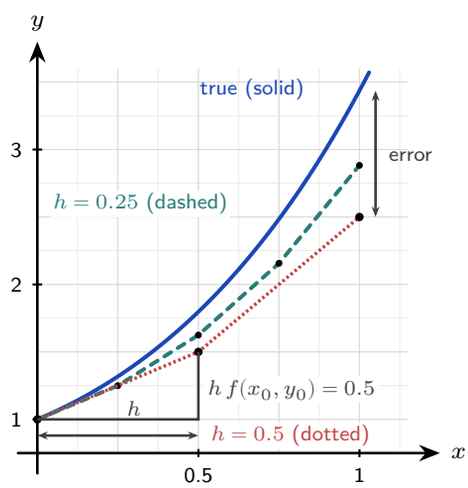


Figure 16.10 Euler's method on $\frac{dy}{dx} = x + y$, $y(0) = 1$: the true curve (solid), steps of $h = 0.5$ (dotted) and $h = 0.25$ (dashed). The curve bends up away from each step, so the estimate is too small. Halving h reduces the error at $x = 1$ from about 0.94 to 0.55.

The *error* is the difference between the true value and the estimate. In the figure it is the gap at $x = 1$. The dashed path takes steps half as long, and it ends closer to the curve.

EXT A **slope field** draws, at many points, a short segment with the gradient that the equation gives there. Every solution curve follows the segments, so one picture shows the whole family of solutions, and each Euler step moves along one segment.

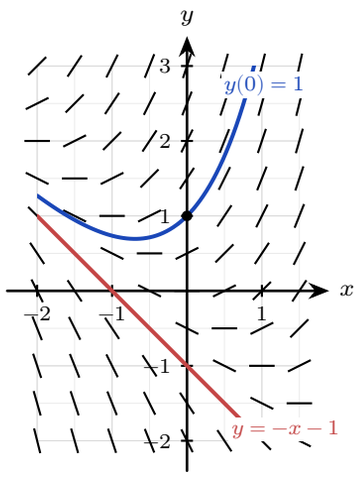


Figure 16.11 The slope field of $\frac{dy}{dx} = x + y$, with the solution through $(0, 1)$ and the straight-line solution $y = -x - 1$.

EXT The direction of the error in a *single* step follows from the concavity. A tangent drawn at a point of the curve lies below the curve where the curve is concave up, and above it where the curve is concave down. So the first step underestimates or overestimates accordingly. Later steps start from points that are already off the curve. So after many steps the same direction of error is usual when the step length is small, but it is not guaranteed. Comparing the results for h and $\frac{h}{2}$ shows how much the step length matters.

Each step makes a small error of its own, and these errors add up. So the estimate usually moves further from the curve with each step. An exception is an equation whose solutions approach a steady value. Keep full calculator precision from step to step and round only the final answer.

EXAMPLE 16.13

For the equation in the figure, $\dfrac{dy}{dx} = x + y$ with $y(0) = 1$, use Euler's method with $h = 0.1$ to estimate $y(0.3)$.

SOLUTION

Apply $y_{n+1} = y_n + h(x_n + y_n)$ three times. A table organises the work and shows each step to whoever marks it.

n	x_n	y_n	gradient $x_n + y_n$	change $h\,f$	next y_{n+1}
0	0	1	1	0.1	1.1
1	0.1	1.1	1.2	0.12	1.22
2	0.2	1.22	1.42	0.142	1.362

Each row adds the change $h\,f$ to y_n, and the result starts the next row. So $y(0.3) \approx \boxed{1.362}$

The exact value is about 1.3997, so the estimate is too small. This is what the concavity predicts: this solution is concave up.

The method matters most when no exact solution is available. Suppose the air resistance on a falling body depends on its shape and on the air density, which changes with height. The equation then usually has no solution that can be written with the functions of this course, but Euler's method still gives a table of values. Weather forecasts are computed in the same way, on a much larger scale. The equations are too hard to solve exactly, so a computer steps them forward numerically, millions of times.

A quantity stays at a steady value only where its rate of change is zero. For an equation $\frac{dv}{dt} = f(v)$, solve $f(v) = 0$ to find the **equilibrium** value (the steady value; for a falling body, its terminal velocity); this does not require solving the differential equation.

