CALCULUS

Chapter 14

Derivative Applications

14

SYLLABUS 5.2 5.7 5.8 5.9 5.14

Throw a ball straight up. It rises, it slows, and at the very top it stops for an instant before it falls back. At that instant its velocity is exactly zero. A moment earlier the ball was moving up; a moment later it is moving down. The top of the flight is the one instant when it is doing neither.

From that instant you can build a method for finding a largest or smallest value. The top of the flight is where the height is greatest, and it is also where the velocity is zero. In many problems, when a smooth quantity reaches its largest or smallest value at an interior point (a point inside the domain, not at an end), its rate of change is zero at that moment, and the curve is flat.

In Ch. 13 you learned to calculate derivatives. In this chapter you use them. By the end of this chapter you will be able to:

- find the stationary points of a curve, and decide whether each is a maximum, a minimum or neither;
- read the shape of a graph, including its points of inflexion, from the signs of f' and f'';
- find the largest or smallest value of a quantity under a constraint (optimisation);
- describe motion along a straight line (kinematics);
- find the gradient of a curve that is not written as $y = f(x)$ (implicit differentiation), and link the rates of quantities that change together (related rates).

The derivative measures how fast a quantity changes, so it lets you model and interpret real situations: the largest volume a box can hold, the moment a drug concentration is greatest, how fast the top of a sliding ladder falls. Several of these start where the ball problem starts: find the moment the derivative is zero.

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14.1 Stationary Points and Their Nature

When a quantity reaches its largest or smallest value, it has stopped increasing and has not yet started decreasing, or the reverse. In many subjects you find that moment by solving $f'(x) = 0$. A ball thrown straight up is at its greatest height when its velocity is zero (§14.4). A business makes its largest profit where the rate of change of profit with respect to the number of items made is zero. In physics a system is in **equilibrium** (balance) where its potential energy is stationary. In statistics the mode of a continuous distribution is at a maximum of its density function §12.4.2. To use the condition, first read what the sign of f' says about a curve.

14.1.1 Increasing and decreasing

The derivative gives the gradient, so its sign shows which way the curve is going. A function is **increasing** on an interval when a larger x gives a larger $f(x)$, and **decreasing** when a larger x gives a smaller $f(x)$. Where $f'(x) > 0$ the tangent rises from left to right, so the function is increasing. Where $f'(x) < 0$ the function is decreasing. Where $f'(x) = 0$ the tangent is horizontal.

> **KEY 14.1** **Increasing and decreasing functions**
>
> If $f'(x) > 0$ for every x in an interval, then f is **increasing** on that interval.
> If $f'(x) < 0$ for every x in an interval, then f is **decreasing** on that interval.

To find where f is increasing, differentiate, factorise $f'(x)$, and solve the inequality $f'(x) > 0$. Solve $f'(x) < 0$ for where it is decreasing. If f has a derivative everywhere, it can change from increasing to decreasing only where $f'(x) = 0$. Those points are studied in the rest of this section.

EXAMPLE 14.1

Find the intervals on which $f(x) = x^3 - 9x^2$ is increasing, and those on which it is decreasing.

SOLUTION

Differentiate and factorise:

$$f'(x) = 3x^2 - 18x = 3x(x - 6).$$

The product is zero at $x = 0$ and $x = 6$. It is positive when both factors have the same sign, so $f'(x) > 0$ for $x < 0$ and for $x > 6$. Between the roots the factors have opposite signs, so $f'(x) < 0$ for $0 < x < 6$.
So f is increasing for $x < 0$ or $x > 6$, and decreasing for $0 < x < 6$

14.1.2 Stationary points

A **stationary point** is a point where the gradient is zero: the tangent is horizontal. Since the derivative gives the gradient, stationary points are exactly the solutions of $f'(x) = 0$. A stationary point may be a maximum, a minimum or neither: $y = x^3$ is stationary at the origin and goes on increasing through it.

For a smooth curve, every local maximum and minimum inside the domain is a stationary point, so the condition $f'(x) = 0$ finds them all. Endpoints are checked separately, as in the closed-interval method of §14.1.5. The condition does not find an extreme value at a corner, where the derivative does not exist: $f(x) = |x|$ has a minimum at $x = 0$ and no stationary point there. See §13.2.

For a polynomial this gives a count. If f has degree $n \geq 1$, then f' has degree $n - 1$ and so at most $n - 1$ zeros. So f has at most $n - 1$ turning points §6.1.

DEFINITION 14.1 **Stationary points**

A **stationary point** of f occurs where $f'(x) = 0$. At a **local maximum** the curve is higher than at all nearby points; at a **local minimum**, lower.

The word local means highest or lowest among nearby points only. A local maximum need not be the largest value of the function: the curve $y = x^3 - 3x$ below has a local maximum at $(-1, 2)$ and still increases without limit to the right.

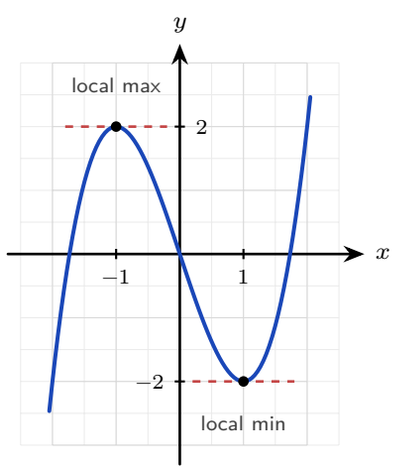


Figure 14.1 $y = x^3 - 3x$. At each stationary point the tangent is horizontal.

A stationary point is a candidate, not an answer. A zero gradient says only that the curve is flat there. Flat can mean a local maximum, a local minimum, or neither. A stationary point that is a local maximum or minimum is a turning point §4.1.3; a stationary point of inflexion (§14.2) is not. You can decide which case you have with either of two tests: the first derivative test or the second derivative test.

14.1.3 The first derivative test

The sign of f' just to the left and just to the right of a stationary point shows what the curve does there. Just before a maximum the curve rises, so $f' > 0$; just after it, the curve falls, so $f' < 0$. A minimum is the reverse. If f' has the same sign on both sides, the curve keeps rising, or keeps falling, through the point. It is then a stationary point of inflexion (§14.2).

Sign of f'	Shape of the curve	Nature of the point
$+$ **then** $-$	rising, then falling	local maximum
$-$ **then** $+$	falling, then rising	local minimum
same sign	rising throughout, or falling throughout	stationary point of inflexion

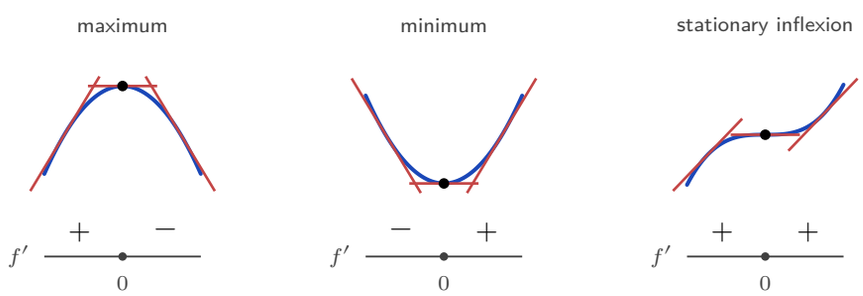


Figure 14.2 The first derivative test. The red strokes are tangents: rising, flat, then falling at a maximum, and the reverse at a minimum. At a stationary inflexion (right) the tangent is flat for an instant, but f' has the same sign on both sides. The strip below records the sign of f'.

