CALCULUS

Chapter 15

Integral Calculus

15

SYLLABUS 5.5 5.9 5.10 5.11 5.15 5.16 5.17

A speedometer shows your speed, the rate at which distance changes: a derivative Ch. 13. Now reverse the question. Suppose you record the speedometer for an entire journey. Can you work out how far you travelled, using only the speed?

Over a tiny slice of time, your speed barely changes, so the distance covered in that slice is roughly speed times the slice of time. Add up all the slices and you have the total distance. As the slices become narrower, the rough sum approaches the exact distance. What you are computing is the area under the speed-time graph.

This is the second main idea of calculus, and it has two parts. The first is **reversing a derivative**: given a rate of change, recover the quantity it came from. The second is **accumulating**: adding up a great many thin slices to get a total. The area under a graph is exactly this kind of total.

They have no obvious connection. Reversing a derivative is a problem in algebra. Adding slices is a problem in geometry. Yet in the car journey the two meet: the distance travelled is the quantity whose derivative is the speed, and it is also the area under the speed-time graph.

In §15.1 you meet the antiderivative and the area separately, and then use the result that connects them: the Fundamental Theorem of Calculus, proved at the end of the chapter. By the end of this chapter you will be able to find antiderivatives of the standard functions and evaluate definite integrals. You will also integrate by substitution and by parts, find areas between curves and volumes of revolution, and find displacement and distance travelled from velocity.

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15.1 Antiderivatives and Integrals

Many operations have an opposite: division undoes multiplication, and a logarithm undoes an exponential. Differentiation has one too. It takes a function and gives its derivative, so the opposite task is to start from a function f and find a function F whose derivative is f. Such an F is called an **antiderivative** of f.

15.1.1 Antiderivatives and indefinite integrals

Read any known derivative backwards to get an antiderivative:

- x^2 is an antiderivative of $2x$;
- $\sin x$ is an antiderivative of $\cos x$;
- e^x is an antiderivative of e^x;
- $\ln x$ is an antiderivative of $\frac{1}{x}$, for $x > 0$.

A function has more than one antiderivative. The derivative of a constant is zero, so adding any constant to an antiderivative gives another one: $x^2 + 5$ and $x^2 - 1$ are also antiderivatives of $2x$. So antiderivatives come in families, written $F(x) + C$.

DEFINITION 15.1 **Antiderivative**

F is an **antiderivative** of f when $F'(x) = f(x)$. If F is one antiderivative, then $F(x) + C$ is an antiderivative for every constant C, and on an interval (one unbroken range of x-values) these are all of them.

The whole family of antiderivatives of f is written $\int f(x)\,dx = F(x) + C$. This is the **indefinite integral** of f, and C is the **constant of integration**. To **integrate** f is to find its integral. The next subsection explains where the sign $\int$ comes from.

CAUTION The $+C$ is easy to forget, and an answer written without it names a single antiderivative, not the whole family.

To integrate a power of x, reverse the power rule for differentiation: raise the power by one and divide by the new power.

FORMULA 15.1 **Power rule for integration** FORMULA BOOKLET · 5.5

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \qquad n \neq -1$$

The exception $n = -1$ is handled in §15.2. Constant multiples and sums are integrated term by term, just as they are differentiated term by term.

EXAMPLE 15.1

Find $\int (3x^2 - 4x + 1)\,dx$.

SOLUTION

Integrate each term, raising each power by one:

$$\int (3x^2 - 4x + 1)\,dx = \boxed{x^3 - 2x^2 + x + C}$$

Check by differentiating: $\frac{d}{dx}(x^3 - 2x^2 + x + C) = 3x^2 - 4x + 1$. ✓

The rule holds for negative and fractional powers too. As with differentiation, write each root and each reciprocal as a power of x first.

EXAMPLE 15.2

Find $\int \left(\frac{8}{x^5} + 4\sqrt[3]{x}\right) dx$.

SOLUTION

Write the integrand as $8x^{-5} + 4x^{1/3}$. Raise each power by one and divide by the new power:

$$\int \left(8x^{-5} + 4x^{1/3}\right) dx = \frac{8x^{-4}}{-4} + \frac{4x^{4/3}}{4/3} + C = \boxed{-\frac{2}{x^4} + 3x^{4/3} + C}$$

Adding one to -5 gives -4, not -6. Dividing by $\frac{4}{3}$ is multiplying by $\frac{3}{4}$. Check by differentiating: $\frac{d}{dx}\left(-2x^{-4} + 3x^{4/3}\right) = 8x^{-5} + 4x^{1/3}$. ✓

Because of the $+C$, an indefinite integral is a family of functions, not a single function. To pick out one member of the family, substitute a known fact, such as a point the curve passes through, to find C. Such a fact is called a **boundary condition**.

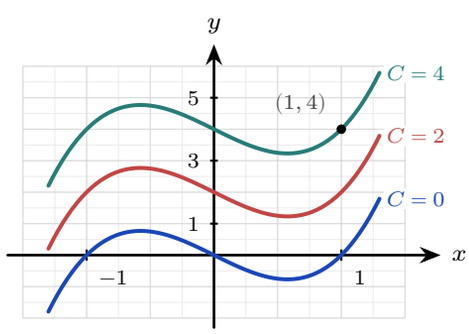


Figure 15.1 Three members of the family $y = 2x^3 - 2x + C$. All have gradient $6x^2 - 2$, but only $C = 4$ passes through $(1, 4)$.

EXAMPLE 15.3

A curve has gradient $\dfrac{dy}{dx} = 6x^2 - 2$ and passes through $(1, 4)$. Find its equation.

SOLUTION

Integrate to recover y:

$$y = \int (6x^2 - 2)\,dx = 2x^3 - 2x + C.$$

Use the point $(1, 4)$: $4 = 2 - 2 + C \implies C = 4$, so $\boxed{y = 2x^3 - 2x + 4}$

15.1.2 The area under a curve

Leave antiderivatives for a moment and look at a different problem: the area under the curve in the figure below. You know formulas for the areas of rectangles, triangles and circles. A region with a curved top, like this one, has no such formula.

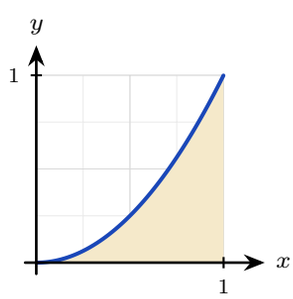


Figure 15.2 The region under $y = x^2$ between $x = 0$ and $x = 1$. No formula from geometry gives its area.

Cut the region into thin vertical strips, and treat each strip as a rectangle. For a curve $y = f(x)$, a strip at x has height $f(x)$ and width Δx, so the area of one strip is

$$f(x)\,\Delta x.$$

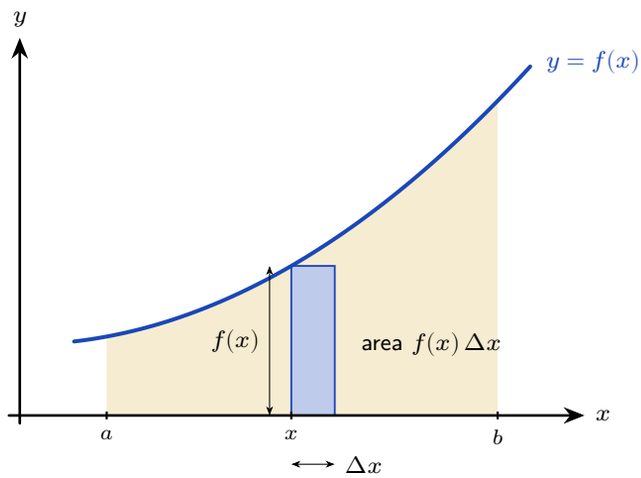


Figure 15.3 One strip of the region under $y = f(x)$ between $x = a$ and $x = b$. The strip at x is treated as a rectangle of height $f(x)$ and width Δx.

To get the area under the curve, add the areas of all the strips. With four strips,

$$\text{area} \approx f(x_1)\,\Delta x + f(x_2)\,\Delta x + f(x_3)\,\Delta x + f(x_4)\,\Delta x,$$

where $x_1, \ldots, x_4$ are the left ends of the strips. For a curve that rises across the interval, such as $y = x^2$ on $[0, 1]$, these rectangles lie under the curve. They do not fit it exactly, so the sum is only an estimate. With more strips, each one narrower, the rectangles fit the curve more closely.

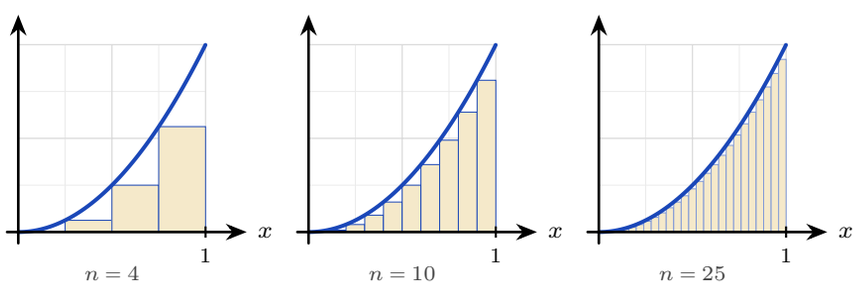


Figure 15.4 Rectangles under $y = x^2$ on $[0, 1]$. Four, ten and twenty-five strips give 0.219, 0.285 and 0.314, closing in on the true area $\frac{1}{3}$.

With many strips the sum is too long to write out, so it is written in sigma notation §2.5. For n strips,

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\,\Delta x.$$

For $y = x^2$ on $[0, 1]$ the sums in the figure above come closer and closer to $\frac{1}{3}$.

Now let the number of strips grow without end, so that each strip becomes narrower and narrower. The value that the sums approach is written with a new sign. For a region from

$x = a$ to $x = b$, the Σ becomes $\int$, a long S for 'sum', and the width Δx becomes dx:

$$\sum_{i=1}^{n} f(x_i)\,\Delta x \quad \longrightarrow \quad \int_a^b f(x)\,dx.$$

This is the **definite integral** of f from a to b. It is what we mean by the area under the curve, and finding it is called **integration**.

DEFINITION 15.2 **Definite integral**

The **definite integral** of f from a to b is the limit of the rectangle sums as the strips become narrower and narrower:

$$\int_a^b f(x)\,dx = \lim_{\Delta x \to 0} \sum_i f(x_i)\,\Delta x.$$

For $f(x) \geq 0$ it is the area between the curve and the x-axis from a to b.

Read the notation one piece at a time. The $\int$ stands for the limit of the strip sums from $x = a$ to $x = b$. Inside it, $f(x)\,dx$ recalls the area $f(x)\,\Delta x$ of one strip, and the dx also names x as the variable of integration. The numbers a and b are the **limits** of integration (a different use of the word from §13.1), and $f(x)$ is the **integrand** (the function being integrated). The definite integral is the value approached by sums of more and more strips, each narrower than the last. It is not a sum of rectangles of zero width.

15.1.3 Evaluating a definite integral

The definition says what the area is, but not how to calculate it. It is a limit, not a finite sum, and adding a thousand rectangles by hand is not practical.

A definite integral can be evaluated with an antiderivative. So the area can be found without working out a single strip. The reason, in one line: the height of the curve is the rate at which the area grows. This result is the **Fundamental Theorem of Calculus**. It is discussed in full and proved at the end of this chapter, after the Reflections.

KEY 15.2 **Evaluating a definite integral**

If f is continuous on $a \leq x \leq b$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a).$$

To use the formula, find an antiderivative F of the integrand. Substitute the upper limit into F, then subtract the value of F at the lower limit. No $+C$ is needed, because the same constant would appear in $F(b)$ and in $F(a)$ and cancel in the subtraction. Check first that the integrand has no break or asymptote between the limits, because the formula does not apply across one.

Try it on the region under $y = x^2$. The function $\frac{x^3}{3}$ is an antiderivative of x^2, since its derivative is x^2. So

$$\int_0^1 x^2\,dx = \left[\tfrac{x^3}{3}\right]_0^1 = \tfrac{1}{3} - 0 = \tfrac{1}{3},$$

the value that the rectangle sums in §15.1.2 were approaching, found in one line.

EXAMPLE 15.4

Evaluate $\int_{-1}^{2} (3x^2 - 4x + 1)\,dx$.

SOLUTION

Use the antiderivative $x^3 - 2x^2 + x$ found above. Put each limit in brackets, so that the signs of a negative limit are kept:

$$\int_{-1}^{2} (3x^2 - 4x + 1)\,dx = \Big[x^3 - 2x^2 + x\Big]_{-1}^{2} = (8 - 8 + 2) - (-1 - 2 - 1) = 2 + 4 = \boxed{6}$$

- A definite integral and an area are closely related, but **they are not always equal**. Where the curve lies above the x-axis, the definite integral is the area under it, as in the calculation above. Where the curve lies below the axis, the integral is negative. A definite integral gives a signed area: positive above the axis and negative below it. Finding the area of a region, including one where the curve crosses the axis, is the subject of §15.6.

Five properties of definite integrals are used throughout the chapter.

KEY 15.3 **Properties of definite integrals**

$\int_a^a f(x)\,dx = 0$	An interval of zero width contributes zero.
$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	Reversing the limits changes the sign.
$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	Contributions from adjacent intervals add.
$\int_a^b k\,f(x)\,dx = k\int_a^b f(x)\,dx$	A constant factor can be taken outside.
$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	Sums and differences are integrated term by term.

The first three can be checked from the formula. Equal limits give $F(a) - F(a) = 0$. Swapping the limits gives $F(a) - F(b)$, the negative of $F(b) - F(a)$. Joining two intervals gives $(F(b) - F(a)) + (F(c) - F(b)) = F(c) - F(a)$, because $F(b)$ cancels. The last two

hold because if F and G are antiderivatives of f and g, then kF is an antiderivative of kf, and $F \pm G$ is an antiderivative of $f \pm g$.

EXAMPLE 15.5

Given that $\int_1^4 f(x)\,dx = 7$ and $\int_1^6 f(x)\,dx = 10$, find (a) $\int_4^6 f(x)\,dx$, (b) $\int_6^4 3f(x)\,dx$.

SOLUTION

(a) Joining intervals,

$$\int_1^4 f(x)\,dx + \int_4^6 f(x)\,dx = \int_1^6 f(x)\,dx, \qquad \text{so} \qquad \int_4^6 f(x)\,dx = 10 - 7 = \boxed{3}$$

(b) Swapping the limits and taking out the constant,

$$\int_6^4 3f(x)\,dx = -3\int_4^6 f(x)\,dx = \boxed{-9}$$

CAUTION The sign $\int$ now has two uses, and they give different kinds of answer. The indefinite integral $\int f(x)\,dx$ is a family of functions, $F(x) + C$. The definite integral $\int_a^b f(x)\,dx$ is a number, $F(b) - F(a)$.

YOUR TURN 15.1 Antiderivatives and Integrals

Integrating term by term

1. Find each indefinite integral.

(a) $\int (4x^3 - 6x)\,dx$

(b) $\int (x^2 + 3x - 5)\,dx$

(c) $\int \frac{6}{x^3}\,dx$

(d) $\int \sqrt{x}\,dx$

(e) $\int \left(x^2 - \frac{1}{x^2}\right)dx$

(f) $\int \frac{x^3 - 2x}{x}\,dx$

(g) $\int x^{-1/2}\,dx$

(h) $\int (2x)^3\,dx$

Boundary conditions

2.