EXAMPLE 16.14

A skydiver falls from rest, and air resistance gives $\frac{dv}{dt} = 9.8 - 0.1v^2$ (in m s^{-1} and seconds). Use Euler's method with $h = 0.5$ to estimate the speed after 1.5 seconds, and compare with the terminal velocity.

SOLUTION

Apply $v_{n+1} = v_n + 0.5(9.8 - 0.1v_n^2)$ three times from $v_0 = 0$:

$$v_1 = 0 + 0.5(9.8) = 4.9$$
$$v_2 = 4.9 + 0.5(9.8 - 0.1(4.9)^2) = 4.9 + 0.5(7.399) = 8.5995$$
$$v_3 = 8.5995 + 0.5(9.8 - 0.1(8.5995)^2) = 8.5995 + 0.5(2.404859975) = 9.8019\ldots$$

So $v(1.5) \approx \boxed{9.80\ \text{m s}^{-1}}$

The terminal velocity is where the acceleration is zero:
$9.8 - 0.1v^2 = 0 \implies v = \sqrt{98} \approx \boxed{9.90\ \text{m s}^{-1}}$

The exact solution gives $v(1.5) \approx 8.93\ \text{m s}^{-1}$, so this estimate is about 10% too high. This is what the concavity predicts: the graph of v against t is concave down, so the tangent steps tend to lie above the curve and Euler's method overestimates. Each bracket $9.8 - 0.1v_n^2$ is the net force per unit mass, and it decreases towards zero as the speed increases.

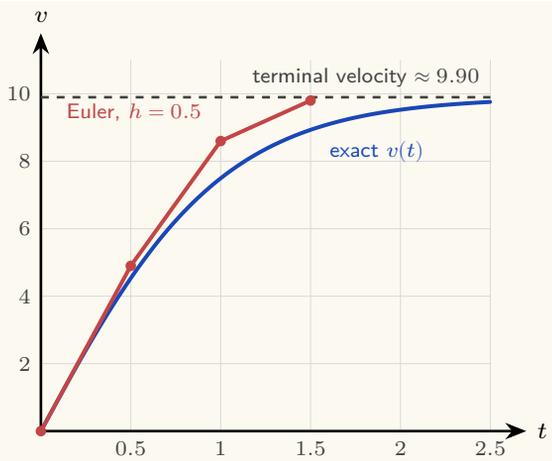


Figure 16.12 The skydiver. The Euler steps (red) lie above the exact curve, because the curve is concave down, and both approach the terminal velocity $\sqrt{98} \approx 9.90$ (dashed).

YOUR TURN 16.5 Euler's Method

Carrying out Euler's method

31. GDC Use Euler's method for each of the following.

(a) $\dfrac{dy}{dx} = 2x$, $y(0) = 1$; use $h = 0.5$ to estimate $y(1)$.

(b) $\dfrac{dy}{dx} = y$, $y(0) = 1$; use $h = 0.1$ to estimate $y(0.2)$.

(c) $\dfrac{dy}{dx} = x - y$, $y(0) = 1$; use $h = 0.2$ to estimate $y(0.4)$.

(d) $\dfrac{dy}{dx} = xy$, $y(1) = 1$; use $h = 0.1$ to estimate $y(1.2)$.

32. GDC

(a) For $\dfrac{dy}{dx} = xy$ with $y(1) = 2$, take one step with $h = 0.2$ to estimate $y(1.2)$.

(b) For $\dfrac{dy}{dx} = x^2 + y$ with $y(0) = 0$, take two steps with $h = 0.5$ to estimate $y(1)$.

(c) For $\dfrac{dy}{dx} = x + y$, find the gradient of the slope-field segments at $(0, 0)$, $(1, 1)$ and $(-1, 0)$. Show that $y = -x - 1$ is a solution, and explain why the segments at points on this line all lie along it.

Accuracy and step length

33. GDC This question uses the equation $\dfrac{dy}{dx} = y$ with $y(0) = 1$ from Question 31(b).

(a) Solve the equation exactly, and compare the exact $y(0.2)$ with your answer to Question 31(b).