The method has three steps.

1. Solve $f'(x) = 0$ to find the stationary points.
2. Find the sign of f' just to the left and just to the right of each one. If $f'(x)$ is factorised, the sign of each factor gives the sign of f' without substituting any numbers.
3. Read the nature of each point from the table.

EXAMPLE 14.2

Find and classify the stationary points of $f(x) = 3x^4 - 4x^3$.

SOLUTION

Differentiate and factorise:

$$f'(x) = 12x^3 - 12x^2 = 12x^2(x - 1).$$

So $f'(x) = 0$ at $x = 0$ and $x = 1$.

The factor $12x^2$ is positive on both sides of 0 and on both sides of 1. So the sign of f' is the sign of $x-1$: negative for $x<1$, positive for $x>1$.
At $x=0$ the sign of f' is negative, then negative again. The curve falls through the point, so $(0,0)$ is a stationary point of inflexion.
At $x=1$ the sign changes from negative to positive, and $f(1)=3-4=-1$, so $(1,-1)$ is a local minimum.

A squared factor such as x^2 in $f'(x)$ does not change sign. At its root, f' changes sign only if another factor does. So a squared factor on its own gives a stationary point of inflexion, as at $x=0$ here.

14.1.4 Concavity

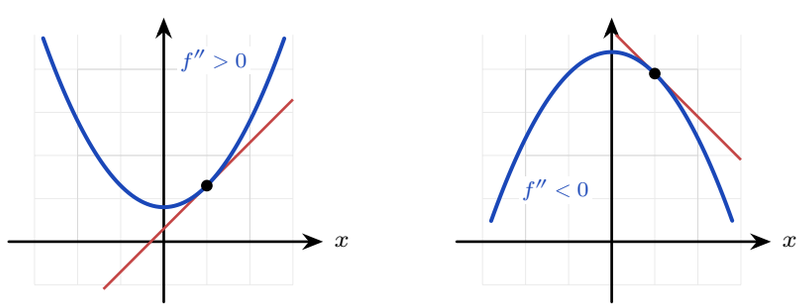
The first derivative says whether a curve is going up or down. It does not say how the curve bends. Two curves can both rise from the same point, one more and more steeply, the other less and less steeply. You can tell them apart with the second derivative, because f'' is the rate of change of the gradient §13.7.

Where $f''(x)>0$ the gradient is increasing, and the curve is **concave up**: it bends upward, like a cup that holds water. Where $f''(x)<0$ the gradient is decreasing, and the curve is **concave down**: it bends downward, like a cup turned upside down.

DEFINITION 14.2 **Concavity**

The curve $y=f(x)$ is **concave up** on an interval where $f''(x)>0$, and **concave down** on an interval where $f''(x)<0$.

The tangent line gives a picture of the same fact. Where a curve is concave up it lies *above* its tangents, so a tangent touches the curve and the curve then rises away from it. The increasing gradient is the reason. To the right of the point of contact, the curve is steeper than the tangent. So y increases by more on the curve than on the line. To the left, the curve is less steep than the tangent. So between a point on the left and the point of contact, y increases by less on the curve than on the line, and at that left point the curve is above the line. Where the curve is concave down, it lies below its tangents.



concave up: curve *above* its tangent concave down: curve *below* its tangent

Figure 14.3 The tangent picture of concavity.

- Notice what f'' describes: **the gradient, not the height**. Concave up means the gradient is increasing, and that can happen while the curve falls, because a gradient that goes from -5 to -1 has increased. So the signs of f' and f'' answer two separate questions: is the curve going up or down, and which way does it bend?

Sign	f': going up or down?	f'': bending which way?
positive	**increasing**	**concave up** (bends upward)
negative	**decreasing**	**concave down** (bends downward)
zero	**stationary point**: the curve is flat there	**point of inflexion** (§14.2), *if* the concavity also changes

A curve can also be increasing and concave down at the same time: it rises by less and less.

EXAMPLE 14.3

Find where the curve $f(x) = x^3 - 9x^2$ is concave up and where it is concave down.

SOLUTION

Differentiate twice and find the sign of f'':

$$f'(x) = 3x^2 - 18x, \qquad f''(x) = 6x - 18 = 6(x - 3).$$

So $f''(x) < 0$ for $x < 3$ and $f''(x) > 0$ for $x > 3$. The curve is concave down for $\boxed{x < 3}$ and concave up for $\boxed{x > 3}$

Compare this with the earlier result for the same function: it is decreasing for $0 < x < 6$, and across that interval it is first concave down, then concave up.

14.1.5 The second derivative test

At a stationary point the tangent is horizontal. A curve that is concave up there lies above its horizontal tangent, so the point is a local minimum. A curve that is concave down lies below it, so the point is a local maximum. One substitution into f'' therefore decides the nature of the point. This is usually faster than the first derivative test.

KEY 14.2 **Second derivative test**

At a stationary point where $f'(x) = 0$:

$$f''(x) > 0 \implies \text{local minimum}, \qquad f''(x) < 0 \implies \text{local maximum}.$$

If $f''(x) = 0$ this test gives no answer. Use the first derivative test instead.

- When $f''(x) = 0$ the test gives no answer, and **this does not mean the point is an inflexion**. Both $y = x^4$ and $y = x^3$ have $f'(0) = 0$ and $f''(0) = 0$. Yet x^4 has a minimum at the origin, and x^3 has a stationary point of inflexion there. The first derivative test separates them: for x^4 the sign of f' is negative then positive, while for x^3 it is positive on both sides.

EXAMPLE 14.4

Find and classify the stationary points of $f(x) = x^3 + 3x^2 - 9x + 2$.

SOLUTION

Differentiate and solve $f'(x) = 0$:

$$f'(x) = 3x^2 + 6x - 9 = 3(x+3)(x-1) = 0 \implies x = -3 \text{ or } x = 1.$$

The second derivative is $f''(x) = 6x + 6$. At $x = -3$, $f''(-3) = -12 < 0$ and $f(-3) = 29$, so $\boxed{(-3, 29)}$ is a local maximum. At $x = 1$, $f''(1) = 12 > 0$ and $f(1) = -3$, so $\boxed{(1, -3)}$ is a local minimum.

On a closed interval $a \le x \le b$ the greatest and least values need not be at stationary points. List the candidates: the stationary points inside the interval, and the two endpoints. Evaluate f at each one and compare. This assumes f has a derivative at every point inside the interval; a corner would be one more candidate.

EXAMPLE 14.5

Find the greatest and least values of the same cubic, $f(x) = x^3 + 3x^2 - 9x + 2$, on $-4 \le x \le 4$.