(a) A curve has $\frac{dy}{dx} = 4x - 3$ and passes through $(2, 5)$. Find y.

(b) A curve has $\frac{dy}{dx} = 3\sqrt{x}$ and passes through $(4, 20)$. Find y.

(c) A curve has $\frac{d^2y}{dx^2} = 6x$, with $\frac{dy}{dx} = 1$ and $y = 2$ when $x = 0$. Find y. *(Hint: integrate twice, and use one condition after each integration)*

(d) A curve has $\frac{d^2y}{dx^2} = 12x^2 - 4$. Its gradient is 0 at the point $(1, 3)$. Find y.

(e) Explain why a gradient function alone does not determine a curve, and what extra information does.

(f) Explain why parts (c) and (d) needed two pieces of information rather than one.

Concept check

3. A student writes $\int \frac{1}{x^2}\,dx = \frac{1}{x^3/3} + C = \frac{3}{x^3} + C$. Identify the error, and find the correct integral.

Evaluating definite integrals

4. Evaluate each definite integral.

(a) $\int_1^2 4x\,dx$

(b) $\int_0^3 (x^2 + 1)\,dx$

(c) $\int_1^2 \frac{6}{x^3}\,dx$

(d) $\int_1^4 \frac{1}{\sqrt{x}}\,dx$

(e) $\int_0^2 (x + 1)(x - 3)\,dx$

(f) $\int_{-1}^2 (3x^2 - 2x)\,dx$

Area under a curve

5. Each region lies on one side of the x-axis. Find its area.

(a) The region under $y = x^2$ between $x = 1$ and $x = 3$.

(b) The region enclosed between $y = x^2 - 1$ and the x-axis. *(Hint: the region lies below the axis, so the integral is negative and the area is its modulus; see §15.6.1)*

Properties of definite integrals

6. It is given that $\int_1^5 f(x)\,dx = 6$ and $\int_1^3 f(x)\,dx = 2$. Find the value of each integral.

(a) $\int_3^5 f(x)\,dx$

(b) $\int_5^1 f(x)\,dx$

(c) $\int_1^5 4f(x)\,dx$

(d) $\int_1^5 (f(x) + 2x)\,dx$

(e) $\int_3^3 f(x)\,dx + \int_5^3 f(x)\,dx$

Concept check

7. A student writes

$$\int_1^3 \frac{2}{x^3}\,dx = \left[-\frac{1}{x^2}\right]_1^3 = -\frac{1}{9} - 1 = -\frac{10}{9}.$$

Explain why a negative answer cannot be right here, identify the error, and find the correct value.

15.2 Standard Integrals

Every derivative established in Ch. 13, and the inverse trigonometric derivatives proved in §14.5, can be read in reverse. Doing so turns a table of derivatives into a table of integrals. The question "what is the integral of this function?" becomes the easier question "what was differentiated to produce it?"

FORMULA 15.4 **Standard integrals** FORMULA BOOKLET · 5.10

$$\int \frac{1}{x}\,dx = \ln|x| + C, \qquad \int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C, \qquad \int e^x\,dx = e^x + C$$

The results for $\sin x$ and $\cos x$ hold only for x in radians.

Each should be understood rather than memorised.

- $\int \frac{1}{x}\,dx = \ln|x| + C$ is the one case the power rule does not cover. Raising $n = -1$ by one gives 0, and the rule would then divide by 0, which is impossible. So an antiderivative of x^{-1} cannot be a power at all. The logarithm is one, because $\frac{d}{dx}\ln x = \frac{1}{x}$. The modulus is there because $\frac{1}{x}$ is also defined for negative x. On that side the chain rule gives $\frac{d}{dx}\ln(-x) = \frac{-1}{-x} = \frac{1}{x}$, the same derivative as for positive x.
- $\int e^x\,dx = e^x + C$ because e^x is equal to its own derivative. Since e^x is unchanged by differentiating, it is one of its own antiderivatives, and the others are $e^x + C$.
- $\int \cos x\,dx = \sin x + C$ is the direct reversal of $\frac{d}{dx}\sin x = \cos x$, with no change of sign.
- $\int \sin x\,dx = -\cos x + C$ has a minus sign that is easy to forget. Differentiating cosine gives one: $\frac{d}{dx}\cos x = -\sin x$. To finish with $+\sin x$ you must therefore start from $-\cos x$.

At HL, the formula booklet gives three more standard integrals, the reverses of the exponential and inverse-trigonometric derivatives.

FORMULA 15.5 **Further standard integrals** FORMULA BOOKLET · 5.15 HL

$$\int a^x\,dx = \frac{a^x}{\ln a} + C, \qquad \int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \arcsin\frac{x}{a} + C, \qquad |x| < a$$

In all three $a > 0$, and in the first $a \neq 1$, since $\ln 1 = 0$.

Read these three in the same way: each one reverses a derivative. Differentiating a^x produces an extra factor: writing $a^x = e^{x\ln a}$ and applying the chain rule gives $\frac{d}{dx}a^x = a^x \ln a$. Dividing by $\ln a$ removes it. When $a = e$ the factor $\ln e$ is 1, and the formula becomes the standard integral $\int e^x\,dx = e^x + C$.

The two inverse-trigonometric forms look alike, but only the arctangent form has a factor $\frac{1}{a}$ in front. This difference is a common source of error. Here is where it comes from.

- Both forms come from derivatives you know: $\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$ and $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}}$. The formulas replace x by $\frac{x}{a}$.
- Differentiating $\frac{x}{a}$ gives a chain-rule factor $\frac{1}{a}$ in both cases.
- Arcsine: $\dfrac{d}{dx}\arcsin\dfrac{x}{a} = \dfrac{1}{a}\cdot\dfrac{1}{\sqrt{1-x^2/a^2}} = \dfrac{1}{a}\cdot\dfrac{a}{\sqrt{a^2-x^2}} = \dfrac{1}{\sqrt{a^2-x^2}}$, since $\sqrt{a^2} = a$ for $a > 0$. The a cancels the $\frac{1}{a}$ exactly, so the formula needs no factor in front.
- Arctangent: $\dfrac{d}{dx}\arctan\dfrac{x}{a} = \dfrac{1}{a}\cdot\dfrac{1}{1+x^2/a^2} = \dfrac{1}{a}\cdot\dfrac{a^2}{a^2+x^2} = \dfrac{a}{a^2+x^2}$. A factor a is left over, so the formula needs $\frac{1}{a}$ in front to remove it.

The booklet gives no integral that leads to $\arccos$, and none is needed. From §13.5, $\arccos\frac{x}{a} = \frac{\pi}{2} - \arcsin\frac{x}{a}$ for $|x| \le a$, so

$$\int \frac{-1}{\sqrt{a^2-x^2}}\,dx = \arccos\frac{x}{a} + C \quad \text{or} \quad -\arcsin\frac{x}{a} + C.$$

The two answers differ by the constant $\frac{\pi}{2}$, which the $+C$ includes, so both are correct.

Reading the rest of the HL derivative table backwards gives four more. They must be learned, because the formula booklet gives only the derivatives:

KEY 15.6 **Integrals from the reciprocal trigonometric derivatives** HL

$$\int \sec^2 x\,dx = \tan x + C, \qquad \int \sec x \tan x\,dx = \sec x + C,$$

$$\int \operatorname{cosec}^2 x\,dx = -\cot x + C, \qquad \int \operatorname{cosec} x \cot x\,dx = -\operatorname{cosec} x + C$$

Here x is in radians.

EXAMPLE 15.6

Find (a) $\displaystyle\int \frac{1}{x^2+25}\,dx$, (b) $\displaystyle\int \frac{1}{\sqrt{9-x^2}}\,dx$.

SOLUTION

Match the standard forms with $a = 5$ and $a = 3$:

$$\text{(a)}\ \boxed{\tfrac{1}{5}\arctan\frac{x}{5} + C}, \qquad \text{(b)}\ \boxed{\arcsin\frac{x}{3} + C}$$

The only work is recognising a^2 as 25 and as 9. After that, write down the standard form.

Definite integrals of these standard forms are evaluated exactly as in §15.1: find an antiderivative, then substitute the limits.

EXAMPLE 15.7

Evaluate (a) $\displaystyle\int_0^{\pi/2} \sin x\,dx$, (b) $\displaystyle\int_1^{e^2} \frac{1}{x}\,dx$.

SOLUTION

(a) An antiderivative of $\sin x$ is $-\cos x$, so

$$\begin{aligned}\int_0^{\pi/2} \sin x\,dx &= \Big[-\cos x\Big]_0^{\pi/2} \\ &= -\cos\frac{\pi}{2} + \cos 0 \\ &= 0 + 1 \\ &= \boxed{1}\end{aligned}$$

The area under the sine curve from 0 to $\frac{\pi}{2}$ is exactly 1, with no π in the answer.

(b) An antiderivative of $\dfrac{1}{x}$ is $\ln|x|$, and both limits are positive, so

$$\int_1^{e^2} \frac{1}{x}\,dx = \Big[\ln x\Big]_1^{e^2} = \ln e^2 - \ln 1 = 2 - 0 = \boxed{2}$$

Stopping at $x = e$ instead would give an area of exactly one. This gives one way to define e: it is the upper limit that makes the area under $y = \frac{1}{x}$, starting at $x = 1$, equal to one.

15.2.1 Linear composites

Sometimes the function inside is linear, as in e^{2x} or $\cos 3x$. Integrate as usual, then divide by the coefficient of x inside. This reverses the chain rule, whose extra factor was that same coefficient. Writing down an antiderivative directly in this way, by recognising the result of a chain rule differentiation, is called **integration by inspection**.

KEY 15.7 **Linear composites**

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b) + C, \qquad a \neq 0, \text{ where } F' = f$$

- This works **only when the inner function is linear**. For e^{x^2} the inner function x^2 is not linear, and dividing by its derivative gives a wrong answer: the derivative of $\dfrac{e^{x^2}}{2x}$ is not e^{x^2}.

EXAMPLE 15.8

Find (a) $\displaystyle\int e^{2x}\,dx$, (b) $\displaystyle\int (2x+1)^5\,dx$, (c) $\displaystyle\int \cos 3x\,dx$.

SOLUTION

Each has a linear function inside, so integrate and then divide by the coefficient of x:

(a) $\boxed{\frac{1}{2}e^{2x} + C}$, (b) $\boxed{\dfrac{(2x+1)^6}{12} + C}$, (c) $\boxed{\frac{1}{3}\sin 3x + C}$

In (b), reversing the power rule gives $\frac{(2x+1)^6}{6}$, then dividing by the coefficient 2 gives the 12.

The linear-composite rule extends to the whole table. For instance, $\displaystyle\int \sec^2(2x+5)\,dx = \tfrac{1}{2}\tan(2x+5) + C$ and $\displaystyle\int \frac{1}{2x+1}\,dx = \tfrac{1}{2}\ln|2x+1| + C$. The second is used for partial fractions in §15.3.3.

The inverse trigonometric forms need x^2 with coefficient 1. When x^2 has another positive coefficient k^2, write that term as $(kx)^2$ and use the linear-composite rule, which divides by k. The coefficient need not be a perfect square: $3x^2 = (\sqrt{3}\,x)^2$.

EXAMPLE 15.9

HL Find (a) $\displaystyle\int \frac{1}{4x^2+25}\,dx$, (b) $\displaystyle\int \frac{1}{\sqrt{16-9x^2}}\,dx$.

SOLUTION

(a) Write $4x^2+25=(2x)^2+5^2$. The arctangent form with $a=5$ gives $\frac{1}{5}\arctan\frac{2x}{5}$, and the linear-composite rule divides it by the inner coefficient 2:

$$\int \frac{1}{(2x)^2+5^2}\,dx=\frac{1}{2}\cdot\frac{1}{5}\arctan\frac{2x}{5}+C=\boxed{\frac{1}{10}\arctan\frac{2x}{5}+C}$$

Check by differentiating: $\frac{d}{dx}\left(\frac{1}{10}\arctan\frac{2x}{5}\right)=\frac{1}{10}\cdot\frac{2}{5}\cdot\frac{1}{1+\frac{4x^2}{25}}=\frac{1}{25+4x^2}$. ✓

(b) In the same way $16-9x^2=4^2-(3x)^2$, so

$$\int \frac{1}{\sqrt{16-9x^2}}\,dx=\boxed{\frac{1}{3}\arcsin\frac{3x}{4}+C}$$

YOUR TURN 15.2 Standard Integrals

Reversing the standard derivatives

8. Find each indefinite integral.

(a) $\displaystyle\int (2\cos x-\sin x)\,dx$

(b) $\displaystyle\int \left(3e^x+\frac{1}{x}\right)dx$

Linear composites

9. Find each indefinite integral.

(a) $\displaystyle\int e^{3x}\,dx$

(b) $\displaystyle\int (3x-2)^4\,dx$

(c) $\displaystyle\int \sin 2x\,dx$

(d) $\displaystyle\int \sec^2(4x)\,dx$

(e) $\displaystyle\int \cos(3x+1)\,dx$

(f) $\displaystyle\int \frac{1}{2x+5}\,dx$

The HL standard forms

10. Find each indefinite integral, matching it to a standard form.

(a) $\displaystyle\int \frac{1}{x^2+9}\,dx$

(b) $\displaystyle\int \frac{1}{\sqrt{16-x^2}}\,dx$

(c) $\displaystyle\int 5^x\,dx$

(d) $\displaystyle\int \frac{2}{4+x^2}\,dx$

(e) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$

(f) $\displaystyle\int \frac{-1}{\sqrt{25-x^2}}\,dx$, in two forms

11. Find each indefinite integral by reading a derivative in reverse.

(a) $\int \operatorname{cosec}^2 x \, dx$ (b) $\int \sec x \tan x \, dx$

(c) $\int 3 \operatorname{cosec} 2x \cot 2x \, dx$

Definite integrals of standard forms

12.

(a) Evaluate $\int_1^{e^3} \frac{2}{x} \, dx$.

(b) Evaluate $\int_0^{\pi/6} \cos x \, dx$.

(c) Find the area under $y = \sin x$ from $x = 0$ to $x = \pi$.

(d) Evaluate $\int_0^{2\pi} \sin x \, dx$, and explain why the answer is not the area between the curve and the axis over that interval.

(e) A student writes $\int_{-1}^{1} \frac{1}{x} \, dx = \left[\ln |x|\right]_{-1}^{1} = 0$. Explain why this working is not valid.

Concept check

13. A student writes $\int e^{x^2} \, dx = \frac{e^{x^2}}{2x} + C$, dividing by the derivative of the inner function as for $\int e^{3x} \, dx = \frac{1}{3}e^{3x} + C$. Show, by differentiating, that the answer is wrong, and explain why the method works for e^{3x} but not for e^{x^2}.

15.3 Rewriting Before Integrating

Many integrands do not match a standard form as they stand, but can be rewritten until they do. Four rewritings cover the cases met in this course: divide, complete the square, split into partial fractions, or replace a squared trigonometric function using a double angle. The table in §15.5.2 summarises how to choose between all the methods of this chapter.

15.3.1 Divide first

When the numerator has degree at least that of the denominator, the fraction is improper §6.6 and no standard form applies. Divide before anything else. For instance

$$\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}, \qquad \text{so} \qquad \int \frac{x^2}{x+1} \, dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

Only when the numerator has the lower degree are the next two methods available.

15.3.2 Completing the square HL

A denominator is not always in the form $a^2 + x^2$. A quadratic such as $x^2 + 2x + 5$ has that shape, but only after you complete the square.