(b) State whether Euler's method over- or under-estimates the solution, and explain why in terms of the shape of the curve.

(c) Estimate $y(0.2)$ again, using $h = 0.05$.

(d) Compare the errors of the two estimates, and state the effect of halving the step length.

Equilibrium values

34. GDC

(a) A falling object has $\frac{dv}{dt} = 10 - 0.4v$, with v in m s^{-1}. Find its terminal velocity.

(b) A skydiver falls from rest, with $\frac{dv}{dt} = 9.8 - 0.2v^2$ and v in m s^{-1}. Find the terminal velocity.

(c) For the skydiver in part (b), use Euler's method with $h = 0.5$ to estimate $v(1)$.

(d) Explain why the estimate in part (c) cannot be accurate.

Concept check

35. GDC A student says: "*Euler's method underestimates, because each step follows a tangent.*" Use Euler's method on $\frac{dy}{dx} = -2x$, $y(0) = 0$, with $h = 0.5$, to estimate $y(1)$. Hence show that the claim is false, and state when Euler's method does underestimate.

Chapter 16 · Differential Equations

Reflections

TOK Many laws of physics are differential equations: Newton's second law, radioactive decay, the spread of an epidemic. For the equations of this chapter, each with one starting value, suitable conditions on the equation ensure that a rule for change together with one starting value determines a unique local solution. This is mathematical determinism. Yet more general systems of such equations can be so sensitive to their starting values that long-term prediction becomes impossible in practice, the phenomenon of *chaos*. It is why a weather forecast is useful for days, not months. Determined but unpredictable: what does that tell us about the limits of knowledge?

IM Three of the series of this chapter were found first in India. Mathematicians of the Kerala school, beginning with Madhava around 1400, found the expansions of $\sin x$, $\cos x$ and $\arctan x$ well over two centuries before Newton, Leibniz and Maclaurin. In the 15th century Jamshīd al-Kāshī, working at the observatory in Samarkand, needed $\sin 1^\circ$ to great precision for his astronomical tables. No formula gave it, so he wrote down an equation the value satisfied and solved it by repeated approximation: the answer of each round was the starting value of the next. Each idea was found more than once, in different places and centuries.

RW A detective's question, "when did it happen?", can be answered with a differential equation. Newton's law of cooling says a body cools at a rate proportional to its temperature above the room's: $dT/dt = -k(T - T_s)$, so $T - T_s = (T_0 - T_s)e^{-kt}$. Forensic scientists use this, with care, to estimate a time of death. Under a simplified model with a constant room temperature, suppose a body is found at 29°C in a room at 20°C, and an hour later it has cooled to 27°C. The two readings determine k; starting from a normal body temperature of 37°C then determines t. *Try it:* show that death occurred about two and a half hours before the body was found.

EXT The standard series do not all converge for every x (§16.4). Each series has a *radius of convergence* $(\text{the distance from } 0 \text{ within which the series converges})$, and outside it the infinite series does not converge at all, however many terms are taken. Why is the radius finite for $\ln(1 + x)$ but infinite for e^x? The answer lies in complex numbers: the radius depends on where the function, extended to complex inputs, is undefined. That theory is beyond this course. Series can also be used to solve exactly some differential equations that the methods of this course cannot solve. You assume that the answer is a power series and match the coefficients.

End-of-Chapter Problems

1. GDC A population grows so that $\frac{dP}{dt} = 0.2P$, with $P = 500$ when $t = 0$.

(a) Solve the differential equation for $P(t)$. [4]

(b) Find the population when $t = 10$. [2]

2. GDC A body cools according to $\frac{dT}{dt} = -0.1(T - 25)$, with $T(0) = 90$ (in °C).

(a) Solve for $T(t)$. [4]

(b) Find the temperature after 10 minutes. [2]

(c) State the temperature the body approaches in the long run. [1]

3. GDC A bowl of miso soup at 90°C is left in a 25°C room and obeys $\frac{dT}{dt} = -k(T - 25)$. After 10 minutes it has cooled to 65°C.