SOLUTION

Both stationary points lie inside the interval, so there are four values to compare:

$$f(-4) = 22, \qquad f(-3) = 29, \qquad f(1) = -3, \qquad f(4) = 78.$$

The greatest value is $\boxed{78}$, at the endpoint $x = 4$, where the gradient is not zero. The least value is $\boxed{-3}$, at the local minimum. The local maximum 29 is not the greatest value on this interval. Endpoints are used again in §14.3.

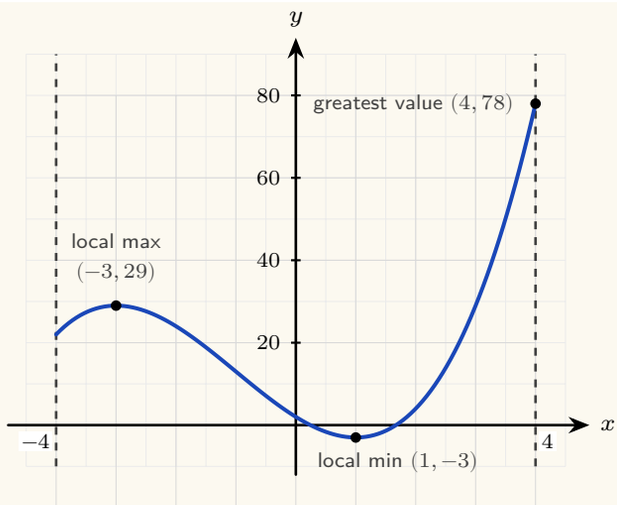


Figure 14.4 $f(x) = x^3 + 3x^2 - 9x + 2$ on $-4 \le x \le 4$. The local maximum 29 is not the greatest value: the curve is higher at the endpoint $x = 4$.

The method does not depend on the function being a polynomial. Any function you can differentiate twice can be tested in the same way. State the domain whenever the function restricts it, as $\ln x$ requires $x > 0$. If $f''(x) > 0$ across the whole domain, and the domain is one interval, the curve is concave up everywhere. A stationary point is then the lowest point of the entire curve, not only a local minimum.

EXAMPLE 14.6

Find and classify the stationary point of $f(x) = x \ln x$, for $x > 0$.

SOLUTION

The product rule gives

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1.$$

Set $f'(x) = 0$: $\ln x = -1$, so $x = e^{-1} \approx 0.368$.

The second derivative is $f''(x) = \dfrac{1}{x}$, which is positive for every x in the domain. So the point is a local minimum, and

$$f\left(\tfrac{1}{e}\right) = \tfrac{1}{e} \ln \tfrac{1}{e} = -\tfrac{1}{e} \approx -0.368.$$

The minimum is at $\left(\frac{1}{e}, -\frac{1}{e}\right)$

The domain $x > 0$ comes from the logarithm, and since $f'' > 0$ on all of it, this is the lowest point of the entire curve.

YOUR TURN 14.1 Stationary Points and Their Nature

Increasing and decreasing

1. Find the intervals on which each function is increasing, and those on which it is decreasing.

(a) $f(x) = x^2 - 4x$
(b) $f(x) = x^4 - 2x^2$
(c) $f(x) = \sin x$, for $0 \leq x \leq 2\pi$
(d) $f(x) = xe^{-x}$

Concavity

2. Find the intervals on which each curve is concave up, and those on which it is concave down.

(a) $y = x^3$
(b) $y = x^3 - 6x^2$
(c) $y = \cos x$, for $0 \leq x \leq 2\pi$
(d) $y = \ln x$, for $x > 0$

Finding and classifying stationary points

3. Find the stationary points of each curve.

(a) $y = x^2 - 6x + 5$
(b) $y = x^3 - 12x$
(c) $y = 2x^3 - 3x^2 - 12x + 1$
(d) $y = x^4 - 2x^2$

4.

(a) Use the second derivative to classify the stationary point of $y = x^2 - 6x + 5$.
(b) Use the second derivative to classify both stationary points of $y = x^3 - 12x$.
(c) Find the stationary point of $y = x + \dfrac{1}{x}$ for $x > 0$, and state its nature.
(d) A curve has $\dfrac{dy}{dx} = (x-1)(x-4)$. Write down the x-coordinates of its stationary points, and state the nature of each without differentiating again.

5. Find the stationary point of each curve and determine its nature.

(a) $y = e^x - 3x$
(b) $y = \dfrac{\ln x}{x}$, for $x > 0$
(c) $y = \sin x + \cos x$, for $0 \leq x \leq 2\pi$ (there are two)
(d) $y = xe^{-2x}$

The first derivative test

6. The curve $y = x^3 - 3x^2 + 3x$ has one stationary point.

(a) Show that $x = 1$ is a stationary point.
(b) Explain why the second derivative test cannot classify it.
(c) Use the sign of $\dfrac{dy}{dx}$ on either side to classify it.

7.

(a) A curve has $\frac{dy}{dx} = (x+1)^2(x-2)$. Find the x-coordinates of its stationary points, and use the first derivative test to classify each one.

(b) Find and classify the stationary points of $f(x) = x^4 - 4x^3$.

Local, greatest and least values

8. Let $f(x) = 2x^3 - 3x^2 - 12x$ for $-2 \le x \le 4$.

(a) Find the local maximum and the local minimum of f.

(b) Find the greatest and least values of f on the interval, and explain why the local maximum is not the greatest value.

Concept check

9.

(a) A student writes: "$f'(a) = 0$, *so* f *has a maximum or a minimum at* $x = a$." Give a counter-example, and state what must be checked before this conclusion can be drawn.

(b) Explain why the graph of a quartic function has at most three turning points, and give a quartic whose graph has exactly one turning point.

14.2 Points of Inflexion

The concavity of a curve can change, and the point where it changes is often where a rate of change is greatest or least. Suppose $f(t)$ is the total number of cases in an epidemic $(\text{a disease spreading quickly through a population})$ after t days. Then $f'(t)$ is the number of new cases per day. While $f'' > 0$ the daily count is growing. Once $f'' < 0$ the daily count is falling, even though the total still rises. The change from one to the other is the day on which the daily count is highest.

A **point of inflexion** is a point where the curve changes the direction of its bend: from concave up to concave down, or the reverse.

DEFINITION 14.3 **Point of inflexion**

A **point of inflexion** is a point on a curve where the concavity changes. For a function that can be differentiated twice, $f''(x) = 0$ at every point of inflexion. But $f''(x) = 0$ alone does not guarantee one: the concavity must actually change.

An inflexion can occur at any gradient, and the syllabus names two kinds. At a **stationary point of inflexion** the gradient is zero, as for $y = x^3$ at the origin. At a **non-stationary point of inflexion** the gradient is not zero, and the curve changes its bend while it is still rising or falling.