Any quadratic $x^2 + bx + c$ with no real zeros can be written as $(x + p)^2 + a^2$. The integral is then the arctangent form in the variable $x + p$. No extra factor appears, because the derivative of $x + p$ is 1.

EXAMPLE 15.10

Find $\displaystyle\int \frac{1}{x^2 + 2x + 5}\,dx$.

SOLUTION

Complete the square: $x^2 + 2x + 5 = (x + 1)^2 + 4$. This is the arctangent shape with $a = 2$ and the translated variable $x + 1$:

$$\int \frac{dx}{(x+1)^2 + 4} = \boxed{\frac{1}{2}\arctan\left(\frac{x+1}{2}\right) + C}$$

15.3.3 Partial fractions HL

When the quadratic *does* factorise, use a different method. Split the fraction into partial fractions, as in Ch. 6. When the factors are distinct, each piece is a constant over a linear factor, $\dfrac{A}{ax + b}$, and integrates to the logarithm $\dfrac{A}{a}\ln|ax + b|$. A repeated factor behaves differently: $\int \frac{1}{(x+1)^2}\,dx = -\frac{1}{x+1} + C$, a power and not a logarithm.

EXAMPLE 15.11

Find $\displaystyle\int \frac{1}{x^2 + 3x + 2}\,dx$.

SOLUTION

Factorise and split into partial fractions: $\dfrac{1}{(x+1)(x+2)} = \dfrac{1}{x+1} - \dfrac{1}{x+2}$. Then

$$\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln|x+1| - \ln|x+2| + C = \boxed{\ln\left|\frac{x+1}{x+2}\right| + C}$$

So look at the denominator to choose the method; the table in §15.5.2 sets out every case.

When the numerator is linear and the quadratic has no real zeros, split the numerator into a multiple of the denominator's derivative plus a constant. For example, $\dfrac{2x+3}{x^2+2x+5} = \dfrac{2x+2}{x^2+2x+5} + \dfrac{1}{(x+1)^2+4}$, so

$$\int \frac{2x+3}{x^2+2x+5}\,dx = \ln\left(x^2+2x+5\right) + \frac{1}{2}\arctan\frac{x+1}{2} + C.$$

15.3.4 Using a double angle

There is no standard integral for $\sin^2 x$ or $\cos^2 x$, and neither is the derivative of anything simple. The method is to remove the square before integrating, using the double-angle identity from Ch. 8:

$$\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1.$$

Rearranged, these give the two rewritings that matter here:

KEY 15.8 **Squares of sine and cosine**

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2}$$

Each right-hand side is a constant plus a linear composite, and both of those integrate immediately.

EXAMPLE 15.12

Find (a) $\displaystyle\int \sin^2 x\,dx$, (b) $\displaystyle\int_0^{\pi} \sin^2 x\,dx$.

SOLUTION

(a) Replace the square first:

$$\begin{aligned}\int \sin^2 x\,dx &= \int \frac{1 - \cos 2x}{2}\,dx \\ &= \frac{1}{2}\int (1 - \cos 2x)\,dx \\ &= \boxed{\frac{x}{2} - \frac{\sin 2x}{4} + C}\end{aligned}$$

The $\frac{1}{4}$ comes from the linear composite: integrating $\cos 2x$ gives $\frac{1}{2}\sin 2x$, and the outer $\frac{1}{2}$ halves it again.

(b) Now substitute the limits:

$$\int_0^{\pi} \sin^2 x\,dx = \left[\frac{x}{2} - \frac{\sin 2x}{4}\right]_0^{\pi} = \left(\frac{\pi}{2} - 0\right) - (0 - 0) = \boxed{\frac{\pi}{2}}$$

The answer is a simple multiple of π because, from 0 to π, $\sin^2 x$ has average value exactly $\frac{1}{2}$, so the area is half the width π. The oscillating part, $\cos 2x$, integrates to zero over a whole number of its periods.

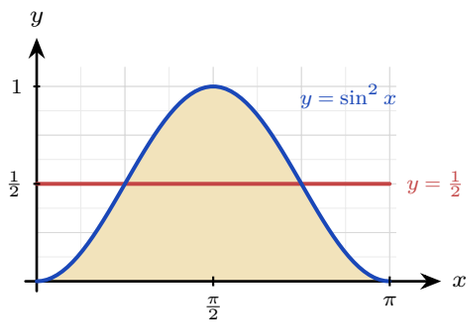


Figure 15.5 $y = \sin^2 x$ on $[0, \pi]$. The parts of the curve above $y = \frac{1}{2}$ exactly fill the gaps below it, so the area is $\frac{1}{2} \times \pi$.

The same rewriting works for $\cos^2 x$:

$$\int \cos^2 x \, dx = \frac{x}{2} + \frac{\sin 2x}{4} + C.$$

Use it for the square of any sine or cosine, including those met in §15.4 and §15.7.

A squared tangent is rewritten with the Pythagorean identity instead: $\tan^2 x = \sec^2 x - 1$, so $\int \tan^2 x \, dx = \tan x - x + C$.

YOUR TURN 15.3 Rewriting Before Integrating

Dividing first

14. Rewrite each integrand before integrating.

(a) $\displaystyle\int \frac{x^2 + 3x}{x} \, dx$

(b) $\displaystyle\int \frac{2x + 1}{x} \, dx$

(c) $\displaystyle\int \frac{x^3 - 1}{x^2} \, dx$

(d) $\displaystyle\int \frac{2x + 3}{x + 1} \, dx$

(e) $\displaystyle\int \frac{x^2 + 1}{x + 1} \, dx$

Partial fractions

15.

(a) Find $\displaystyle\int \frac{1}{x^2 - 9} \, dx$ using partial fractions.

(b) Find $\displaystyle\int \frac{4x + 5}{(x + 1)(x + 2)} \, dx$.

(c) Find $\displaystyle\int \frac{3}{(x - 2)^2} \, dx$, and explain why the answer has no logarithm.

Completing the square

16.

(a) Find $\displaystyle\int \frac{1}{x^2+4x+13}\,dx$.

(b) Find $\displaystyle\int \frac{1}{2x^2+4x+10}\,dx$.

(c) Explain how you decide, from the denominator alone, whether to use partial fractions or to complete the square.

(d) Find $\displaystyle\int \frac{2x+7}{x^2+4x+13}\,dx$ by writing the numerator as the derivative of the denominator plus a constant.

Using an identity

17.

(a) Find $\displaystyle\int \cos^2 x\,dx$.

(b) Find $\displaystyle\int \sin^2 3x\,dx$.

(c) Evaluate $\displaystyle\int_0^{\pi/6} \sin^2 x\,dx$, giving an exact answer.

(d) Find $\displaystyle\int \tan^2 x\,dx$.

Concept check

18. A student writes $\displaystyle\int \sin^2 x\,dx = \frac{\sin^3 x}{3} + C$. Identify the error, and find the correct integral.

19. A student writes $\displaystyle\int \frac{1}{x^2+4}\,dx = \ln(x^2+4) + C$. Identify the error, and find the correct integral.

15.4 Integration by Substitution

Substitution reverses the chain rule. Look for a composite function whose inner function has its derivative, up to a constant multiple, as a factor outside. In $2x(x^2+1)^3$ the inner function is x^2+1, and its derivative $2x$ is the factor outside. Put u equal to the inner function, and the integral in x becomes a simpler one in u.

The method has four steps.

1. Let u be the inner function.
2. Replace dx using $du = \dfrac{du}{dx}\,dx$, so that no x remains.
3. Integrate with respect to u.

4. Substitute back to write the answer in terms of x.

Step 2 is the chain rule read in reverse. If F is an antiderivative of f, the chain rule gives $\frac{d}{dx}F(u) = f(u)\,\frac{du}{dx}$, so

$$\int f(u)\,\frac{du}{dx}\,dx = F(u) + C = \int f(u)\,du.$$

Writing du in place of $\frac{du}{dx}\,dx$ records this result.

EXAMPLE 15.13

Find $\int 2x(x^2+1)^3\,dx$.

SOLUTION

Let $u = x^2 + 1$, so $\frac{du}{dx} = 2x$, giving $du = 2x\,dx$. The integral becomes

$$\int u^3\,du = \frac{u^4}{4} + C = \boxed{\frac{(x^2+1)^4}{4} + C}$$

The factor $2x$ outside is exactly the $\frac{du}{dx}$ required. Without it, an x would remain after the substitution, and the integral could not be written in u alone.

The substitution $u = f(x)$ works for every fraction whose numerator is the derivative of its denominator, because $du = f'(x)\,dx$ turns it into $\int \frac{1}{u}\,du$:

KEY 15.9 **Integral of $f'(x)/f(x)$**

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

Once the pattern is recognised, the substitution need not be written out. This is integration by inspection again §15.2.1.

EXAMPLE 15.14

Find $\int \tan x\,dx$.

SOLUTION

Write $\tan x = \frac{\sin x}{\cos x}$. The derivative of the denominator $\cos x$ is $-\sin x$, so the numerator is -1 times that derivative:

$$\int \tan x\,dx = -\int \frac{-\sin x}{\cos x}\,dx = \boxed{-\ln|\cos x| + C}$$

This result is not in the formula booklet. It is needed again in the problems on integration by parts, and in §16.3, where it gives the integrating factor $\sec x$.

CAUTION For a definite integral by substitution, either convert the limits to u-values, or substitute back to x before applying the original limits. Applying the x-limits to a u-expression gives a wrong answer.

The limits can be converted because $F(u(x))$ is an antiderivative of the original integrand in x. Evaluating it from $x = a$ to $x = b$ gives $F(u(b)) - F(u(a))$, which is the integral of $f(u)$ from $u(a)$ to $u(b)$.

EXAMPLE 15.15

Evaluate $\displaystyle\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx$.

SOLUTION

Let $u = x^2 + 1$, so $du = 2x\,dx$ and $x\,dx = \frac{1}{2}\,du$. Convert the limits too: $x = 0 \implies u = 1$ and $x = \sqrt{3} \implies u = 4$. Then

$$\begin{aligned}\int_0^{\sqrt{3}} x\sqrt{x^2+1}\,dx &= \frac{1}{2}\int_1^4 \sqrt{u}\,du \\ &= \frac{1}{2}\cdot\frac{2}{3}\Big[u^{3/2}\Big]_1^4 \\ &= \frac{1}{3}(8-1) \\ &= \boxed{\frac{7}{3}}\end{aligned}$$

Once the limits are converted, you never return to x at all; the whole calculation is done in u.

Substitution is also the method for many exponential and trigonometric integrands that match no standard form.

EXAMPLE 15.16

Find (a) $\displaystyle\int xe^{-x^2}\,dx$, (b) $\displaystyle\int \sin^4 x\cos x\,dx$.

SOLUTION

(a) The inner function is $-x^2$, and its derivative $-2x$ appears outside up to the constant -2. Let $u = -x^2$, so $du = -2x\,dx$ and $x\,dx = -\frac{1}{2}\,du$:

$$\begin{aligned}\int xe^{-x^2}\,dx &= -\frac{1}{2}\int e^u\,du \\ &= -\frac{1}{2}e^u + C \\ &= \boxed{-\frac{1}{2}e^{-x^2} + C}\end{aligned}$$

On its own, e^{-x^2} cannot be integrated in terms of the functions of this course. The factor x is what makes the integral possible.

(b) Here the inner function is $\sin x$, and its derivative $\cos x$ is a factor outside. Let $u = \sin x$, so $du = \cos x\,dx$:

$$\int \sin^4 x \cos x\,dx = \int u^4\,du = \frac{u^5}{5} + C = \boxed{\frac{\sin^5 x}{5} + C}$$

In both parts the test is the same: is the derivative of the inner function present as a factor? If it is, substitution usually works. If it is not, another method is usually quicker.

15.4.1 A substitution that is given HL

In every integral so far, u was an inner function whose derivative was present as a factor. Some integrals need a substitution that is not of this kind, and the question then states it. The steps are the same, with one change: write x in terms of the new variable, and differentiate that to replace dx.

EXAMPLE 15.17

Use the substitution $u = \sqrt{x}$ to find $\displaystyle\int \frac{1}{x + \sqrt{x}}\,dx$.

SOLUTION

Write $x = u^2$, so $\dfrac{dx}{du} = 2u$ and $dx = 2u\,du$. Then

$$\int \frac{1}{x + \sqrt{x}}\,dx = \int \frac{2u}{u^2 + u}\,du = \int \frac{2}{u + 1}\,du = 2\ln|u + 1| + C = \boxed{2\ln\left(\sqrt{x} + 1\right) + C}$$

The modulus sign is dropped at the end because $\sqrt{x} + 1$ is positive.

A trigonometric substitution works the same way. To evaluate $\displaystyle\int_0^1 \sqrt{1 - x^2}\,dx$ with $x = \sin\theta$, write $dx = \cos\theta\,d\theta$. The limits become $\theta = 0$ and $\theta = \frac{\pi}{2}$, and on that interval $\sqrt{1 - \sin^2\theta} = \cos\theta$, because $\cos\theta \geq 0$ there. The double-angle rewriting of §15.3

completes the integral:

$$\int_0^1 \sqrt{1-x^2}\,dx = \int_0^{\pi/2} \cos^2\theta\,d\theta = \left[\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{4}.$$

The answer can be checked without calculus: the region is a quarter of a circle of radius 1, with area $\frac{\pi}{4}$.

YOUR TURN 15.4 Integration by Substitution

Indefinite integrals by substitution

20. Find each integral by substitution.

(a) $\displaystyle\int 3x^2(x^3+1)^4\,dx$

(b) $\displaystyle\int xe^{x^2}\,dx$

(c) $\displaystyle\int 2x\sqrt{x^2+1}\,dx$

(d) $\displaystyle\int \frac{x}{(x^2+4)^2}\,dx$

(e) $\displaystyle\int \frac{\ln x}{x}\,dx$

(f) $\displaystyle\int \sin^3 x\cos x\,dx$

Integrals of the form $\frac{f'(x)}{f(x)}$

21.

(a) Find $\displaystyle\int \frac{3x^2+2}{x^3+2x}\,dx$.

(b) Find $\displaystyle\int \frac{e^x}{e^x+3}\,dx$.

(c) By writing $\cot x = \dfrac{\cos x}{\sin x}$, show that $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$.

(d) Evaluate $\displaystyle\int_0^1 \frac{x+1}{x^2+2x+3}\,dx$, giving an exact answer.

Definite integrals by substitution

22.

(a) Evaluate $\displaystyle\int_0^2 x^2\sqrt{x^3+1}\,dx$ by converting the limits.

(b) Evaluate the same integral by substituting back to x first, and confirm that the answers agree.

(c) Evaluate $\displaystyle\int_0^{\pi/2} \sin^2 x\cos x\,dx$ by converting the limits.

(d) Using the substitution $x = 2\sin\theta$, evaluate $\displaystyle\int_0^1 \sqrt{4-x^2}\,dx$, giving an exact answer.

Concept check

23. A student evaluates $\displaystyle\int_0^1 2x\left(x^2+1\right)^3 dx$ with $u = x^2+1$, and writes

$$\int_0^1 u^3\,du = \left[\frac{u^4}{4}\right]_0^1 = \frac{1}{4}.$$

Identify the error, and find the correct value.

15.5 Integration by Parts HL

Substitution reverses the chain rule; **integration by parts** reverses the product rule. It handles products that substitution cannot, such as xe^x. It also handles some single functions, such as $\ln x$, once they are written as a product with 1.