(a) Solve the differential equation, expressing T in terms of k and t. [4]

(b) Find k. [2]

(c) Find how long, from the start, the soup takes to reach 40°C. [2]

4. GDC A fish population P (in thousands) in a lake satisfies the logistic equation $\frac{dP}{dt} = 0.4P(2 - P)$, where t is the time in years and $P(0) = 0.5$.

(a) Using partial fractions, show that the solution is $P = \frac{2}{1 + 3e^{-0.8t}}$. [6]

(b) Find the time at which the population reaches 1500 fish. [3]

5. Consider $\frac{dy}{dx} = \frac{3x^2 + y^2}{2xy}$ for $x > 0$, $y > 0$.

(a) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = \frac{3 - v^2}{2v}$. [4]

(b) Hence find the general solution, giving y^2 in terms of x. [3]

6. Consider $\frac{dy}{dx} = \frac{y^2 + 3xy}{x^2}$ for $0 < x < \sqrt{3}$, with $y(1) = 1$.

(a) Use the substitution $y = vx$ to show that $x\frac{dv}{dx} = v^2 + 2v$. [3]

(b) Hence show that $y = \frac{2x^3}{3 - x^2}$. [5]

7. Consider $\frac{dy}{dx} + \frac{1}{x}y = 3x$ for $x > 0$.

(a) Find the integrating factor. [2]

(b) Solve the equation given $y(1) = 2$. [4]

8. Solve each linear equation.

(a) $\frac{dy}{dx} - \frac{y}{x} = x^2$ for $x > 0$, given $y(1) = 1$. [6]

(b) $\frac{dy}{dx} + 2y = e^{-x}$ with $y(0) = 0$. [6]

9. GDC Solve $\frac{dy}{dx} + (\tan x)\, y = \sec x$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, given $y(0) = 2$. [7]

10. GDC Let $f(x) = \cos x$.

(a) Find $f(0), f'(0), f''(0), f'''(0), f^{(4)}(0)$. [3]

(b) Hence write the Maclaurin series of $\cos x$ up to the term in x^4. [2]

(c) Use it to estimate $\cos(0.2)$. [2]

11. EXACT

(a) Write down the Maclaurin series for $\sin x$ up to the term in x^5. [2]

(b) Hence evaluate $\lim_{x \to 0} \frac{x - \sin x}{x^3}$. [3]

12. GDC

(a) Write down the Maclaurin series for e^x up to the term in x^3. [1]

(b) By substituting $-x^2$, find the series for e^{-x^2} up to x^4. [2]

(c) Hence estimate $\int_0^1 e^{-x^2}\, dx$ using terms up to x^4. [4]

13. Find the Maclaurin series of $\ln(1 - 3x)$ up to the term in x^3

(a) by substitution into a standard series; [3]

(b) by computing derivatives from the definition. [3]

14. GDC By setting $x = 1$ in the Maclaurin series for $\arctan x$:

(a) write down a series for $\frac{\pi}{4}$; [2]

(b) estimate π using the first four terms; [2]

(c) comment on the rate of convergence. [1]

15. EXACT Use Maclaurin series to evaluate $\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$. [5]

16. GDC

(a) Find the Maclaurin series of $e^{-x}\sin x$ up to the term in x^3. [4]

(b) Use your series to estimate $e^{-0.1}\sin 0.1$, and compare it with the value from your calculator. [3]

17. A function satisfies $\frac{dy}{dx} = x + y^2$ with $y(0) = 1$. Find the Maclaurin series of y up to the term in x^3. [7]

18. GDC Consider $\frac{dy}{dx} = y - x^2$ with $y(0) = 1$.

(a) Use Euler's method with $h = 0.25$ to estimate $y(0.5)$. [4]

(b) The exact solution is concave up on $0 \le x \le 0.5$. State, with a reason, whether your estimate is above or below the true value. [2]

19. GDC A rumour spreads through a school. The number of students who have heard it, P hundred, satisfies $\frac{dP}{dt} = P(4 - P)$, where t is the time in days and $P(0) = 1$.