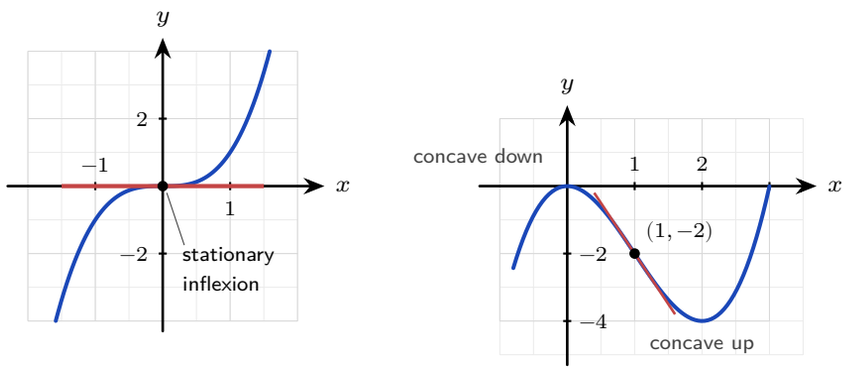


Figure 14.5 Left: a stationary inflexion, tangent horizontal. Right: a non-stationary inflexion of $y = x^3 - 3x^2$ at $(1, -2)$, where the tangent has gradient -3.

To find the points of inflexion of a curve:

1. Find $f''(x)$ and solve $f''(x) = 0$.
2. Check that f'' changes sign at each solution, by testing a value on each side.
3. Substitute each confirmed value of x into f to find the point.
4. To name the kind, evaluate f' there: zero means stationary, non-zero means non-stationary.

EXAMPLE 14.7

Find the point of inflexion of $f(x) = x^3 - 3x^2$, and decide whether it is stationary.

SOLUTION

Differentiate twice:

$$f'(x) = 3x^2 - 6x, \qquad f''(x) = 6x - 6.$$

Set $f''(x) = 0$: $6x - 6 = 0 \implies x = 1$. For $x < 1$, $f'' < 0$ (concave down); for $x > 1$, $f'' > 0$ (concave up). The sign changes, so $x = 1$ gives a point of inflexion. Since $f(1) = -2$, the point is $\boxed{(1, -2)}$

The gradient there is $f'(1) = 3 - 6 = -3 \neq 0$, so this is a non-stationary inflexion: the curve changes its bend while it is falling.

14.2.1 Inflexions of other functions

Polynomials are not the only curves that bend both ways. Take $f(x) = \sin x$ and differentiate twice:

$$f'(x) = \cos x, \qquad f''(x) = -\sin x.$$

So $f''(x) = 0$ whenever $\sin x = 0$, that is at $x = 0, \pm\pi, \pm 2\pi, \ldots$. At each of these the concavity changes, because $-\sin x$ changes sign every time the sine crosses the axis. The sine curve therefore has a point of inflexion at every multiple of π. None is stationary, since $f'(x) = \cos x = \pm 1$ there. Each inflexion of the sine curve is also the steepest point on its

part of the curve. In general, an inflexion of f is where f'' changes sign, so it is **a turning point of** f': the gradient there is a local maximum or a local minimum. For the sine curve these are the steepest points. For $y = x^3 + x$, whose gradient $3x^2 + 1$ is least at $x = 0$, the inflexion is the least steep point.

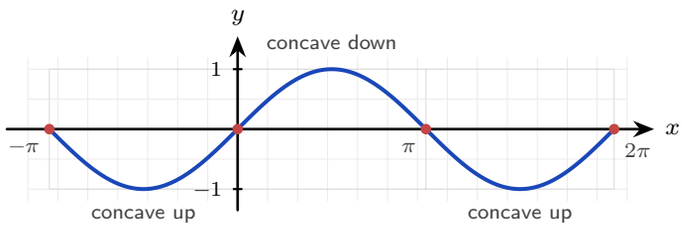


Figure 14.6 The inflexions of $y = \sin x$ (red points) are where the curve crosses the x-axis.

Other familiar curves have no point of inflexion at all. For $y = e^x$, $f''(x) = e^x > 0$ for every x, so the curve is concave up everywhere. For $y = \ln x$, $f''(x) = -\dfrac{1}{x^2} < 0$ for every $x > 0$, so it is concave down across its whole domain. For $y = x^2$, $f''(x) = 2 > 0$ everywhere. In all three f'' never changes sign, so there is no inflexion to find.

EXAMPLE 14.8

Find the points of inflexion of $f(x) = e^{-x^2}$.

SOLUTION

The chain rule gives $f'(x) = -2xe^{-x^2}$. Differentiating again needs the product rule:

$$f''(x) = -2e^{-x^2} + (-2x)(-2x)e^{-x^2} = (4x^2 - 2)\,e^{-x^2}.$$

Since $e^{-x^2} > 0$ for every x, the sign of f'' is decided by $4x^2 - 2$ alone. This is zero when $x^2 = \frac{1}{2}$, so $x = \pm\frac{1}{\sqrt{2}} \approx \pm 0.707$, and it changes sign at each: negative between the two values, positive outside them. Both are points of inflexion, and $f\left(\pm\frac{1}{\sqrt{2}}\right) = e^{-1/2} \approx 0.607$, so they are $\left(\pm\frac{1}{\sqrt{2}},\, e^{-1/2}\right)$

So the curve is concave down over the central peak and concave up in each tail. This is the bell shape of §12.5. If you repeat the calculation for $e^{-x^2/2}$, a constant multiple of the standard normal density, you find inflexions at $x = \pm 1$: exactly one standard deviation from the mean.

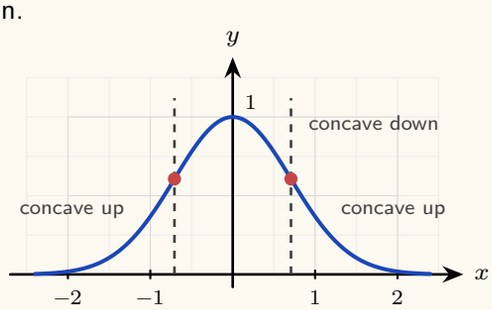


Figure 14.7 $y = e^{-x^2}$ changes from concave up to concave down at $x = -\frac{1}{\sqrt{2}}$, and back at $x = \frac{1}{\sqrt{2}}$.

14.2.2 Reading f, f' and f'' together

A common examination question shows one of these three graphs and asks about another. Below are $f(x) = x^3 - 3x$ and its two derivatives, drawn one above the other on the same x-axis. The vertical scales are not the same; only the zeros and the signs matter here.

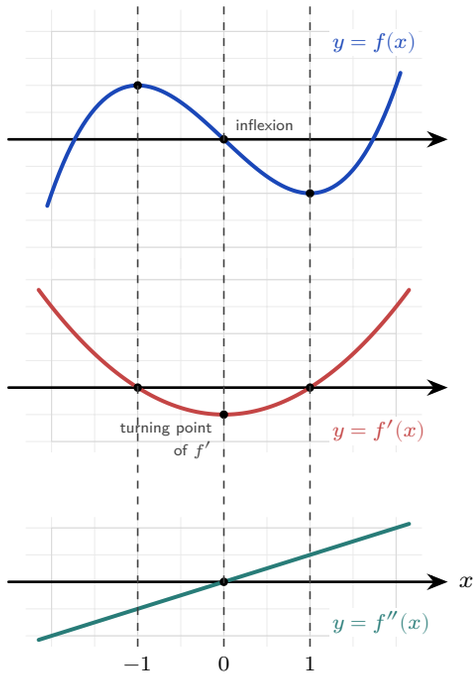


Figure 14.8 f, f' and f'' for $f(x) = x^3 - 3x$, with the vertical axes omitted.