EXAMPLE 15.18

Use the product rule to show that $\displaystyle\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx$.

SOLUTION

By the product rule §13.6, $\dfrac{d}{dx}(uv) = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$. Integrate both sides with respect to x. The left side returns uv:

$$uv = \int u\frac{dv}{dx}\,dx + \int v\frac{du}{dx}\,dx.$$

Subtracting the last integral from both sides gives the result, the formula in the card below.

FORMULA 15.10 **Integration by parts** FORMULA BOOKLET · 5.16 HL

$$\int u\frac{dv}{dx}\,dx = uv - \int v\frac{du}{dx}\,dx \qquad \text{or} \qquad \int u\,dv = uv - \int v\,du$$

These are the two forms printed in the formula booklet. The second is the first with dv written for $\frac{dv}{dx}\,dx$ and du for $\frac{du}{dx}\,dx$, as in §15.4. In working it is quicker to name the two factors of the integrand directly. Call them u and w, with u the one you differentiate and w the one you integrate. In the booklet's letters, w is $\frac{dv}{dx}$ and v is $\int w\,dx$, so the working form below is the same statement as the booklet card.

KEY 15.11 **Integration by parts, working form** HL

$$\int u\,w\,dx = u\int w\,dx \;-\; \int\left(\int w\,dx\right)u'\,dx$$

The first function times the integral of the second, minus the integral of that integral times the derivative of the first.

The first factor, u, is copied once and differentiated once. The second, w, is only ever integrated, and the same $\int w\,dx$ appears twice. Work it out once and write it in both places.

	factor	after one step
differentiate	u	u'
integrate	w	$\int w\,dx$

Choose as u the factor that becomes simpler when differentiated, so that the second integral is easier than the one you started with. The order **ILATE** is a useful guide: whichever type appears first in this list should be u.

I **Inverse trigonometric**: $\arcsin x$, $\arctan x$

L **Logarithmic**: $\ln x$

A **Algebraic**: x, x^2, $3x-1$

T **Trigonometric**: $\sin x$, $\cos x$

E **Exponential**: e^x, 2^x

In $\int xe^x\,dx$ the factors are algebraic and exponential, so A comes before E and $u = x$. In $\int x\ln x\,dx$ they are logarithmic and algebraic, so L comes before A and $u = \ln x$. The order works because differentiating reduces the first three types to something simpler, while trigonometric and exponential functions keep the same form however often you differentiate them.

TIP ILATE is a guide, not a theorem, and it usually gives a good choice. The test behind it always applies: is the second integral easier than the first?

For a definite integral by parts, find the whole antiderivative first, then substitute the limits once at the end.

EXAMPLE 15.19

Find $\int xe^x\,dx$.

SOLUTION

Let $u = x$ (so $u' = 1$) and $w = e^x$ (so $\int w\,dx = e^x$):

$$\int xe^x\,dx = xe^x - \int e^x \cdot 1\,dx = xe^x - e^x + C = \boxed{e^x(x-1) + C}$$

Choosing $u = x$ is what makes the new integral, $\int e^x\,dx$, a standard integral. The other choice, $u = e^x$, would give $\int \frac{x^2}{2}e^x\,dx$, which has a higher power of x.

EXAMPLE 15.20

Find (a) $\int (x+1)\sin x\,dx$, (b) $\int_0^{\pi/2} (x+1)\cos x\,dx$.

SOLUTION

(a) The factors are algebraic and trigonometric, so ILATE gives $u = x+1$ and $w = \sin x$, hence $\int w\,dx = -\cos x$ and $u' = 1$:

$$\int (x+1)\sin x\,dx = -(x+1)\cos x - \int (-\cos x)\,dx = \boxed{-(x+1)\cos x + \sin x + C}$$

Watch the two minus signs. Integrating $\sin x$ gives $-\cos x$, and the formula then subtracts the integral of $-\cos x$, so the sign changes back to $+$.

(b) Same choice, now with $w = \cos x$ and $\int w\,dx = \sin x$:

$$\int (x+1)\cos x\,dx = (x+1)\sin x - \int \sin x\,dx = (x+1)\sin x + \cos x + C.$$

Now substitute the limits:

$$\int_0^{\pi/2} (x+1)\cos x\,dx = \Big[(x+1)\sin x + \cos x\Big]_0^{\pi/2} = \left(\frac{\pi}{2} + 1 + 0\right) - (0+1) = \boxed{\frac{\pi}{2}}$$

A single function that has no standard integral, such as $\ln x$ or $\arctan x$, can still be integrated by parts. Write it as the function times 1: take u as the function and $w = 1$, so $\int w\,dx = x$.

EXAMPLE 15.21

Find $\int \ln x\,dx$.

SOLUTION

Write $u = \ln x$ and $w = 1$, so $u' = \dfrac{1}{x}$ and $\int w\,dx = x$:

$$\int \ln x\,dx = x\ln x - \int x \cdot \frac{1}{x}\,dx = \boxed{x\ln x - x + C}$$

EXAMPLE 15.22

Find $\int \arcsin x \, dx$.

SOLUTION

Take $u = \arcsin x$ and $w = 1$, so $u' = \dfrac{1}{\sqrt{1-x^2}}$ and $\int w \, dx = x$:

$$\int \arcsin x \, dx = x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} \, dx.$$

The new integral is a substitution §15.4. Let $t = 1 - x^2$ (the letter t, because u is already in use), so $dt = -2x \, dx$ and $x \, dx = -\frac{1}{2} \, dt$:

$$\int \frac{x}{\sqrt{1-x^2}} \, dx = -\frac{1}{2} \int t^{-1/2} \, dt = -\sqrt{t} + C = -\sqrt{1-x^2} + C.$$

Subtracting this from $x \arcsin x$ gives

$$\int \arcsin x \, dx = \boxed{x \arcsin x + \sqrt{1-x^2} + C}$$

Parts and substitution can both be needed in one question, as here.

15.5.1 Repeated integration by parts

If one application of parts leaves an integral that still needs parts, apply it again. For x^n times an exponential, a sine or a cosine, each round lowers the power by one, so x^n needs n rounds.

EXAMPLE 15.23

Find $\int x^2 e^{-x} \, dx$.

SOLUTION

Apply parts twice. First round, $u = x^2$ and $w = e^{-x}$, so $\int w \, dx = -e^{-x}$ and $u' = 2x$:

$$\int x^2 e^{-x} \, dx = -x^2 e^{-x} + 2 \int x e^{-x} \, dx.$$

The remaining integral needs parts a second time, with $u = x$:

$$\int x e^{-x} \, dx = -x e^{-x} + \int e^{-x} \, dx = -e^{-x}(x+1) + C.$$

Substituting that back,

$$\int x^2 e^{-x} \, dx = -x^2 e^{-x} - 2e^{-x}(x+1) + C = \boxed{-e^{-x}\left(x^2 + 2x + 2\right) + C}$$

Sometimes *neither* factor becomes simpler when differentiated. For $e^x \sin x$, two rounds of parts give the original integral again. Call it I, and solve the resulting equation for I. Take u **of the same type in both rounds** (here the trigonometric factor). If you change type in the second round, the working returns to $I = I$ and gives nothing. Add the constant of integration at the end.

EXAMPLE 15.24

Find $I = \int e^x \sin x \, dx$.

SOLUTION

Parts with $u = \sin x$:

$$I = e^x \sin x - \int e^x \cos x \, dx.$$

Parts again on the new integral, $u = \cos x$:

$$\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx = e^x \cos x + I.$$

Substitute back:

$$I = e^x \sin x - e^x \cos x - I \implies 2I = e^x(\sin x - \cos x) \implies \boxed{I = \tfrac{1}{2}e^x(\sin x - \cos x) + C}$$

15.5.2 Choosing a method

The table lists the methods of this chapter, and starting the wrong one wastes time. Choose by looking at the integrand before writing anything.

Work down the table from top to bottom, because the earlier tests are quicker to apply and the earlier methods are shorter. If a substitution leaves an integral no easier than the one you started with, the substitution was the wrong one. Return to the start and try another method.

What you see	Method	Example
A fraction whose numerator has the same or higher degree as the denominator	divide first	$\dfrac{x^2}{x+1}$
A standard form exactly, or with a linear function inside	use the standard integral, dividing by the inner coefficient	$\cos 3x$, e^{2x}, $(2x+1)^5$
A composite function, with the derivative of its inner function (or a multiple of it) as a factor	substitution	$2x(x^2+1)^3$, $\dfrac{2x}{x^2+1}$
A product of two different types of function, one of which becomes simpler when differentiated; or a single function with no standard integral	parts	xe^x, $x\sin x$, $\ln x$
$\sin^2$ or $\cos^2$ of a linear function	double angle, then linear composites	$\sin^2 x$, $\sin^2 2x$
A substitution stated in the question	substitute as given	$\dfrac{1}{x+\sqrt{x}}$, $u=\sqrt{x}$
A fraction whose denominator has distinct linear factors	partial fractions, then logarithms	$\dfrac{1}{x^2+3x+2}$
A fraction whose denominator is a quadratic with no real zeros	complete the square, then arctangent	$\dfrac{1}{x^2+2x+5}$

YOUR TURN 15.5 Integration by Parts HL

One application of parts

24. Find each integral by parts.

(a) $\displaystyle\int x\cos x\,dx$

(b) $\displaystyle\int xe^{2x}\,dx$

(c) $\displaystyle\int x\ln x\,dx$

(d) $\int \arctan x \, dx$

(e) $\int x^2 \ln x \, dx$

(f) $\int x \sec^2 x \, dx$ *(Hint: $\int \tan x \, dx = -\ln|\cos x| + C$, from §15.4 of the notes)*

(g) $\int \arcsin x \, dx$

Definite integrals and the choice of u

25.

(a) Evaluate $\int_0^{\pi} x \sin x \, dx$.

(b) Evaluate $\int_0^{1} xe^x \, dx$.

(c) State which factor you chose as u in part (b), and why the other choice would not help.

(d) Use ILATE to say which factor should be u in each of $\int x^2 e^x \, dx$, $\int x \arctan x \, dx$ and $\int e^x \sin x \, dx$.

(e) Explain why the third integral in part (d) behaves differently from the first two.

(f) Without integrating, name the method you would use for each of $\int \frac{x^3}{x-1} \, dx$, $\int x \cos(x^2) \, dx$, $\int x \cos x \, dx$ and $\int \frac{1}{x^2-4} \, dx$, and give a reason for each.

Applying parts twice

26. Find each integral. Each needs integration by parts twice.

(a) $\int x^2 e^x \, dx$

(b) $\int x^2 \sin x \, dx$

When the original integral returns

27. Find each integral. After two applications of parts the original integral appears again; solve for it.

(a) $\int e^x \cos x \, dx$

(b) $\int e^{2x} \sin x \, dx$

Concept check

28. A student writes $\int xe^x \, dx = \frac{x^2}{2} e^x + C$, integrating each factor separately. Show, by differentiating, that this is wrong, and find the correct integral.

15.6 Areas

A definite integral gives a signed area (§15.1). Here it gives the area of a region between a curve and the x-axis, and then the area between two curves.

15.6.1 Area under a curve

For a curve above the axis, the area is one definite integral.

FORMULA 15.12 **Area under a curve** FORMULA BOOKLET · 5.5

$$A = \int_a^b y\,dx$$

For a curve that lies above the x-axis between a and b.

TIP Your GDC evaluates definite integrals directly. On a calculator paper, unless the question asks for analytic working, its value can be the answer. Write the integral, with its limits, before every GDC value, because the written integral shows the method. When analytic working is required, the GDC value is a quick check of your antiderivative.

EXAMPLE 15.25

Find the area under $y = \sqrt{1+x^3}$ from $x = 0$ to $x = 2$.

SOLUTION

This area cannot be found with the antiderivatives of this course, so it is a calculator question. Write the integral first, then the value the GDC gives:

$$A = \int_0^2 \sqrt{1+x^3}\,dx \approx \boxed{3.24}$$

The written integral is the method, and it is what earns the method mark.

Where the curve lies below the x-axis, the integral is negative (§15.1.3).

FORMULA 15.13 **Area enclosed by a curve and the x-axis** FORMULA BOOKLET · 5.11

$$A = \int_a^b |y|\,dx$$

Without a calculator, find where $y = 0$ between a and b, integrate over each piece, and add the moduli. With a GDC, evaluate $\int_a^b |y|\,dx$ directly.

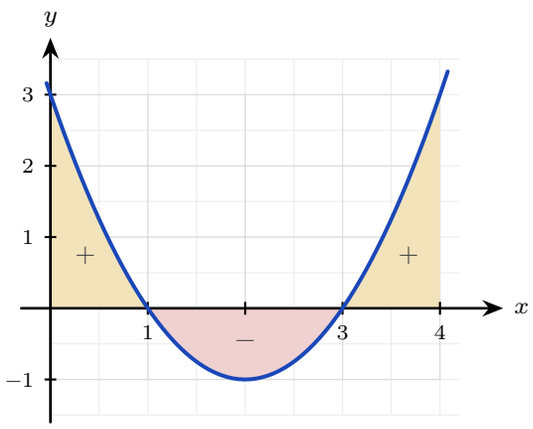


Figure 15.6 The curve $y = x^2 - 4x + 3$. A definite integral adds *signed* area: the gold regions above the axis count positively, the red one below negatively. One integral across the whole interval gives less than the total area.

EXAMPLE 15.26

Find the total area between the curve $y = x^2 - 4x + 3$ and the x-axis from $x = 0$ to $x = 4$.

SOLUTION

The curve, shown in the figure, is $y = (x-1)(x-3)$, so it crosses the x-axis at $x = 1$ and $x = 3$. Split the interval at the crossings. An antiderivative is $F(x) = \frac{1}{3}x^3 - 2x^2 + 3x$, with $F(0) = 0$, $F(1) = \frac{4}{3}$, $F(3) = 0$ and $F(4) = \frac{4}{3}$.

Interval	Definite integral	Area
$[0, 1]$	$F(1) - F(0) = \frac{4}{3}$	$\frac{4}{3}$
$[1, 3]$	$F(3) - F(1) = -\frac{4}{3}$	$\frac{4}{3}$
$[3, 4]$	$F(4) - F(3) = \frac{4}{3}$	$\frac{4}{3}$

The middle integral is negative because the red region lies below the axis. The area is the sum of the moduli, $\frac{4}{3} + \frac{4}{3} + \frac{4}{3} = \boxed{4}$

A single integral from 0 to 4 gives $F(4) - F(0) = \frac{4}{3}$, because the negative integral over the red region cancels the positive integral over one gold region.

CAUTION Before finding an area, check where the curve crosses the x-axis. Taking the modulus of one integral across the whole interval does not recover the area lost through cancellation.

15.6.2 Area between two curves

To find the area between two curves, integrate the difference: the upper curve minus the lower. No modulus is needed, because $y_{\text{top}} - y_{\text{bottom}}$ is positive wherever the upper curve is above the lower one. This holds even where both curves lie below the x-axis.

KEY 15.14 **Area between two curves**

$$A = \int_a^b (y_{\text{top}} - y_{\text{bottom}})\,dx$$

Valid only where $y_{\text{top}} \geq y_{\text{bottom}}$ on the whole interval. If the curves cross between a and b, split the integral at each crossing.