(a) Use Euler's method with $h = 0.5$ to estimate $P(1.5)$, showing each step. [4]

(b) Find the non-zero value of P at which $\frac{dP}{dt} = 0$, and comment on the behaviour of the Euler estimates around it. [3]

20. GDC Consider $\frac{dy}{dx} = y - 2x$, with $y(0) = 3$.

(a) Use Euler's method with $h = 0.25$ to estimate $y(0.5)$, showing each step. [4]

(b) Solve the equation exactly (integrating factor) and find the true $y(0.5)$. [4]

(c) Comment on the size and direction of the error. [2]

21. For each equation, name a method of solution, then find the general solution.

(a) $\frac{dy}{dx} = \frac{e^{2x}}{y}$, giving y^2 in terms of x [3]

(b) $\frac{dy}{dx} = \frac{x + 3y}{x}$, for $x > 0$ [4]

(c) $\frac{dy}{dx} + y\cot x = 2\cos x$, for $0 < x < \pi$ [4]

22. GDC A drug is infused into the bloodstream at a rate that raises its concentration by r mg L^{-1} per hour, and the body removes it at a rate proportional to the concentration c, so that

$$\frac{dc}{dt} + kc = r,$$

where t is measured in hours and $c(0) = 0$.

(a) Explain why this equation is linear, and write down its integrating factor. [3]

(b) Solve the equation to show that $c = \frac{r}{k}(1 - e^{-kt})$. [4]

(c) A patient is given $r = 6$ and has $k = 0.3$. Find the concentration after 4 hours. [2]

(d) Find the concentration the patient approaches in the long run, and explain what this means for the treatment. [3]

(e) Find how long it takes to reach 90% of that level. [2]

(f) When 90% of the limiting level is reached the infusion is stopped, so that $\frac{dc}{dt} = -kc$. Solve this equation and find how long the concentration then takes to halve. [3]

23. GDC Water drains from a tank through a hole in its base. The depth h metres at time t minutes satisfies

$$\frac{dh}{dt} = -k\sqrt{h},$$

where k is a positive constant.

(a) Separate the variables to show that $2\sqrt{h} = C - kt$ for some constant C. [3]

(b) Given that the depth is h_0 when $t = 0$, show that, until the tank is empty, $h = \left(\sqrt{h_0} - \frac{1}{2}kt\right)^2$. [2]

(c) A tank starts with $h_0 = 1.44$ and has $k = 0.2$. Find how long it takes to empty. [3]

(d) Find the depth after 6 minutes. [2]

(e) Use Euler's method with steps of 3 minutes to estimate the depth after 6 minutes, and find the error in this estimate. [4]

(f) Describe the shape of the graph of h against t, and use it to explain why the Euler estimate lies below the exact value. [2]

24. GDC In this question you will build a series for logarithms that converges quickly.

(a) Find the Maclaurin series of $\ln(1 + x)$ up to the term in x^4. [4]

(b) Write down the Maclaurin series of $\ln(1 - x)$ up to the term in x^4. [2]

(c) Hence show that $\ln\left(\frac{1+x}{1-x}\right) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right)$. [3]

(d) By choosing a suitable value of x, use the three terms shown to estimate $\ln 3$. [3]

(e) The series for $\ln(1 + x)$ is valid only for $-1 < x \leq 1$. State the value of x that would be needed to reach $\ln 3$ from that series alone, and explain why the series in part (c) is the more useful one here. [2]

25. Consider the differential equation $\frac{dy}{dx} = 2x(y - 3)^2$.

(a) Find the general solution, giving y in terms of x. [3]

(b) Find the particular solution with $y(0) = 2$. [2]

(c) Find the particular solution with $y(0) = 3$. [2]

(d) Find the particular solution with $y(0) = 4$, and state the largest interval containing $x = 0$ on which it is valid. [3]

26. Solve $(x+1)\dfrac{dy}{dx} - y = (x+1)^3$ for $x > -1$, given $y(0) = 2$. [6]

27. Consider $\dfrac{dy}{dx} = \dfrac{x^2 + 2y^2}{xy}$ for $x > 0$, with $y(1) = -1$.

(a) Use the substitution $y = vx$ to show that $x\dfrac{dv}{dx} = \dfrac{1+v^2}{v}$. [3]

(b) Hence find y in terms of x, and state the largest interval containing $x = 1$ on which the solution is valid. [6]