At every dashed line a feature of one graph lines up with a feature of another. The links are:

- The links between f and f' are those of §13.5: a **turning point of** f sits above a **zero of** f' where f' changes sign, and an **interval where** f **increases** sits above an **interval where** f' **is positive**. Here they occur at $x = \pm 1$.
- An **inflexion of** f sits above a **turning point of** f', and above a **zero of** f'' where f'' changes sign. In this example all three occur at $x = 0$.

Each link works in both directions, from f to f' and from f' back to f. So from any one of the three graphs you can sketch the shape of the other two. Going from f to f' to f'' the sketch is fully determined. Going back, the sketch is not fully determined. The graph of f' fixes the shape of f but not its height, so f is known only up to a vertical shift. In the same way, f'' fixes f' only up to a vertical shift.

YOUR TURN 14.2 Points of Inflexion

Finding points of inflexion

10. Find the point of inflexion of each curve, and state whether it is a stationary or a non-stationary point of inflexion.

(a) $y = x^3 - 6x^2 + 5$

(b) $y = x^3 + 6x^2 + 12x$

(c) $y = 2x^3 - 9x^2 + 12x$

11.

(a) Find the coordinates of the point of inflexion of $y = xe^x$.

(b) Find the x-coordinates of the points of inflexion of $y = 2x + \cos x$ for $0 \leq x \leq 2\pi$.

(c) Show that $y = \dfrac{x}{1 + x^2}$ has three points of inflexion, and find their coordinates.

When $f''(x) = 0$ is not enough

12.

(a) Show that $y = x^4 - 4x + 1$ has $\dfrac{d^2y}{dx^2} = 0$ at $x = 0$, and explain why there is no point of inflexion there.

(b) Determine, for $y = x^5$, whether the point where $f'' = 0$ is a point of inflexion, and if so whether it is stationary.

(c) Explain why $y = e^x + x^2$ has no point of inflexion.

Reading f, f' and f'' together

13. The graph of $y = f'(x)$ crosses the x-axis only at $x = -1$, from above to below, and at $x = 3$, from below to above. It has a single turning point, a minimum at $x = 1$. For the curve $y = f(x)$, state

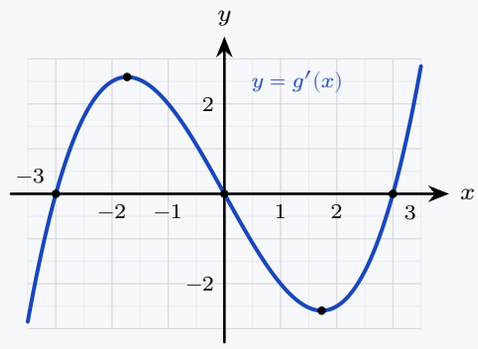
(a) the x-coordinates of its stationary points, and the nature of each

(b) the interval on which f is decreasing

(c) the x-coordinate of its point of inflexion

(d) the interval on which it is concave up.

(e) The graph of $y = g'(x)$, the derivative of a second function g, is shown for $-3.5 \leq x \leq 3.5$. It meets the x-axis at $x = -3$, 0 and 3, and its turning points are at $x = -\sqrt{3}$ and $x = \sqrt{3}$.



Find the x-coordinates of the stationary points of g, and state the nature of each.

(f) For the function g in part (e), write down the x-coordinates of the points of inflexion of the graph of g, and the intervals on which the graph of g is concave up.

14. A function has $f'(x) = x^2(x-3)$. Without finding $f(x)$,

(a) classify the stationary points of f

(b) find the x-coordinates of the points of inflexion of f, and state whether each is stationary or non-stationary.

Concept check

15. A student says: "*a point of inflexion is a point where the gradient is zero.*" Give a counter-example, and state the condition that defines a point of inflexion.

14.3 Optimisation

Optimisation means finding the largest or smallest value of a quantity in a real situation: the greatest volume, the least cost, the shortest time.

Without the derivative, you could only try inputs one at a time and compare the results. That can suggest an answer but cannot settle it, because between any two inputs you tried there are infinitely many you did not. With the derivative, you solve an equation instead of searching. At a maximum or a minimum inside the domain the curve is flat, so $f'(x) = 0$. Solving that one equation, together with checking any endpoints, gives the only candidates.

In practice, setting up the function is usually more work than the calculus.

The method has five steps.

1. **Name the quantity to be optimised** and write an expression for it. At this stage it may contain more than one variable.

2. **Find the constraint**, the second equation linking those variables. It is given in the question, often as a fixed volume, a fixed length of material, or a fixed perimeter.
3. **Use the constraint to eliminate variables**, until the quantity is a function of one variable only. State the values that variable may take.
4. **Differentiate, set the derivative to zero, and solve.**
5. **Justify the nature of the point**, then answer the question.

TIP An optimisation question is not finished when you solve $f'(x) = 0$. If the question asks for the maximum volume, give the volume, not the value of x that produces it.

EXAMPLE 14.9

An open box is made from a 20×20 cm square of card. A square of side x is cut from each corner, and the sides are folded up, as in the figure below. Find the value of x that maximises the volume, and the maximum volume.

SOLUTION

After cutting, the base is $(20 - 2x)$ by $(20 - 2x)$ and the height is x, so

$$V(x) = x(20 - 2x)^2, \qquad 0 < x < 10.$$

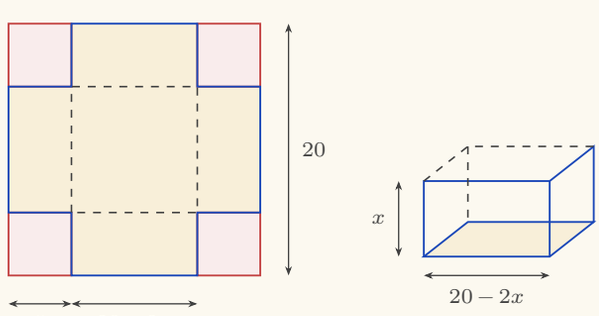


Figure 14.9 Cut out a square of side x (red) at each corner, then fold up along the dashed lines. In the folded box, dashed lines are hidden edges.

Differentiate with the product rule, then take out the common factor $(20 - 2x)$:

$$V'(x) = (20 - 2x)^2 + x \cdot 2(20 - 2x)(-2) = (20 - 2x)(20 - 6x).$$

Setting $V'(x) = 0$ gives $x = 10$ (rejected, no box) or $x = \frac{10}{3}$. On $0 < x < 10$ the factor $20 - 2x$ is positive, so V' has the sign of $20 - 6x$: positive before $\frac{10}{3}$ and negative after. So this is a maximum by the first derivative test, at $\boxed{x = \frac{10}{3}}$

The maximum volume is

$$V\left(\tfrac{10}{3}\right) = \boxed{\frac{16000}{27} \approx 593 \text{ cm}^3}$$

The restriction $0 < x < 10$ comes from the geometry. If x is 0 the box has no height; if x is 10 it has no base.