To find the area between two curves:

1. Find where the curves meet: solve the equation formed by setting the two expressions for y equal. These x-values are usually the limits.
2. Between each pair of intersections, test one convenient point to find which curve is on top.
3. Integrate $y_{\text{top}} - y_{\text{bottom}}$ over each interval, and add the results.

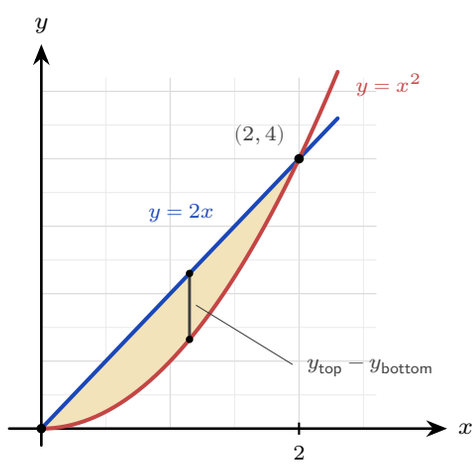


Figure 15.7 The region enclosed between $y = 2x$ and $y = x^2$.

EXAMPLE 15.27

Find the area of the region enclosed between $y = 2x$ and $y = x^2$, shown in the figure.

SOLUTION

The curves meet where $2x = x^2$, that is at $x = 0$ and $x = 2$. Test $x = 1$: the line gives 2 and the curve gives 1. So on $(0, 2)$ the line $y = 2x$ is above, and

$$\begin{aligned} A &= \int_0^2 (2x - x^2)\,dx \\ &= \left[x^2 - \tfrac{x^3}{3}\right]_0^2 \\ &= 4 - \tfrac{8}{3} \\ &= \boxed{\tfrac{4}{3}} \end{aligned}$$

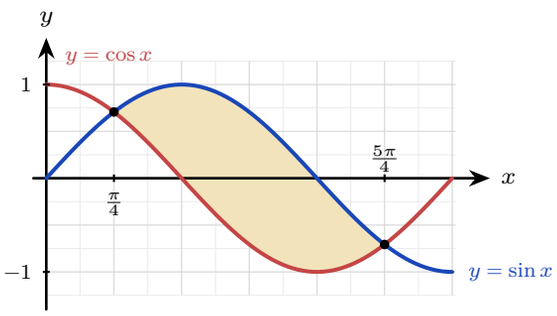


Figure 15.8 $y = \sin x$ and $y = \cos x$ meet at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$. Between them the sine curve is on top, and the gold region is the area found below.

EXAMPLE 15.28

Find the area enclosed between $y = \sin x$ and $y = \cos x$ between their first two intersections for $x \geq 0$.

SOLUTION

They meet where $\sin x = \cos x$, that is $\tan x = 1$, so the first two intersections are at $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$.
On $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$ the sine is above: at $x = \frac{\pi}{2}$, $\sin\frac{\pi}{2} = 1$ while $\cos\frac{\pi}{2} = 0$. So

$$A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\, dx = \Big[-\cos x - \sin x\Big]_{\pi/4}^{5\pi/4}.$$

At the upper limit both $\cos x$ and $\sin x$ equal $-\frac{\sqrt{2}}{2}$, so the bracket is $\sqrt{2}$; at the lower limit both equal $\frac{\sqrt{2}}{2}$ and the bracket is $-\sqrt{2}$:

$$A = \sqrt{2} - \left(-\sqrt{2}\right) = \boxed{2\sqrt{2}} \approx 2.83$$

When the curves cross between the limits, the curve on top changes at each crossing. A single integral of the difference then adds a positive part and a negative part, and they cancel. Split the interval at every crossing, integrate the upper curve minus the lower one on each piece, and add the results. With a GDC, $\int_a^b |y_1 - y_2|\, dx$ gives the total in one step, whichever curve is on top.

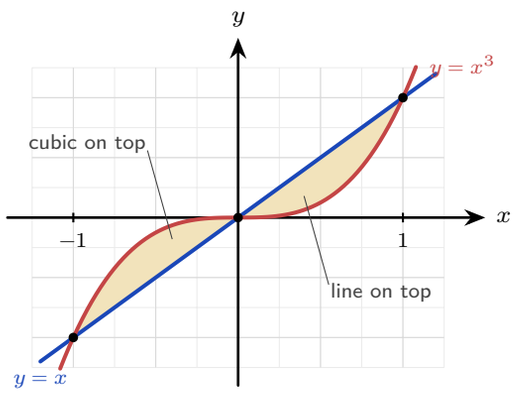


Figure 15.9 $y = x^3$ and $y = x$ cross at $x = -1$, 0 and 1. The cubic is on top in the left region and the line in the right one, so the area is found in two pieces.

EXAMPLE 15.29

Find the total area enclosed between $y = x^3$ and $y = x$.

SOLUTION

The curves meet where $x^3 = x$, that is $x(x-1)(x+1) = 0$, so at $x = -1$, 0 and 1. There are two regions, so test a point in each. At $x = -\frac{1}{2}$ the cubic gives $-\frac{1}{8}$ and the line gives $-\frac{1}{2}$, so the cubic is on top on $(-1, 0)$. At $x = \frac{1}{2}$ the line gives $\frac{1}{2}$ and the cubic gives $\frac{1}{8}$, so the line is on top on $(0, 1)$.

$$A_1 = \int_{-1}^{0} (x^3 - x)\,dx = \left[\tfrac{x^4}{4} - \tfrac{x^2}{2}\right]_{-1}^{0} = 0 - \left(\tfrac{1}{4} - \tfrac{1}{2}\right) = \tfrac{1}{4},$$

$$A_2 = \int_{0}^{1} (x - x^3)\,dx = \left[\tfrac{x^2}{2} - \tfrac{x^4}{4}\right]_{0}^{1} = \tfrac{1}{2} - \tfrac{1}{4} = \tfrac{1}{4}.$$

So the total area is $A = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$

A single integral $\int_{-1}^{1} (x - x^3)\,dx$ gives 0, not $\frac{1}{2}$. The upper curve changes at $x = 0$, so the two equal pieces have opposite signs and cancel.

15.6.3 Area between a curve and the y-axis HL

Some regions are bounded by the y-axis rather than the x-axis, or their curve is given as x in terms of y. For these it is shorter to integrate with respect to y. Express x as a function of y, and add horizontal strips of length $|x|$ and thickness dy. The limits a and b are then y-values. The modulus has the same role as in §15.6.1: where the curve lies to the left of the y-axis, x is negative, so split the integral at each crossing of the y-axis.

FORMULA 15.15 **Area enclosed by a curve and the y-axis** FORMULA BOOKLET · 5.17 HL

$$A = \int_a^b |x|\, dy$$

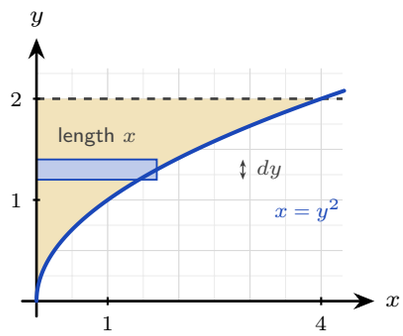


Figure 15.10 The region between $x = y^2$ and the y-axis for $0 \le y \le 2$. Each horizontal strip has length x and thickness dy.

EXAMPLE 15.30

Find the area between the curve $x = y^2$ and the y-axis from $y = 0$ to $y = 2$.

SOLUTION

Integrate with respect to y:

$$A = \int_0^2 y^2\, dy = \left[\frac{y^3}{3}\right]_0^2 = \frac{8}{3}$$

In x the same area needs $\int_0^4 (2 - \sqrt{x})\, dx$: the line $y = 2$ minus the curve. Integrating in y is shorter.

YOUR TURN 15.6 Areas

Curves that cross the x-axis

29. Each curve crosses the x-axis inside the interval. Find the total area between the curve and the x-axis over the interval given.

(a) $y = x^3$, from $x = -2$ to $x = 2$.

(b) $y = x^2 - x - 2$, from $x = 0$ to $x = 3$. State also the value of $\int_0^3 (x^2 - x - 2)\, dx$, and explain why it is not the area.

(c) $y = x^3 - x$, from $x = 0$ to $x = 2$.

(d) $y = 2\cos x - 1$, from $x = 0$ to $x = \frac{\pi}{2}$.

(e) $y = e^x - 2$, from $x = 0$ to $x = \ln 4$.

(f) $y = \dfrac{1}{x} - 1$, from $x = \frac{1}{2}$ to $x = 2$.

Finding an enclosed area

30. Consider the region enclosed by $y = x^2$ and $y = x + 2$.

(a) Find the x-coordinates where the two curves meet.

(b) State which curve is the upper boundary between those values.

(c) Find the area of the enclosed region.

(d) State what answer you would get by integrating $x^2 - (x + 2)$ instead, and explain what has gone wrong.

(e) Sketch the curve and the line on the same axes. Label the points where they meet, and shade the enclosed region.

31. Find the area of the region enclosed by each pair of curves.

(a) $y = x$ and $y = x^2$.

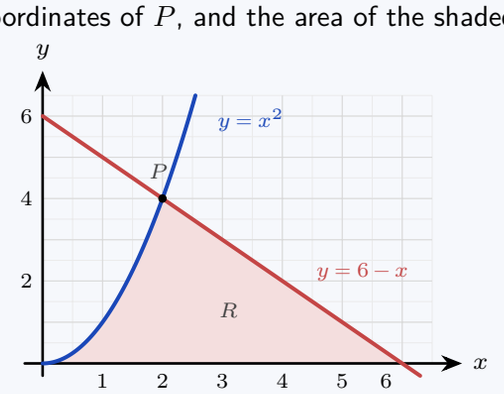
(b) $y = 6 - x^2$ and $y = 2$.

(c) $y = x^2$ and $y = 2 - x^2$.

(d) $y = x^2$ and $y = 3x + 4$.

(e) $y = e^x$ and $y = e^{-x}$, between $x = 0$ and $x = 1$.

(f) The diagram shows the curve $y = x^2$ and the line $y = 6 - x$, which meet at the point P. Find the coordinates of P, and the area of the shaded region R.



Curves that cross inside the interval

32. GDC

(a) Without a GDC, find the total area between $y = x^2$ and $y = x$ from $x = 0$ to $x = 2$.

(b) Without a GDC, find the total area of the two regions between $y = x^3$ and $y = 4x$.

(c) The curves $y = e^x$ and $y = 4 - x^2$ cross once between $x = 0$ and $x = 2$. Use your GDC to find where they cross, and to find the total area between the curves from $x = 0$ to $x = 2$.

(d) A student evaluates $\int_0^2 (e^x - 4 + x^2)\,dx$ and gets about 1.06. Explain why this is not the area in part (c).

Trigonometric boundaries

33.

(a) Find the area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{4}$.
(b) Find the total area between $y = \sin x$ and $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$.
(c) Find the area of the region enclosed by $y = \sin 2x$ and $y = \sin x$ for $0 \le x \le \frac{\pi}{3}$.

Integrating with respect to y

34.

(a) Find the area between the curve $x = y^2 + 1$ and the y-axis, from $y = 0$ to $y = 3$.
(b) Find the area between the curve $y = x^3$ and the y-axis, from $y = 1$ to $y = 8$.

Concept check

35. A student is finding the area enclosed by $y = x$ and $y = x^2 - 2$, and says: "*part of the region is below the x-axis, so I must split the integral where the curves cross the x-axis.*" Explain why no split is needed, and find the area.

15.7 Volumes of Revolution HL

Many solids, such as a bowl, a bottle or a lens, have a curved outline and round cross-sections. Geometry has no formula for their volume, but slicing gives one. Rotate the region under a curve through $360°$ about the x-axis, and it forms such a solid. Slice the solid into thin discs perpendicular to the axis. Each disc is a cylinder of radius y (the curve's height) and thickness dx, with volume $\pi y^2\,dx$. Adding the discs and letting their thickness tend to zero turns the sum into an integral. About the y-axis the same argument uses radius x and thickness dy.

FORMULA 15.16 **Volume of revolution** FORMULA BOOKLET · 5.17 HL

$$V = \int_a^b \pi y^2\,dx \quad \text{(about the } x\text{-axis)}, \qquad V = \int_a^b \pi x^2\,dy \quad \text{(about the } y\text{-axis)}$$

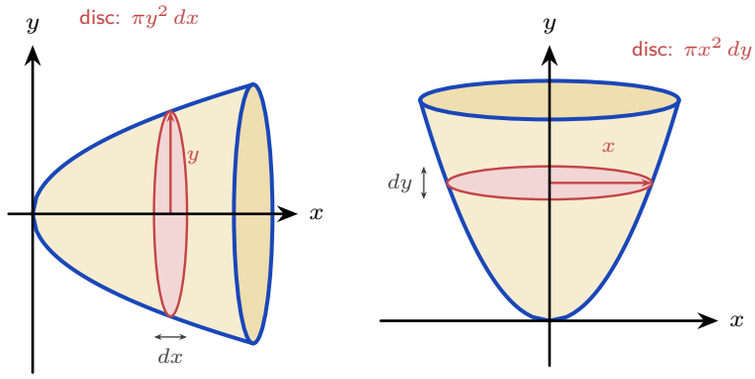


Figure 15.11 The same curve rotated about each axis. About the x-axis each disc has radius y and thickness dx, so volume $\pi y^2\,dx$; about the y-axis, radius x and thickness dy, so volume $\pi x^2\,dy$.

About the y-axis, each disc comes from a horizontal strip that runs from the y-axis out to the curve. So the region rotated is the region between the curve and the y-axis, not the region under the curve. The two regions give different volumes in general. Write x^2 in terms of y and use y-values as the limits. Square y (or x) before integrating. Integrating first and then squaring the result gives a different number. For the region under $y = x^3$ from 0 to 2, that would give $\pi\left(\int_0^2 x^3\,dx\right)^2 = 16\pi$ instead of the correct $\frac{128\pi}{7}$.

EXAMPLE 15.31

The region under $y = x^3$, for $0 \le x \le 2$, is rotated through 360° about the x-axis. Find the volume.

SOLUTION

Use $V = \pi \int_a^b y^2\,dx$ with $y = x^3$:

$$\begin{aligned} V &= \int_0^2 \pi (x^3)^2\,dx \\ &= \pi \int_0^2 x^6\,dx \\ &= \pi \left[\frac{x^7}{7}\right]_0^2 \\ &= \boxed{\frac{128\pi}{7}} \end{aligned}$$

EXAMPLE 15.32

The region between $y = x^2$ and the y-axis, from $y = 0$ to $y = 1$, is rotated about the y-axis. Find the volume.

SOLUTION

About the y-axis, slice into discs of radius x and thickness dy, with $x^2 = y$:

$$\begin{aligned} V &= \int_0^1 \pi x^2 \, dy \\ &= \pi \int_0^1 y \, dy \\ &= \pi \left[\tfrac{y^2}{2}\right]_0^1 \\ &= \boxed{\frac{\pi}{2}} \end{aligned}$$

Because the formula squares the function, a sine or cosine curve gives $\sin^2$ or $\cos^2$ in the integral. Rewrite these with a double angle before integrating §15.3.

EXAMPLE 15.33

The region under $y = \sin 2x$ from $x = 0$ to $x = \frac{\pi}{4}$ is rotated $360°$ about the x-axis. Find the volume.