Challenge Questions

C1. GDC **Coffee in a room that warms and cools.** In Section 16.1 a cup of coffee cooled in a room held at $20°$C. A real room is not held fixed. Suppose instead that the room's temperature is $20 + 5\sin\omega t$ degrees, where t is the time in minutes and $\omega > 0$ is a constant, so that the room moves between $15°$ and $25°$ once every $\frac{2\pi}{\omega}$ minutes. Newton's law of cooling then gives

$$\frac{dT}{dt} = -k\left(T - 20 - 5\sin\omega t\right),$$

where T is the temperature of the coffee and $k > 0$ is a constant. The coffee is poured at $80°$, so $T(0) = 80$.

(a) Let $u = T - 20$. Show that $\dfrac{du}{dt} + ku = 5k\sin\omega t$, and explain why this equation cannot be solved by separating the variables. [2]

(b) Using an integrating factor, and given that

$$\int e^{kt}\sin\omega t\,dt = \frac{e^{kt}\left(k\sin\omega t - \omega\cos\omega t\right)}{k^2+\omega^2} + C,$$

show that $u = \dfrac{5k\left(k\sin\omega t - \omega\cos\omega t\right)}{k^2+\omega^2} + Ce^{-kt}$, and find C. [5]

(c) Show that $k\sin\omega t - \omega\cos\omega t = \sqrt{k^2+\omega^2}\,\sin(\omega t - \alpha)$, where $\tan\alpha = \dfrac{\omega}{k}$ and $0 < \alpha < \frac{\pi}{2}$. Hence show that, once t is large, $T \approx 20 + R\sin(\omega t - \alpha)$, where $R = \dfrac{5k}{\sqrt{k^2+\omega^2}}$, and explain why the coffee's temperature reaches each maximum $\frac{\alpha}{\omega}$ minutes after the room's. [5]

(d) Take $k = 0.05$. Find R and the delay $\frac{\alpha}{\omega}$ for a room that completes one cycle a day, $\omega = \frac{2\pi}{1440}$, and for a room whose heater switches on and off every 10 minutes, $\omega = \frac{2\pi}{10}$. [4]

(e) Using your answers to part (d), conjecture what happens to R and to the delay $\frac{\alpha}{\omega}$ as $\omega \to 0$ and as $\omega \to \infty$. Prove your conjectures for R, and use the Maclaurin series $\arctan z = z - \frac{z^3}{3} + \cdots$ to show that the delay tends to $\frac{1}{k}$ as $\omega \to 0$. Explain both results in terms of the coffee and the room. [4]

(f) A real room's temperature is not a single sine wave. Explain why, because the equation in part (a) is linear, a room temperature of 20 plus a sum of sine waves produces a long-run coffee temperature that is 20 plus the sum of the separate responses. Describe how you could use this to predict, and then test, the temperature of a drink left in a real room. [2]

C2. The ferry and the current. A river flows in the positive y-direction, and one bank is the line $x = 0$, with a dock at the origin. A ferry sets off from $(1, 0)$, moving through the water at a constant speed of 1 unit per minute and always aiming straight at the dock, while the current carries it in the positive y-direction at k units per minute, where $k > 0$. Its velocity therefore has components

$$\frac{dx}{dt} = -\frac{x}{\sqrt{x^2+y^2}}, \qquad \frac{dy}{dt} = -\frac{y}{\sqrt{x^2+y^2}} + k,$$

and its path satisfies

$$\frac{dy}{dx} = \frac{y - k\sqrt{x^2+y^2}}{x}, \qquad x > 0.$$

(a) Show the equation is homogeneous, and that $y = vx$ leads to $x\dfrac{dv}{dx} = -k\sqrt{1+v^2}$. [3]

(b) Given that $\displaystyle\int \frac{dv}{\sqrt{1+v^2}} = \ln\left(v + \sqrt{1+v^2}\right) + C$, show that $v + \sqrt{1+v^2} = x^{-k}$. [3]

(c) Hence show that $y = \frac{1}{2}\left(x^{1-k} - x^{1+k}\right)$. [4]

(d) Taking $k = \frac{1}{2}$, find the point that the ferry approaches as $x \to 0$. [2]