EXAMPLE 14.10

A closed cylindrical can must hold $355\,\text{cm}^3$. Find the radius that minimises the surface area.

SOLUTION

The volume condition gives the height: $\pi r^2 h = 355$, so $h = \dfrac{355}{\pi r^2}$. The surface area is two circles plus the curved side:

$$S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + \frac{710}{r}.$$

Differentiate and set to zero:

$$\frac{dS}{dr} = 4\pi r - \frac{710}{r^2} = 0 \implies r^3 = \frac{710}{4\pi} \implies \boxed{r \approx 3.84 \text{ cm}}$$

Since $\dfrac{d^2S}{dr^2} = 4\pi + \dfrac{1420}{r^3} > 0$, this is a minimum by the second derivative test. Either test is a valid justification; choose the one with shorter working. Using the constraint to remove h is the step that reduces two variables to one.
The height is then $h = \dfrac{355}{\pi r^2} \approx 7.67$ cm. This is $2r$: since $r^3 = \dfrac{355}{2\pi}$, $h = \dfrac{355}{\pi r^2} = \dfrac{2r^3}{r^2} = 2r$, so the best can is as tall as it is wide.

CAUTION If the variable is restricted, for example $0 < r \leq 3$, the best value may be at an **endpoint**, where the derivative need not be zero. HL List every stationary point in the interval and every endpoint that belongs to it, then compare their values.

EXAMPLE 14.11

For the can above, the manufacturer's machines cannot make a can with a radius above 3 cm, so $0 < r \leq 3$. Find the radius that minimises the surface area now.

SOLUTION

The unconstrained minimum at $r \approx 3.84$ is no longer allowed. Consider the derivative on $(0, 3]$:

$$\frac{dS}{dr} = 4\pi r - \frac{710}{r^2}.$$

Both terms increase as r increases, so $\dfrac{dS}{dr}$ is increasing on this interval. So it is enough to test the largest value of r: at $r = 3$, $12\pi - \frac{710}{9} \approx -41 < 0$. The derivative is negative at the right-hand end, and smaller still to the left, so it is negative throughout and S is decreasing across the whole interval. The minimum is therefore at the endpoint $\boxed{r = 3}$, with $S = 18\pi + \frac{710}{3} \approx 293$ cm^2, even though $S'(3) \neq 0$.

Optimisation is not restricted to areas and volumes. Any quantity that can be written as a differentiable function of one variable can be maximised or minimised the same way, and in applied work that function is often exponential.

EXAMPLE 14.12

After a dose is given, the concentration of a drug in the blood is modelled by $C(t) = 20t\,e^{-0.5t}$ mg L^{-1}, where $t \geq 0$ is the time in hours. Find when the concentration is greatest, and the maximum concentration.

SOLUTION

The product rule gives

$$C'(t) = 20e^{-0.5t} + 20t(-0.5)e^{-0.5t} = 20e^{-0.5t}\,(1 - 0.5t)\,.$$

Since $e^{-0.5t}$ is never zero, $C'(t) = 0$ requires $1 - 0.5t = 0$, so $\boxed{t = 2 \text{ hours}}$
To justify that this is a maximum, look at the sign of C': the factor $e^{-0.5t}$ is always positive, so the sign is that of $1 - 0.5t$, which is positive for $t < 2$ and negative for $t > 2$. The concentration rises, then falls. The maximum is

$$C(2) = \boxed{40e^{-1} \approx 14.7 \text{ mg L}^{-1}}$$

The shape can be read from the two terms of $C'(t) = 20e^{-0.5t}(1 - 0.5t)$. The 1 comes from differentiating the factor t, and the $-0.5t$ from differentiating $e^{-0.5t}$. For $t < 2$ the first is larger, so C rises. For $t > 2$ the second is larger, so C falls. At $t = 2$ they cancel, which is what $C'(t) = 0$ records.

YOUR TURN 14.3 Optimisation

Setting up and solving

16.

(a) Two numbers have a sum of 20. Find the maximum value of their product, and justify that it is a maximum.

(b) A rectangle has a perimeter of 40 cm. Find the dimensions giving the maximum area, and justify that it is a maximum.

(c) Find the minimum value of $S = x^2 + \dfrac{16}{x}$ for $x > 0$, justifying that it is a minimum.

17. A rectangular field is fenced on three sides, the fourth being a wall, using 100 m of fencing. Find the maximum area, and justify that it is a maximum.

Using a constraint to remove a variable

18. A closed box has a square base of side x cm and height h cm. Its total surface area is 150 cm^2.

(a) Show that the volume is $V = \dfrac{75x - x^3}{2}$, and state the possible values of x.

(b) Find the value of x that maximises the volume, and justify that it is a maximum.

(c) Find the maximum volume, and describe the shape of the box.

19. An open-topped cylindrical tank must hold 1000 cm^3. Show that its surface area is $S = \pi r^2 + \dfrac{2000}{r}$, and find the radius that minimises it.

20. Find the coordinates of the point on the curve $y = \sqrt{x}$ that is closest to the point $(4, 0)$, and the least distance. *(Hint: minimise the square of the distance)*

Exponential, logarithmic and trigonometric models

21.

(a) Find the minimum value of $y = x - \ln x$ for $x > 0$, justifying that it is a minimum.

(b) The concentration of a drug in the blood is $c(t) = 12t\,e^{-t/4}$ mg L^{-1}, where $t \geq 0$ is measured in hours. Find the time at which the concentration is greatest, and the maximum concentration.

22. GDC A rectangle has its base on the x-axis and its two upper vertices on the curve $y = \cos x$, symmetric about the y-axis, with $0 < x < \frac{\pi}{2}$. Show that its area is $A = 2x\cos x$, and find the value of x that maximises A.

Endpoints of a restricted domain

23. A rectangle has a perimeter of 40 cm, and one side, of length x cm, may be at most 8 cm long. Find the greatest possible area, and explain why it does not occur where $\dfrac{dA}{dx} = 0$.

24. For the drug in Question 21(b), blood samples are taken only during the first three hours, so $0 \leq t \leq 3$. Find the greatest concentration recorded in that time, and state when it occurs.

Concept check

25. A student is asked for the greatest value of $f(x) = x^3 - 3x$ for $0 \leq x \leq 3$, and writes: $f'(x) = 3x^2 - 3 = 0$, so $x = 1$ (as $x = -1$ is outside the interval), and the greatest value is $f(1) = -2$. Identify the two errors and find the greatest value.

14.4 Kinematics

Motion is change of position, so you can describe it with derivatives. Let $s(t)$ be a particle's **displacement** from a fixed origin at time t. Its **velocity** is the rate of change of displacement, $v = \dfrac{ds}{dt}$, and its **acceleration** is the rate of change of velocity.

FORMULA 14.3 **Acceleration** FORMULA BOOKLET · 5.9

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$$

It helps to read the chain from s to a in ordinary words:

$$s \xrightarrow{d/dt} v \xrightarrow{d/dt} a$$

Displacement answers *where is it?* Velocity answers *how fast is the position changing?* Acceleration answers *how fast is the velocity changing?* In a car on a straight road, s is the signed distance from the origin, here the starting point. (The odometer $(\text{the meter that counts the distance driven})$ shows the distance travelled, which is different.) The speedometer shows $|v|$, and the push you feel in your seat is a.