SOLUTION

Apply the formula, squaring the function first:

$$V = \int_0^{\pi/4} \pi y^2 \, dx = \pi \int_0^{\pi/4} \sin^2 2x \, dx.$$

There is no standard integral for $\sin^2 2x$, so replace it using $\sin^2 2x = \dfrac{1 - \cos 4x}{2}$:

$$\pi \int_0^{\pi/4} \frac{1 - \cos 4x}{2} \, dx = \pi \left[\frac{x}{2} - \frac{\sin 4x}{8}\right]_0^{\pi/4} = \pi \left(\frac{\pi}{8} - 0\right) = \boxed{\frac{\pi^2}{8}} \approx 1.23$$

Two πs appear for different reasons, and they must be kept apart: one comes from the volume formula, the other from the upper limit.

Now let the region lie between two curves, with y_{outer} further from the x-axis than y_{inner} across $a \le x \le b$. Each slice is then a disc with a disc-shaped hole in it, called a **washer**. Its volume is the volume of the outer disc minus that of the hole, $\pi y_{\text{outer}}^2 \, dx - \pi y_{\text{inner}}^2 \, dx$, so

$$V = \pi \int_a^b (y_{\text{outer}}^2 - y_{\text{inner}}^2) \, dx.$$

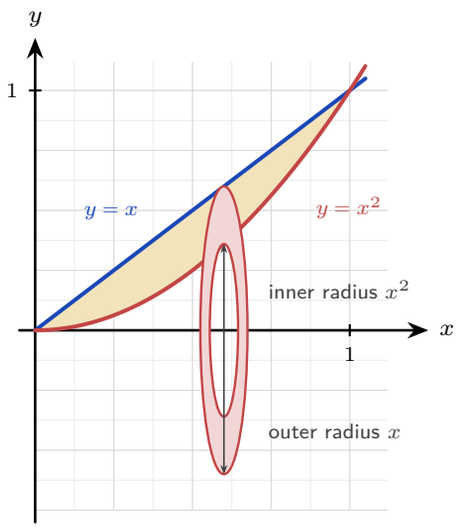


Figure 15.12 Rotating the region between $y = x$ and $y = x^2$: each slice is a washer, a disc with a disc-shaped hole.

This needs both curves on the same side of the axis across the interval, so that each slice really has a hole. Where the region reaches the axis there is no hole, and the inner radius is 0. If the region crosses the axis, the slice has no hole. Its radius is given by whichever curve is further from the axis.

EXAMPLE 15.34

The region enclosed by $y = x$ and $y = x^2$ is rotated $360°$ about the x-axis. Find the volume.

SOLUTION

The curves meet at $x = 0$ and $x = 1$, and on $(0, 1)$ the line is further from the axis, since $x > x^2$ there. So $y_{\text{outer}} = x$ and $y_{\text{inner}} = x^2$:

$$V = \pi \int_0^1 (x^2 - x^4)\,dx = \pi \left[\tfrac{x^3}{3} - \tfrac{x^5}{5}\right]_0^1 = \boxed{\frac{2\pi}{15}}$$

CAUTION Square each function first, then subtract. Squaring the difference, $\pi \int_0^1 (x - x^2)^2\,dx = \frac{\pi}{30}$, gives a different number, which is not the volume.

YOUR TURN 15.7 Volumes of Revolution HL

Rotating about the x-axis

36. Each region under the curve is rotated $360°$ about the x-axis. Find the volume.

(a) $y = x^2$, $0 \le x \le 2$.

(b) $y = \sqrt{x}$, $0 \le x \le 4$.
(c) $y = 2x$, $0 \le x \le 3$.
(d) $y = \dfrac{1}{x}$, $1 \le x \le 3$.
(e) $y = e^{-x}$, $0 \le x \le \ln 2$.
(f) $y = e^{3x}$, $0 \le x \le \frac{1}{3}$.

Squares of sine and cosine

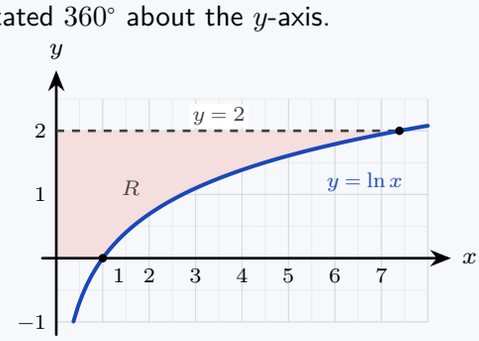
37. Each region under the curve is rotated $360°$ about the x-axis. Find the exact volume.

(a) $y = \cos x$, $0 \le x \le \frac{\pi}{2}$.
(b) $y = \sin x$, $0 \le x \le \frac{\pi}{4}$.
(c) $y = 1 + \cos x$, $0 \le x \le \pi$.
(d) Explain why the volumes in this question need a double-angle rewriting, while those in Question 36(e) and (f) did not.

Rotating about the y-axis

38. Each region is rotated $360°$ about the y-axis. Find the volume.

(a) The region between $y = x^3$ and the y-axis, from $y = 0$ to $y = 8$.
(b) The region between $y = x^2 + 1$ and the y-axis, from $y = 1$ to $y = 3$.
(c) The region between $x = y^2$ and the y-axis, from $y = 0$ to $y = 2$.
(d) Your answer to part (c) equals your answer to Question 36(a). Explain why.
(e) The shaded region R in the diagram is bounded by the curve $y = \ln x$, the y-axis, the x-axis and the line $y = 2$. Find the exact area of R. Find also the exact volume when R is rotated $360°$ about the y-axis.



Washers

39. The region enclosed by each pair of curves is rotated $360°$ about the x-axis. Find the volume.

(a) $y = \sqrt{x}$ and $y = x^2$.
(b) $y = x + 2$ and $y = x^2$.
(c) $y = 4 - x^2$ and $y = 3$.
(d) $y = 4 - x^2$ and $y = \frac{1}{2}x^2 - 2$. Explain why the slices of this solid have no hole, and find the volume.

(e) On one set of axes, sketch $y = \sqrt{x}$ and $y = x^2$ for $0 \le x \le 1$, and shade the region of part (a). Draw one vertical slice of the region, and mark its outer radius and its inner radius.

Concept check

40. For the region enclosed by $y = \sqrt{x}$ and $y = x^2$ in Question 39(a), a student writes $V = \pi \int_0^1 (\sqrt{x} - x^2)^2 \, dx$. Explain the error, and give the correct integral and volume.

15.8 Kinematics by Integration

Cars, planets, blood in an artery and the electrons in a wire all move. To describe a moving object, and to predict where it will be, you need a few quantities, such as position, velocity and acceleration, and the links between them. In §14.4 you started from the position and differentiated. In this section you start from the velocity, which is often what a sensor records, and integrate to find where the object is and how far it has gone.

15.8.1 The quantities of motion

An object is in **motion** when its position changes with time. In this section a particle moves along a straight line. Choose a fixed origin O and a positive direction. The particle's **position** $s(t)$ is its signed distance from O at time t; it is also called its displacement from O. Each other quantity of motion is built from the one before:

- the change in position over an interval is the **displacement**;
- the rate at which displacement changes is the **velocity**;
- the rate at which velocity changes is the **acceleration**.

In symbols,

$$\Delta s = s(t_2) - s(t_1), \qquad v = \frac{ds}{dt}, \qquad a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Differentiating moves along the chain from s to v to a. Integrating moves back, from a to v to s.

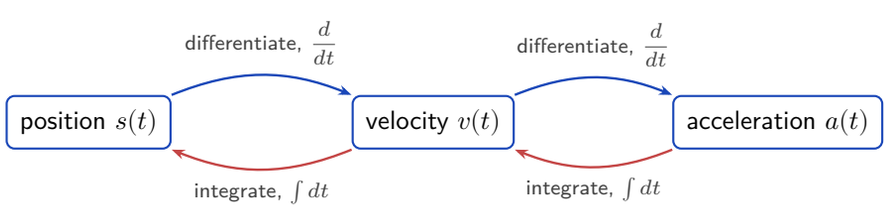


Figure 15.13 Differentiate to go from s to v to a; integrate to come back.

Displacement, velocity and acceleration have a direction, which on a line is a sign. Ignore the direction, and two new quantities appear. The total length of the path, counting forward and backward motion, is the **distance travelled**. The size of the velocity is the **speed**. Keep the six apart, because a question may ask for any one of them; the table below collects them.

Suppose a particle moves from $s = 2$ to $s = 7$, and then back to $s = 4$. Its displacement is $\Delta s = 4 - 2 = 2$ m, but the distance it travels is $D = |7 - 2| + |4 - 7| = 5 + 3 = 8$ m. Displacement depends only on where the motion starts and ends. Distance includes every part of the journey. Whether a particle speeds up or slows down depends on the signs of v and a together, as in §14.4.

Quantity	Meaning	Can it be negative?	Units
position $s(t)$	its signed coordinate, measured from O	yes: positive on the positive side of O, negative otherwise	m
displacement Δs	over an interval, the change in position $s(t_2) - s(t_1)$: end minus start	yes: positive if it ends on the positive side of its start, negative otherwise	m
distance travelled D	the length of the whole path, forwards and backwards	no: it is a length	m
velocity $v = \dfrac{ds}{dt}$	the rate of change of displacement	yes: positive if moving in the positive direction, negative otherwise	$\mathrm{m\,s^{-1}}$
speed $\lvert v \rvert$	the size of the velocity	no: it is the size of v	$\mathrm{m\,s^{-1}}$
acceleration $a = \dfrac{dv}{dt}$	the rate of change of velocity	yes: positive if the velocity is increasing, negative otherwise	$\mathrm{m\,s^{-2}}$

A negative position and a negative displacement are both easy to over-read. A negative position does not show which way the particle is moving, and a negative displacement does not mean it only moved backwards: it may have turned back on the way.

15.8.2 Working from the velocity

Many questions give the velocity $v(t)$ and ask for something else. Differentiate v to find the acceleration. Integrate v to find the position, the displacement or the distance.

The signs of v also describe the motion directly. The particle is at rest where $v = 0$, moves in the positive direction where $v > 0$, and turns back only where v changes sign.

Each indefinite integral has a constant. A known value of v fixes the constant in $v(t)$, and a known value of s fixes the constant in $s(t)$. A value given at the start of the motion is an **initial condition** (the value of v or s you are given at $t = 0$). Questions state it in words:

- "starts from rest" means $v(0) = 0$;
- "starts at the origin" means $s(0) = 0$;
- "starts 5 m from O, on the positive side" means $s(0) = 5$.

CAUTION "Starts 5 m from O" on its own gives only $|s(0)| = 5$. You need the direction to decide between $s(0) = 5$ and $s(0) = -5$.

When a question gives the acceleration instead, first get to the velocity: integrate a once and use the initial velocity. From there, work as before.

EXAMPLE 15.35

A particle starts from rest at the origin, and its acceleration is $a = (6t - 4)\ \text{m s}^{-2}$. Find its velocity and its position at $t = 2$.

SOLUTION

"From rest at the origin" gives $v(0) = 0$ and $s(0) = 0$. First get to the velocity: integrate a and use $v(0) = 0$:

$$v = \int (6t - 4)\,dt = 3t^2 - 4t + C, \qquad 0 = 0 - 0 + C \implies C = 0.$$

Then integrate the velocity, and use $s(0) = 0$:

$$s = \int (3t^2 - 4t)\,dt = t^3 - 2t^2 + D, \qquad D = 0.$$

At $t = 2$, $v = 12 - 8 = \boxed{4\ \text{m s}^{-1}}$ and $s = 8 - 8 = \boxed{0\ \text{m}}$

At $t = 2$ the particle is back at O, but it has moved. The velocity $v = t(3t - 4)$ is negative until $t = \frac{4}{3}$, so the particle first goes in the negative direction and then returns. Its displacement over $0 \le t \le 2$ is zero; the distance it has travelled is not. §15.8.3 shows how to find that distance.

You can also build the known value into a definite integral, with no constant to find:

$$\boxed{v(t) = v(t_0) + \int_{t_0}^{t} a(u)\,du}, \qquad \boxed{s(t) = s(t_0) + \int_{t_0}^{t} v(u)\,du}$$

Here u is the variable of integration, and t is the time at which you want the velocity or the position. In the example above, $s(2) = 0 + \int_0^2 (3u^2 - 4u)\,du = 8 - 8 = 0$, as before.

15.8.3 Displacement and distance

Over an interval from t_1 to t_2, the displacement is $s(t_2) - s(t_1)$. Velocity is the rate of change of s, so integrating v adds up the changes: $\int_{t_1}^{t_2} v\,dt = s(t_2) - s(t_1)$, by the Fundamental Theorem at the end of the chapter. Motion in the negative direction has $v < 0$, so it subtracts. Distance counts that motion as well, so it integrates the speed $|v|$, which is never negative.

FORMULA 15.17 **Displacement and distance** FORMULA BOOKLET · 5.9

$$\Delta s = \int_{t_1}^{t_2} v\,dt, \qquad D = \int_{t_1}^{t_2} |v|\,dt$$

On a velocity–time graph, $\int_{t_1}^{t_2} v\,dt$ is the **signed area** between the graph and the t-axis: area above the axis counts as positive and area below as negative. Distance counts both as positive.

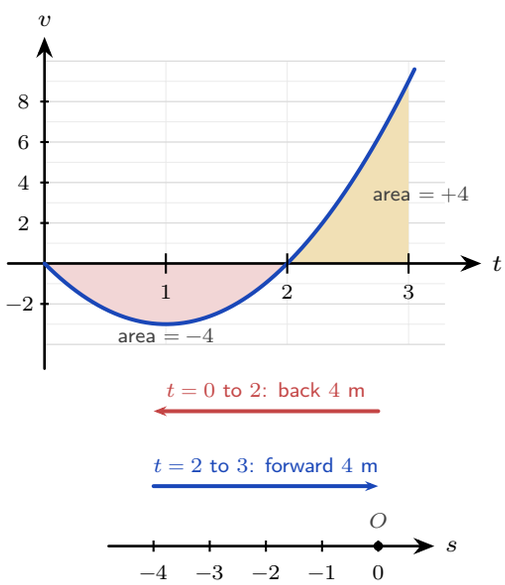


Figure 15.14 The velocity $v = 3t^2 - 6t$ (left) and the same motion on the line (right): 4 m back, then 4 m forward. The signed areas cancel; the distance is 8 m.

To find the distance travelled without a GDC:

1. Find the times in the interval where $v = 0$, and check the sign of v on each side.
2. Split the interval at the times where v changes sign.
3. Integrate v on each piece.
4. Add the sizes of the results, ignoring their signs.

With a GDC, evaluate $\int_{t_1}^{t_2} |v|\,dt$ in one step.

CAUTION Integrating v over the whole interval gives the displacement, not the distance; the two agree only when the particle does not turn back. And $v = 0$ alone does not mean it turns back: $v = (t-1)^2$ is zero at $t = 1$, but never changes sign.

The **average velocity** over an interval is the displacement divided by the time taken. The **average speed** is the distance travelled divided by the time taken. They also differ when the particle turns back.

EXAMPLE 15.36

A particle has velocity $v(t) = (3t^2 - 6t)\,\text{m s}^{-1}$ for $0 \le t \le 3$, as in the figure. Find (a) its displacement, (b) the distance it travels, and (c) its average velocity and average speed over this interval.