(e) Show that, as $x \to 0$, the ferry approaches the dock if and only if $k < 1$. Describe what happens to the path when $k = 1$ and when $k > 1$. [4]

(f) Using part (b), show that $\sqrt{1+v^2} = \frac{1}{2}\left(x^{-k} + x^{k}\right)$. Hence, using the expression for $\dfrac{dx}{dt}$, show that for $0 < k < 1$ the ferry reaches the dock after

$$T = \frac{1}{2}\int_0^1 \left(x^{-k} + x^{k}\right) dx = \frac{1}{1-k^2} \text{ minutes},$$

and explain what the integral shows when $k \geq 1$. [5]

(g) Aiming at the dock is not the only way to steer. Suggest a different strategy for $0 < k < 1$, find the time it takes, and compare it with T. [2]

C3. GDC **How wrong is Euler?** Take $\dfrac{dy}{dx} = y$ with $y(0) = 1$, whose exact solution is $y = e^x$, so that $y(1) = e$.

(a) Show that one Euler step of length h gives $y_{n+1} = (1+h)y_n$, and hence that after n steps $y_n = (1+h)^n$. [3]

(b) Taking $h = \dfrac{1}{n}$, deduce that Euler's estimate of $y(1)$ is $\left(1 + \dfrac{1}{n}\right)^n$. [2]

(c) Evaluate this estimate for $n = 2, 4, 8, 16$, and tabulate the error in each case. [4]

(d) Describe what happens to the error each time h is halved, and state what this says about the accuracy of the method. [3]

(e) Use the Maclaurin series for $\ln(1+x)$ to show that, for $n \geq 2$,

$$n\ln\left(1+\frac{1}{n}\right) = 1 - \frac{1}{2n} + \frac{1}{3n^2} - \cdots.$$

[3]

(f) Hence show that, for large n, the error $e - \left(1+\frac{1}{n}\right)^n$ is approximately $\frac{e}{2n}$. Compare this with your table, and deduce the limit of $\left(1+\frac{1}{n}\right)^n$ as $n \to \infty$. [3]

(g) The error is close to a known multiple of h. Suggest how two Euler estimates, one with n steps and one with $2n$ steps, could be combined into a more accurate estimate, and test your suggestion on the values in your table. [3]

C4. GDC **The series for $\tan x$.** In the notes, the equation $\frac{dy}{dx} = 1 + y^2$ with $y(0) = 0$ gave the first two terms of the Maclaurin series of its solution, $y = \tan x$, by repeated differentiation. This question finds more terms by a quicker method and asks how far the series can be trusted. You may assume that $y = \tan x$ is the only solution with $y(0) = 0$ on $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and that it is equal to its Maclaurin series near $x = 0$.

(a) Show that if $y = g(x)$ satisfies $\frac{dy}{dx} = 1 + y^2$ with $g(0) = 0$, then so does $y = -g(-x)$. Deduce that $\tan x$ is an odd function, so that its series contains only odd powers of x. [3]

(b) Write $y = a_1x + a_3x^3 + a_5x^5 + a_7x^7 + \cdots$. Substitute into the equation and compare the coefficients of x^0, x^2 and x^4 to show that $a_1 = 1$, $a_3 = \frac{1}{3}$ and $a_5 = \frac{2}{15}$. Find a_7. [6]

(c) Use the four terms found in part (b) to estimate $\tan 0.5$, $\tan 1$ and $\tan 1.5$, and compare each estimate with the value from your calculator. [3]

(d) Using the graph of $y = \tan x$, explain why no polynomial can stay close to $\tan x$ as $x \to \frac{\pi}{2}$, and suggest the largest interval centred on 0 on which the series could converge. [3]

(e) Apply the method of part (b) to $\frac{dy}{dx} = 1 - y^2$ with $y(0) = 0$, as far as the term in x^7. Conjecture how its coefficients are related to those of $\tan x$, and explain why the relationship continues for every power. [3]

(f) The coefficients $1, \frac{1}{3}, \frac{2}{15}, \frac{17}{315}, \frac{62}{2835}, \ldots$ follow no obvious pattern. Investigate the ratio of successive coefficients, and suggest how its behaviour is connected to your answer to part (d). [2]