Velocity has a sign as well as a size. On a straight line there are only two directions, and the sign of v tells you which one the particle is moving in. **Speed** is the size of the velocity, written $|v|$, and it is never negative. A particle with $v = -8$ is moving in the negative direction at a speed of 8.

The particle is **momentarily at rest** $(\text{at rest for an instant})$ when $v = 0$. This is the stationary-point condition of §14.1 applied to displacement: a turning point on the graph of $s(t)$ is the instant the particle stops and reverses. The ball at the top of its flight is exactly this point. The particle reverses only if v changes sign. If $v = 0$ but v has the same sign on both sides, the particle stops for an instant and then carries on in the same direction, like a stationary point of inflexion of $s(t)$.

The particle **speeds up when v and a have the same sign** and slows down when the signs differ. Understand this rule rather than memorise it. Think of acceleration as a push. If the push acts in the direction the particle is already moving, the particle gains speed; if it acts against the motion, the particle loses speed.

	$a > 0$	$a < 0$
$v > 0$	speeding up	slowing down
$v < 0$	slowing down	speeding up

Notice that this does not depend on a being positive: a particle moving in the negative direction with a negative acceleration is speeding up, because the push and the motion agree.

CAUTION $a = 0$ does not mean the particle has stopped. Acceleration is the derivative of velocity, so $a = 0$ marks a stationary point of $v(t)$. At that instant the velocity may be greatest or least: it is a candidate, as with $f'(x) = 0$ in §14.1. It is not where the velocity is zero. A car cruising at a steady $100\,\text{km}\,\text{h}^{-1}$ has zero acceleration and is not stopped at all. The particle is at rest only when $v = 0$.

One quantity needs integration rather than differentiation: the **total distance travelled**, which keeps counting when the particle turns and comes back. The change in displacement over an interval is $\int v\,dt$, and the distance travelled is $\int |v|\,dt$ (the sign $\int$ marks an integral). Both belong to the same syllabus item as the derivatives above, and both are developed in Ch. 15.

EXAMPLE 14.13

A particle has displacement $s(t) = t^3 - 12t^2 + 36t$ metres, for $t \geq 0$ seconds. Find when it is at rest and its acceleration at those times, and decide whether it is speeding up at $t = 7$.

SOLUTION

Differentiate for velocity:

$$v(t) = 3t^2 - 24t + 36 = 3(t-2)(t-6).$$

The particle is at rest when $v = 0$: at $t = 2$ s and $t = 6$ s

Acceleration is $a(t) = 6t - 24$. At $t = 2$, $a = -12\,\text{m}\,\text{s}^{-2}$; at $t = 6$, $a = 12\,\text{m}\,\text{s}^{-2}$

Each factor of v changes sign at its root, so v changes sign at $t = 2$ and at $t = 6$, and the particle reverses direction at both.

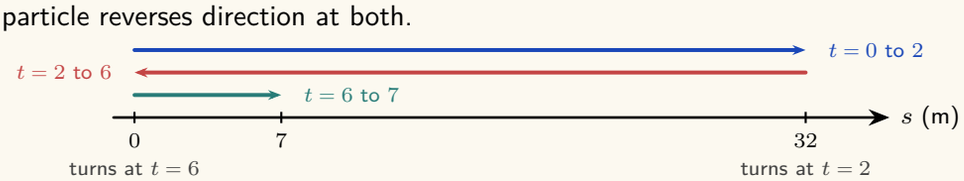


Figure 14.10 The particle moves out to 32 m, back to the origin, then out again. It reverses at $t = 2$ and $t = 6$.

At $t = 7$, $v(7) = 3(5)(1) = 15 > 0$ and $a(7) = 6(7) - 24 = 18 > 0$. The two signs are the same, so the particle is speeding up: it moves in the positive direction and the acceleration acts the same way.

EXT Not all motion is polynomial. Motion that repeats, such as a pendulum (a weight swinging on a string), a mass on a spring, a wave or a vibrating string, is often modelled by a sine or a

cosine. When the acceleration is proportional to the displacement and acts in the opposite direction, so that $a = -k^2 s$ for a constant k, the motion is called **simple harmonic motion**. The particle is at rest at its greatest displacement, where the pull back towards the origin is strongest. It moves fastest as it passes through the origin, where the pull is zero.

EXAMPLE 14.14

A particle moves in a straight line with displacement $s = 4\sin 2t$ metres, for $t \geq 0$ seconds. Find its velocity and acceleration, the first time it is at rest, and show that $a = -4s$.

SOLUTION

Differentiate twice, using the chain rule each time:

$$v = \frac{ds}{dt} = \boxed{8\cos 2t}, \qquad a = \frac{dv}{dt} = \boxed{-16\sin 2t}$$

The particle is at rest when $v = 0$, that is when $\cos 2t = 0$. The first solution is $2t = \frac{\pi}{2}$, so $\boxed{t = \frac{\pi}{4} \approx 0.785 \text{ s}}$

At that instant $s = 4\sin\frac{\pi}{2} = 4$ m, the greatest displacement it reaches.

Finally, since $s = 4\sin 2t$,

$$-4s = -4(4\sin 2t) = -16\sin 2t = a. \qquad \blacksquare$$

So $a = -k^2 s$ with $k = 2$, and the particle moves with simple harmonic motion: the further it is from the origin, the harder it is pulled back.

TIP In kinematics questions, state the units with every answer: m s^{-1} for velocity, m s^{-2} for acceleration. A correct number without its unit may lose a mark.

TIP To find the greatest speed, compare $|v|$, not v. Because speed ignores direction, the greatest speed can occur where the velocity is most negative.

YOUR TURN 14.4 Kinematics

Velocity, acceleration and rest

26. A particle moves in a straight line. Its displacement is s metres and its velocity is v m s^{-1} at time t seconds, $t \geq 0$.

(a) Its displacement is $s = t^2 - 4t + 3$. Find when it is at rest, and its acceleration then.

(b) Its displacement is $s = 2t^3 - 9t^2 + 12t$. Find the times when it is at rest.

(c) Its displacement is $s = t^3 - 3t^2$. Find its velocity and acceleration at $t = 2$.

(d) Its velocity is $v = t^2 - 5t + 6$. Find when it is at rest, and its acceleration at those times.

27. A particle has velocity $v = 3t^2 - 12t + 9$ m s^{-1} at time t seconds, $t \geq 0$.

(a) Find the times when it is at rest.

(b) Find its acceleration at those times.

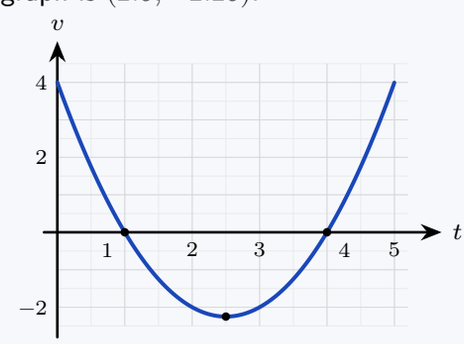
Speeding up or slowing down

28.