SOLUTION

(a) Integrate v directly:

$$\int_0^3 (3t^2 - 6t)\,dt = \Big[t^3 - 3t^2\Big]_0^3 = 27 - 27 = \boxed{0\ \text{m}}$$

(b) Follow the four steps. Here $v = 3t(t-2)$ is zero at $t = 0$ and $t = 2$, and changes sign only at $t = 2$, so split the interval there:

$$\left|\int_0^2 v\,dt\right| + \left|\int_2^3 v\,dt\right| = |-4| + |4| = \boxed{8\ \text{m}}$$

(c) Over 3 seconds, the average velocity is $\frac{0}{3} = \boxed{0\,\text{m s}^{-1}}$ and the average speed is $\frac{8}{3} \approx \boxed{2.67\,\text{m s}^{-1}}$

EXAMPLE 15.37

A particle moves along a line with velocity $v = (2t - 6)\,\text{m s}^{-1}$. At $t = 0$ it is 5 m from O, on the positive side. Find its position at $t = 4$, its displacement from $t = 0$ to $t = 4$, and the distance it travels in that time.

SOLUTION

Integrate v; the constant is the starting position, $s(0) = 5$:

$$s = \int (2t - 6)\,dt = t^2 - 6t + C, \qquad 5 = 0 - 0 + C \implies C = 5.$$

So $s = t^2 - 6t + 5$, and at $t = 4$, $s = 16 - 24 + 5 = \boxed{-3\ \text{m}}$: the particle is 3 m on the negative side of O.

The displacement is end minus start: $s(4) - s(0) = -3 - 5 = \boxed{-8\ \text{m}}$.

For the distance, v changes sign at $t = 3$, so split there:

$$\left|\int_0^3 (2t-6)\,dt\right| + \left|\int_3^4 (2t-6)\,dt\right| = |-9| + |1| = \boxed{10 \text{ m}}$$

It went 9 m back and 1 m forward. The three answers mean:

- where is it? $s(4) = -3$ m;
- how far has its position changed? $\Delta s = \int_0^4 v\,dt = -8$ m;
- how far has it gone? $D = \int_0^4 |v|\,dt = 10$ m.

YOUR TURN 15.8 Kinematics by Integration

Displacement from velocity

41. A particle moves in a straight line, and t is the time in seconds. Find its displacement over the interval given.

(a) $v = (2t+1)$ m s^{-1}, $0 \le t \le 3$.
(b) $v = (t^2-4)$ m s^{-1}, $0 \le t \le 3$.
(c) $v = 3\cos t$ m s^{-1}, $0 \le t \le \frac{\pi}{2}$.
(d) $v = 6e^{-2t}$ m s^{-1}, $0 \le t \le 1$.

Integrating acceleration

42. A particle moves in a straight line, and t is the time in seconds.

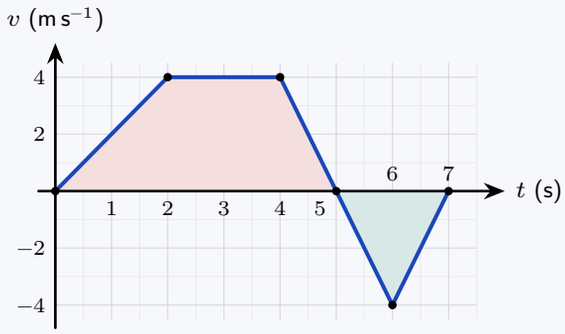
(a) Its acceleration is $a = 6t$ m s^{-2} and $v(0) = 1$ m s^{-1}. Find $v(t)$.
(b) Its acceleration is $a = \sin t$ m s^{-2} and $v(0) = 0$. Find $v(t)$.
(c) Its acceleration is $a = (6t-2)$ m s^{-2}. At $t = 0$ its velocity is 3 m s^{-1} and its displacement from a fixed point O is 1 m. Find its displacement from O at $t = 2$.
(d) Its acceleration is $a = -4\sin 2t$ m s^{-2}. At $t = 0$ it is at O with velocity 2 m s^{-1}. Find its displacement s from O in terms of t.

Distance travelled

43. A particle moves in a straight line, and t is the time in seconds.

(a) Its velocity is $v = (4-2t)$ m s^{-1}. Find the distance travelled over $0 \le t \le 4$.
(b) For the particle in part (a), find the displacement over the same interval, and explain why the two answers differ.
(c) Its velocity is $v = (3t^2-12)$ m s^{-1}. Find the time at which it changes direction, and the distance travelled over $0 \le t \le 3$.
(d) Its velocity is $v = (1-2\sin t)$ m s^{-1}. Find the time in $0 \le t \le \frac{\pi}{2}$ at which it changes direction, and the exact distance travelled over that interval.

(e) The diagram shows the velocity–time graph of a particle for $0 \le t \le 7$. At $t = 0$ the particle is at position $s = 2$ m. Use the signed areas to find its displacement over $0 \le t \le 7$, the distance it travels, and its position at $t = 7$.



(f) For the particle in part (c), sketch the velocity–time graph for $0 \le t \le 3$. Shade the areas that give the displacement and the distance travelled. Hence find the displacement over $0 \le t \le 3$, and state which area counts as negative.

Concept check

44. A particle moves with velocity $v = (t^2 - 4t + 3)$ m s^{-1}. A student finds the distance travelled over $0 \le t \le 3$ as $\left|\int_0^3 v\,dt\right|$. Explain why this is wrong, and find the distance travelled.

Chapter 15 · Integral Calculus

Reflections

TOK The integral has two descriptions, the limit of a sum and the reverse of a derivative, and the theorem proved at the end of this chapter says they agree. It does not explain *why* the world should be arranged that way. Which one is the definition, and does the choice change what an integral is?

IM Long before calculus, Archimedes of Syracuse found the area of a parabolic segment by filling it with smaller and smaller triangles and adding their areas. This method was later called *exhaustion* (filling a region until no area is left over). Each new set of triangles has a quarter of the area of the set before, so the total is a geometric series Ch. 2: $1+\frac{1}{4}+\frac{1}{16}+\cdots=\frac{4}{3}$ times the first triangle. He effectively computed an integral nearly two thousand years before Newton and Leibniz gave the operation a name and a shortcut. The idea of accumulating by slicing is much older than the algebra we use for it. His triangles had the same role as the rectangles in §15.1.

RW In the early 1600s Johannes Kepler, the astronomer who found the laws of planetary motion, bought wine for his wedding. The merchant measured each barrel by pushing a marked rod through the hole in its side, diagonally down to the far edge of one end, and reading the length on the rod. He used the same rule whatever the barrel's shape. Kepler doubted the method and wrote a whole book on the volumes of barrels. He sliced each solid into thin discs and added their volumes. This is a volume of revolution, found before integration existed. (He found that for barrels of the local shape the rod was a fair measure.) Engineers do the same today for fuel tanks, lenses and turbine housings: a shape made by spinning a curve about an axis has a volume given by one integral.

EXT Every continuous function has an antiderivative, but it cannot always be written in familiar functions. For example, $\int e^{-x^2}\,dx$, which is central to statistics, cannot be written in terms of the functions of this course. Its definite integrals still exist as areas, and they are found numerically (by approximation, for example with many thin strips or on a GDC). So 'find the area' applies to more functions than 'reverse the derivative' does.

TOK Each approximation adds rectangles of positive width. As their widths tend to zero, the sums approach the integral. How can these sums approach a positive value even though each rectangle becomes smaller?

The Fundamental Theorem of Calculus

§15.1 named the Fundamental Theorem and used its formula. Here it is stated and proved; the proof is not examined. Barrow, Newton and Leibniz reached this result in the 17th century. It states that if f is continuous on $[a, b]$ and F is any antiderivative of f, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Continuity is part of the theorem, and the proof uses it.

The theorem can also be read the other way round. Write the integrand as a derivative, $f = F'$:

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

Adding up a rate of change across an interval gives the total change: the area under the graph of F' equals the rise of the graph of F. §15.8 uses this reading, where the area under a velocity graph is a displacement.

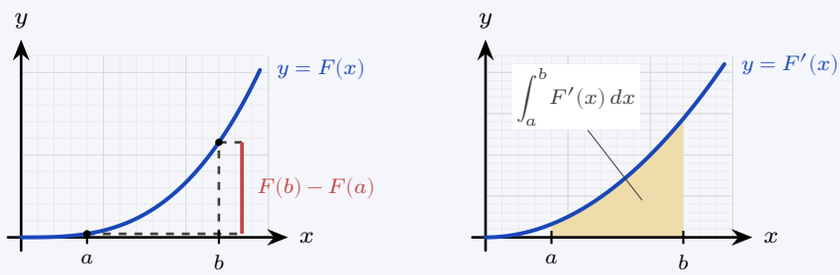


Figure 15.15 Left, $F(x) = \frac{x^3}{3}$ rises by $F(b) - F(a)$ from a to b. Right, the area under $F'(x) = x^2$ over the same interval. For $a = 0.4$, $b = 1.2$ both are 0.555 to 3 s.f.

The proof is given for $f(x) \geq 0$, so that every integral below is an area. The same argument, with signed area, covers a function that takes negative values. Let $A(x) = \int_a^x f(t)\,dt$ be the area under the curve from a to a moving right end x; the letter t is only the variable of integration. When x increases by a small amount h, the extra area $A(x+h) - A(x)$ is one thin strip of width h.

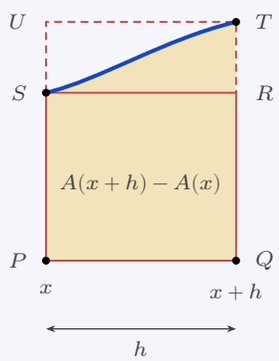


Figure 15.16 One strip, enlarged. Its area $A(x+h) - A(x)$ contains the solid rectangle $PQRS$ and fits inside the dashed rectangle $PQTU$, both of width h.

The short rectangle has height $f(x)$ and the tall one $f(x+h)$, so

$$f(x)\,h \;\le\; A(x+h)-A(x) \;\le\; f(x+h)\,h \quad\Longrightarrow\quad f(x) \;\le\; \frac{A(x+h)-A(x)}{h} \;\le\; f(x+h).$$

Now let $h \to 0$. The left term does not contain h, so it stays $f(x)$. The right term tends to $f(x)$, because f is continuous. The middle term lies between two quantities that both tend to $f(x)$, so it must tend to $f(x)$ as well. This argument is called a **squeeze**. The middle term is the gradient of a chord of the graph of A, so its limit is $A'(x)$, by the definition of the derivative §13.2. So

$$A'(x) = \lim_{h\to 0} \frac{A(x+h)-A(x)}{h} = f(x).$$

So the area function is **an antiderivative of** f. The same squeeze works for $h < 0$, and for a strip on which f decreases or changes direction, using its least and greatest values there.

Now let F be any antiderivative of f. Then $(A-F)' = 0$ on the interval, so $A = F + C$ for a constant C (§15.1). The area from a to a is zero, so $C = -F(a)$, and at $x = b$

$$\int_a^b f(x)\,dx \;=\; A(b) \;=\; F(b) - F(a).$$

The proof also shows that

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Accumulating f and then differentiating gives f again. This statement is called the **first part** of the Fundamental Theorem, and it is also not examined. The evaluation formula $\int_a^b f(x)\,dx = F(b) - F(a)$ is the **second part**, the one used throughout the chapter.

End-of-Chapter Problems
Chapter 15: Integral Calculus

1. A curve has $\frac{dy}{dx} = 3x^2 - 4x + 1$ and passes through $(1, 2)$.

(a) Find the equation of the curve. [3]

(b) Find the value of y when $x = 2$. [2]

2. Evaluate $\int_1^4 \left(\frac{2}{\sqrt{x}} + 1\right) dx$. [4]

3. It is given that $\int_0^2 g(x)\,dx = 5$ and $\int_0^6 g(x)\,dx = -3$.

(a) Find $\int_2^6 g(x)\,dx$. [3]

(b) Find $\int_6^2 2g(x)\,dx$. [2]

(c) Find $\int_0^2 (g(x) + 3x^2)\,dx$. [3]

(d) State, with a reason, whether $g(x)$ can be positive for all x in $2 \leq x \leq 6$. [2]

4. Find the following integrals.

(a) $\int 6e^{1-2x}\,dx$ [2]

(b) $\int (4x - 1)^6\,dx$ [2]

(c) $\int 3\sin\frac{x}{2}\,dx$ [2]

5.

(a) Find $\int \frac{1}{9x^2 + 4}\,dx$. [3]

(b) Evaluate $\int_0^{2/3} \frac{1}{9x^2 + 4}\,dx$, giving an exact answer. [2]

6. Find $\int \frac{x + 7}{(x - 1)(x + 3)}\,dx$. [5]

7. Find $\int \frac{1}{x^2 + 8x + 20}\,dx$. [4]

8.

(a) Using the substitution $u = x^2 + 4$, find $\int \frac{x}{x^2+4}\,dx$. [3]

(b) Hence evaluate $\int_0^2 \frac{x}{x^2+4}\,dx$, giving an exact answer. [2]

9. Using the substitution $u = x^2 + 1$, evaluate $\int_0^2 x(x^2+1)^3\,dx$. [4]

10. Evaluate $\int_0^2 \frac{x^2}{x^3+1}\,dx$, giving your answer in the form $a\ln 3$, where $a \in \mathbb{Q}$. [4]

11.

(a) Find $\int x\sin 3x\,dx$ using integration by parts. [3]

(b) Evaluate $\int_0^{\pi/3} x\sin 3x\,dx$, giving an exact answer. [2]

12. Find the following integrals.

(a) $\int \ln x\,dx$, by writing $\ln x = 1 \times \ln x$ [3]

(b) $\int \sqrt{x}\,\ln x\,dx$ [4]

13.

(a) Use integration by parts twice to find $\int x^2\cos 2x\,dx$. [5]

(b) Hence evaluate $\int_0^{\pi/4} x^2\cos 2x\,dx$, giving an exact answer. [2]

14. This question uses integration by parts throughout.

(a) Find $\int \frac{\ln x}{x^2}\,dx$. [4]

(b) Hence evaluate $\int_1^e \frac{\ln x}{x^2}\,dx$, giving your answer in terms of e. [3]

(c) Evaluate $\int_0^{\pi} x\sin\frac{x}{2}\,dx$. [5]

(d) In part (c), state which factor you took as u, and explain what would go wrong with the other choice. [2]

15. Consider $y = x^2 - 4$.

(a) Find the x-intercepts. [2]

(b) Find the area enclosed between the curve and the x-axis. [4]

16. The curves $y = x^2$ and $y = 8 - x^2$ intersect at two points.

(a) Find the coordinates of the intersection points. [2]
(b) Find the area enclosed between the curves. [4]

17. Consider the curve $y = x^2 - 5x + 4$ on the interval $0 \leq x \leq 2$.

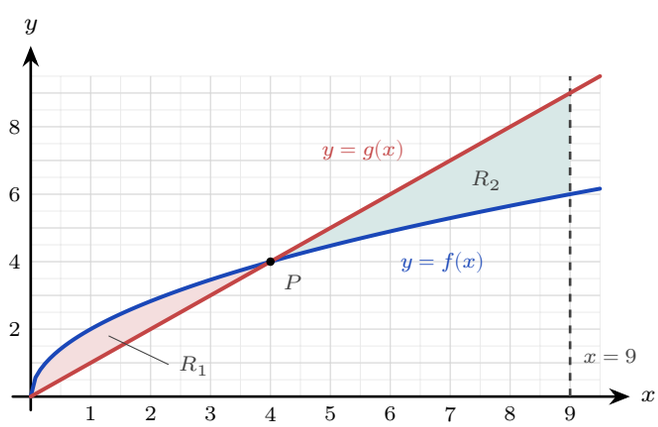
(a) Find where the curve crosses the x-axis in this interval. [1]
(b) Find the total area enclosed between the curve and the x-axis on the interval. [4]

18. Find the area of the region enclosed by the curve $y = 4x - x^2$ and the line $y = x$. [4]

19. The curve $y = x^2$ and the line $y = 2 - x$ enclose a region R.

(a) Find the x-coordinates of the two points of intersection. [3]
(b) Sketch the curve and the line, and shade R. [2]
(c) Show that the area of R is $\frac{9}{2}$. [5]
(d) Explain why integrating $y_{\text{curve}} - y_{\text{line}}$ instead would give the wrong sign. [2]

20. The diagram shows the graphs of $f(x) = 2\sqrt{x}$ and $g(x) = x$ for $0 \leq x \leq 9$. The graphs meet at the origin and at the point P. The shaded regions R_1 and R_2 lie between the two graphs, and R_2 is bounded on the right by the line $x = 9$.



(a) Find the coordinates of P. [2]
(b) Find the area of R_1. [3]
(c) Find the area of R_2. [4]
(d) Hence write down the value of $\int_0^9 (f(x) - g(x))\,dx$. [1]

21. The region under $y = \sqrt{x}$ from $x = 0$ to $x = 4$ is rotated about the x-axis.

(a) Find the volume generated. [3]
(b) The same curve, written $x = y^2$ for $0 \leq y \leq 2$, together with the y-axis, bounds a region. This region is rotated about the y-axis. Find the volume generated. [4]

22. Consider the curve $y = x^3$ for $0 \leq x \leq 1$.

(a) The region between the curve and the x-axis is rotated $360°$ about the x-axis. Show that the volume is $\frac{\pi}{7}$. [4]

(b) The region between the curve and the y-axis is rotated $360°$ about the y-axis. Find this volume, giving an exact answer. [5]

(c) State which of the two solids has the greater volume, and explain the answer by describing the two regions. [3]

23. A particle starts at the origin with velocity $2\,\text{m s}^{-1}$ and has acceleration $a = (6t - 8)\,\text{m s}^{-2}$.

(a) Find the velocity $v(t)$. [2]

(b) Find the displacement $s(t)$. [2]

(c) Find the displacement when $t = 3$, and interpret its sign. [2]

24. The velocity of a particle is $v(t) = (3t^2 - 12t + 9)\,\text{m s}^{-1}$, $t \geq 0$.

(a) Find the times when the particle is at rest. [2]

(b) Find the total distance travelled in the first 3 seconds. [4]

25. A particle moves in a straight line with acceleration $a = (6 - 2t)\,\text{m s}^{-2}$, and is at rest at $t = 0$.

(a) Show that $v = 6t - t^2$. [3]

(b) Find the time at which the particle is next at rest. [2]

(c) Find the maximum velocity, and the time at which it occurs. [3]

(d) Find the distance travelled from $t = 0$ until the particle is next at rest. [4]

(e) Explain why, for this motion, the distance travelled equals the displacement. [2]

26. GDC A particle moves in a straight line with velocity $v(t) = 5\sin\left(\frac{t^2}{4}\right)\,\text{m s}^{-1}$ for $0 \leq t \leq 4$ seconds.

(a) Find the time $t > 0$ at which the particle changes direction. [2]

(b) Find the acceleration of the particle when $t = 3$. [2]

(c) Find the displacement of the particle over $0 \leq t \leq 4$. [2]

(d) Find the total distance travelled over $0 \leq t \leq 4$. [3]

27. A particle moves in a straight line with velocity $v(t) = 6\sin 2t\,\text{m s}^{-1}$, for $0 \leq t \leq \pi$ seconds.

(a) Find the times at which the particle is at rest. [3]

(b) Find the displacement over $0 \leq t \leq \pi$. [3]

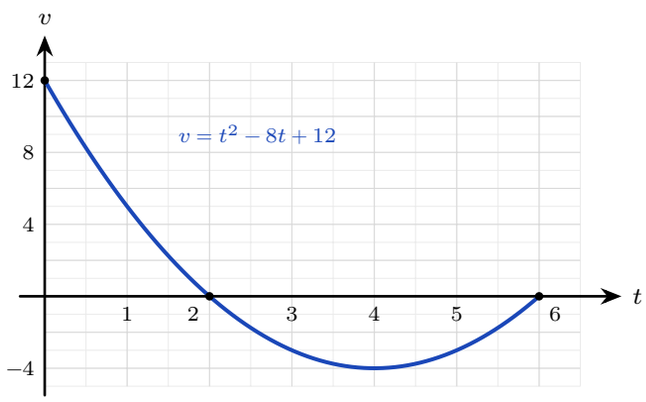
(c) Find the total distance travelled over the same interval. [4]

(d) Explain why the two answers differ. [2]

28. A particle starts from rest at the origin with acceleration $a(t) = 4e^{-2t}$ m s^{-2}.

(a) Show that $v(t) = 2 - 2e^{-2t}$. [3]
(b) Find the displacement over $0 \le t \le 1$. [4]
(c) State the velocity the particle approaches as t grows large, and explain what is happening physically. [2]

29. A drone takes off vertically from the ground at $t = 0$, where t is the time in seconds. Its position s is its height above the ground in metres. Its velocity, in m s^{-1} and positive upwards, is modelled by $v(t) = t^2 - 8t + 12$ for $0 \le t \le 6$. The graph of v is shown.



(a) Write down the time at which the drone is highest above the ground. Give a reason for your answer. [2]
(b) Find the greatest height of the drone. [3]
(c) Show that the displacement of the drone over $0 \le t \le 6$ is zero, and state what this means for the drone at $t = 6$. [3]
(d) Find the distance travelled by the drone over $0 \le t \le 6$. [3]
(e) Find the greatest speed of the drone while it moves downwards. [2]

30. Consider the region in the first quadrant enclosed by $y = x^2$, the y-axis, and $y = 4$.

(a) Find the area of the region. [3]
(b) Find the volume when this region is rotated about the y-axis. [3]

31. Name the method for each integral before using it.

(a) $\displaystyle\int \frac{x^2 + 3}{x - 1}\,dx$ [4]
(b) $\displaystyle\int_1^2 \frac{2x - 1}{x^2 - x + 1}\,dx$ [4]
(c) $\displaystyle\int_0^{\pi/2} \sin x \cos^2 x\,dx$ [4]

(d) $\displaystyle\int \frac{1}{x^2-4x+13}\,dx$ [4]

32. The region R is enclosed by $y=\sin x$, $y=\cos x$ and the y-axis.

(a) Show that the two curves first meet at $x=\frac{\pi}{4}$. [2]
(b) Find the area of R. [4]
(c) R is rotated 360° about the x-axis. Show that the volume is $\frac{\pi}{2}$. [5]

33. Consider the region enclosed by $y=e^x$, $y=e^{-x}$ and the line $x=1$.

(a) Show that the two curves meet at $x=0$. [1]
(b) Find the area of the region. [4]
(c) The region is rotated 360° about the x-axis. Find the volume, giving your answer in terms of e and π. [5]

34. GDC The curves $y=3\cos x$ and $y=x^2$ enclose a region R.

(a) Find the x-coordinates of the two points of intersection, correct to 3 s.f. [2]
(b) Find the area of R. [3]
(c) R is rotated 360° about the x-axis. Find the volume of the solid generated. [3]

35.

(a) Using the substitution $u=\sqrt{x}$, show that $\displaystyle\int_0^4 \frac{1}{1+\sqrt{x}}\,dx = 4-2\ln 3$. [6]
(b) Show that $4x-x^2=4-(x-2)^2$. Hence evaluate $\displaystyle\int_1^2 \frac{1}{\sqrt{4x-x^2}}\,dx$, giving an exact answer. [5]

36. GDC The inside of a vase is modelled by rotating the curve $y=3+\sin(0.5x)$, $0\le x\le 12$, through 360° about the x-axis, where x and y are measured in centimetres. The vase stands on its base, at $x=0$, with the x-axis vertical.

(a) Write down an integral for the volume of the vase. [1]
(b) Find the volume of the vase, in cm^3. [2]
(c) Water is poured into the empty vase until it is exactly half full. Find the depth of the water. [4]
(d) Find the maximum internal radius of the vase, and the height x at which it occurs. [2]

37. Consider the curves $y=2x$ and $y=x^2$ for $0\le x\le 3$.

(a) Show that $\displaystyle\int_0^3 (2x-x^2)\,dx = 0$. [2]
(b) Explain why this does not mean that the area between the curves is zero. [1]

(c) Find the total area between the curves from $x = 0$ to $x = 3$. [4]

38.

(a) Evaluate $\displaystyle\int_{-e^2}^{-1} \frac{1}{x}\,dx$. [3]

(b) A student writes $\displaystyle\int_{-1}^{2} \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_{-1}^{2} = -\frac{1}{2} - 1 = -\frac{3}{2}$. Explain why this answer cannot be correct, and why the method is not valid here. [2]

39.

(a) Find $\displaystyle\int \frac{1}{(2x+1)^2}\,dx$. [2]

(b) Find $\displaystyle\int 2x\cos\left(x^2\right)dx$. [2]

(c) Explain why neither the method used for $\displaystyle\int \cos(3x+1)\,dx$ nor the method of part (b) can be used to find $\displaystyle\int \cos\left(x^2\right)dx$. [2]

Challenge Questions

C1. Solids you already know. Several volume formulas from Ch. 7 can be recovered by rotating a curve about the x-axis. This question derives three of them, and then finds a single formula that contains two of them as special cases.

(a) The line $y = r$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is $\pi r^2 h$, and name the solid. [3]

(b) The line $y = \dfrac{r}{h}x$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is $\frac{1}{3}\pi r^2 h$. [4]

(c) The semicircle $y = \sqrt{r^2 - x^2}$ for $-r \le x \le r$ is rotated about the x-axis. Show that the volume is $\frac{4}{3}\pi r^3$. [4]

(d) Let $m > 0$. The curve $y = r\left(\dfrac{x}{h}\right)^m$ for $0 \le x \le h$ is rotated about the x-axis. Show that the volume is

$$V = \frac{\pi r^2 h}{2m+1}.$$

[4]

(e) Show that the formula in part (d) also gives the results of parts (a) and (b), taking $m = 0$ for the line $y = r$. Use it to explain where the factor $\frac{1}{3}$ in the volume of the cone comes from. [3]

(f) Every solid in part (d) fits inside the cylinder of part (a). Find the value of m for which the solid has exactly half the volume of that cylinder, and describe the curve that is rotated. [2]

(g) Many containers, such as a bowl, a vase or a glass, have a profile close to one of the curves in part (d). Choose one, and describe how you would find a value of m that models it and test the model against a measured volume. [2]

C2. Exact areas from rectangles. The notes estimate the area under $y = x^2$ on $[0, 1]$ with rectangles, and then evaluate it exactly with the Fundamental Theorem of Calculus. This question finds the same exact value from the rectangles alone. Divide $[0, 1]$ into n strips of equal width $\frac{1}{n}$, and let U_n and L_n be the sums of the rectangles whose heights are read at the right and at the left of each strip.

(a) Show that $U_n = \dfrac{1}{n^3}\displaystyle\sum_{r=1}^{n} r^2$, and write L_n as a sum in the same way. [3]

(b) Given that $\displaystyle\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, show that $U_n = \dfrac{(n+1)(2n+1)}{6n^2}$, and find L_n in the same form. Evaluate L_{10} and U_{10}. [4]

(c) Show that $U_n - L_n = \dfrac{1}{n}$. Hence, without using the Fundamental Theorem, show that the area under $y = x^2$ on $[0, 1]$ is exactly $\frac{1}{3}$. [3]

(d) Given that $\displaystyle\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, use the same method to show that the area under $y = x^3$ from $x = 0$ to $x = b$, where $b > 0$, is $\frac{1}{4}b^4$. [4]

(e) Conjecture the area under $y = x^p$ on $[0, 1]$ for a positive integer p. State what the leading term of a formula for $\displaystyle\sum_{r=1}^{n} r^p$ would have to be for the method to give your conjecture. [3]

(f) The method depends on having a formula for the sum. Investigate a curve for which a different kind of sum can be evaluated exactly, for example $y = 2^x$ on $[0, 1]$, whose rectangle sum is a geometric series (Ch. 2). [2]

C3. Damped oscillation. A pendulum swinging in air moves in a way that the graph of $y = e^{-x}\sin x$ describes: it oscillates, and each swing is smaller than the last. This question finds the area under such curves by two methods and then adds up the areas of all the swings. Throughout, a and b are non-zero real constants, and

$$I = \int e^{ax}\cos bx\,dx, \qquad J = \int e^{ax}\sin bx\,dx.$$

(a) Apply integration by parts once to each integral, differentiating e^{ax} each time, to show that, apart from constants of integration,

$$bI = e^{ax}\sin bx - aJ \qquad \text{and} \qquad bJ = -e^{ax}\cos bx + aI.$$

[4]

(b) Solve these two equations simultaneously to show that

$$I = \frac{e^{ax}\,(a\cos bx + b\sin bx)}{a^2 + b^2} + C,$$

and find J in the same form. [4]

(c) A second route uses complex numbers (Ch. 17). You are given that

$$e^{(a+ib)x} = e^{ax}\,(\cos bx + i\sin bx) \qquad \text{and} \qquad \int e^{cx}\,dx = \frac{e^{cx}}{c} + C,$$

where the second result holds when c is a complex constant. Deduce that $I + iJ = \dfrac{e^{(a+ib)x}}{a+ib} + C$. By multiplying the numerator and denominator by $a - ib$, recover both results of part (b) at once. [4]

(d) Let $f(x) = e^{-x}\sin x$ and, for $k = 0, 1, 2, \ldots$, let $A_k = \displaystyle\int_{k\pi}^{(k+1)\pi} f(x)\,dx$, the signed area of the kth swing. Show that $A_0 = \frac{1}{2}\,(1 + e^{-\pi})$, and find A_1 and A_2 exactly. [3]

(e) Conjecture a relationship between A_{k+1} and A_k. Prove it for every $k \geq 0$ by using the substitution $x = u + \pi$ in A_{k+1}. [4]

(f) Hence show that $\displaystyle\int_0^{\infty} e^{-x}\sin x\,dx = \frac{1}{2}$, and confirm this value using your expression for J from part (b). [3]

(g) For the curve $y = e^{-cx}\sin x$ with $c > 0$, investigate how the ratio of the areas of successive swings depends on c. Suggest how timing and measuring the successive swings of a real pendulum could be used to estimate its value of c. [2]

C4. A reduction formula. Let $I_n = \displaystyle\int_0^{\pi/2} \sin^n x\,dx$ for integers $n \geq 0$.

(a) Evaluate I_0 and I_1 directly. [3]

(b) Writing $\sin^n x = \sin^{n-1} x \cdot \sin x$ and using integration by parts, show that

$$I_n = \frac{n-1}{n}\,I_{n-2}, \qquad n \geq 2.$$

[6]

(c) Use the formula to find I_2, I_3, I_4 and I_5 exactly. [4]

(d) Explain why I_n involves π when n is even but not when n is odd. [3]

(e) Explain why $I_{n+1} \leq I_n$ for every $n \geq 0$. Investigate what happens to the ratio $\dfrac{I_{2m}}{I_{2m+1}}$ as m increases, and what this suggests about writing $\frac{\pi}{2}$ as an infinite product of fractions. [3]