(a) For the particle in Question 27, determine whether it is speeding up or slowing down at $t = 1.5$, giving a reason.

(b) A second particle has velocity $v = t - 4$ m s^{-1} at time t seconds. State, with a reason, whether it is speeding up or slowing down at $t = 2$.

(c) The graph shows the velocity v m s^{-1} of a third particle for $0 \leq t \leq 5$ seconds. The minimum point of the graph is $(2.5, -2.25)$.



Write down the times at which the particle is at rest. Hence find the intervals of time on which the particle is speeding up.

Speed

29. A particle has velocity $v = t^2 - 6t + 3$ m s^{-1} at time t seconds, for $0 \leq t \leq 5$.

(a) Find its speed at $t = 1$.

(b) Find the time at which its acceleration is zero. State, with a reason, whether the particle is at rest at that time.

(c) Find the greatest speed of the particle for $0 \leq t \leq 5$.

30. A particle has displacement $s = t^3 - 9t^2 + 24t$ metres at time t seconds, $t \geq 0$.

(a) Find the times when the particle is at rest.

(b) Find the time at which its acceleration is zero, and its speed at that time.

(c) Sketch the velocity–time graph for $0 \leq t \leq 5$. Show the points where it meets the axes and its minimum point.

Other functions of time

31.

(a) A particle has displacement $s = 2\cos 3t$ metres, $t \geq 0$. Find v and a, and show that $a = -9s$.

(b) For the particle in part (a), find the first time after $t = 0$ at which it is at rest, and its displacement at that instant.

(c) A particle has velocity $v = 6e^{-0.5t}\,\text{m s}^{-1}$. Find its acceleration at $t = 0$, and explain why the particle never comes to rest.

(d) A particle has displacement $s = t + 2\cos t$ metres, for $0 \leq t \leq 2\pi$. Find the times at which it is at rest.

(e) A particle has displacement $s = (t-2)^3 + 4$ metres, $t \geq 0$. Show that it is at rest at $t = 2$ but does not change direction there.

Concept check

32. A student says: "*the acceleration is negative, so the particle is slowing down.*" State in one line when this is false.

33. Explain the difference between the *change in displacement* and the *distance travelled* over $0 \leq t \leq 5$, for a particle that reverses direction at $t = 3$.

14.5 Implicit Differentiation HL

So far every curve has been given as $y = f(x)$, with y written on its own. But an equation such as $x^2 + y^2 = 25$ also defines a curve, without y being isolated. **Implicit differentiation** finds the gradient of such a curve directly, without first solving for y. Every tangent and normal method of §13.4 then works on these curves too, including curves that are not the graph of a function.

The method is the chain rule. Treat y as an unknown function of x. Differentiate a term in y as usual, then multiply by an extra factor $\dfrac{dy}{dx}$, because you are differentiating a function of y with respect to x. For example, $\dfrac{d}{dx}(y^2) = 2y\dfrac{dy}{dx}$.

The procedure has four steps.

1. Differentiate every term of the equation with respect to x, using the product rule where x and y are multiplied.
2. Collect the terms containing $\dfrac{dy}{dx}$ on one side.
3. Factorise $\dfrac{dy}{dx}$ out of those terms.
4. Divide to make $\dfrac{dy}{dx}$ the subject.

Before you use a given point, substitute it into the equation to check that it lies on the curve.

EXAMPLE 14.15

For the circle $x^2 + y^2 = 25$, find $\dfrac{dy}{dx}$, and the gradient at $(3, 4)$.

SOLUTION

Differentiate both sides with respect to x:

$$2x + 2y\frac{dy}{dx} = 0 \implies \boxed{\frac{dy}{dx} = -\frac{x}{y}}$$

At $(3, 4)$ the gradient is $\boxed{-\frac{3}{4}}$

The answer depends on both coordinates, as it should, because the gradient changes as you move round the circle.

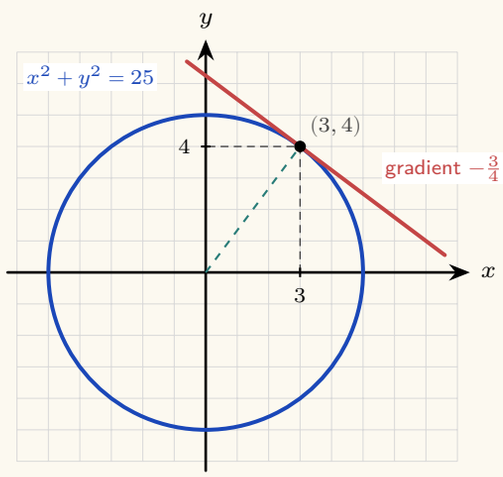


Figure 14.11 The tangent to $x^2 + y^2 = 25$ at $(3, 4)$ has gradient $-\frac{x}{y} = -\frac{3}{4}$. The dashed radius has gradient $\frac{4}{3}$, so the tangent is perpendicular to it, as it must be for a circle.

EXAMPLE 14.16

Find $\dfrac{dy}{dx}$ for $x^2 + xy + y^2 = 7$.

SOLUTION

The middle term needs the product rule. Differentiating term by term:

$$2x + \left(y + x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0.$$

Collect the $\dfrac{dy}{dx}$ terms:

$$\frac{dy}{dx}(x + 2y) = -(2x + y) \implies \boxed{\frac{dy}{dx} = -\frac{2x + y}{x + 2y}}$$

EXAMPLE 14.17

Find the equation of the tangent to the curve $x^2 + xy + y^2 = 7$ at the point $(1, 2)$.

SOLUTION

First confirm the point lies on the curve: $1 + 2 + 4 = 7$. From the previous example,

$$\frac{dy}{dx} = -\frac{2x + y}{x + 2y} = -\frac{2 + 2}{1 + 4} = -\frac{4}{5} \quad \text{at } (1, 2).$$

The tangent is $y - 2 = -\frac{4}{5}(x - 1)$, or $\boxed{4x + 5y = 14}$

You find the gradient by implicit differentiation. The rest is the usual equation of a line through a given point.

Use the same method, without any change, for exponential terms and for the trigonometric terms met later in this section. A term in y is still multiplied by $\frac{dy}{dx}$, whatever function y appears in.

EXAMPLE 14.18

The curve C has equation $e^y + xy = 2$. Show that $(0, \ln 2)$ lies on C, and find the gradient there.

SOLUTION

At $(0, \ln 2)$: $e^{\ln 2} + 0 = 2$, so the point lies on C.

Differentiate every term with respect to x. The first needs the chain rule and the second the product rule:

$$e^y \frac{dy}{dx} + \left(y + x\frac{dy}{dx}\right) = 0.$$

Collect the $\frac{dy}{dx}$ terms and factorise:

$$\frac{dy}{dx}(e^y + x) = -y \implies \frac{dy}{dx} = -\frac{y}{e^y + x}.$$

At $(0, \ln 2)$ this gives $\frac{dy}{dx} = -\frac{\ln 2}{2 + 0} = \boxed{-\frac{\ln 2}{2} \approx -0.347}$

Notice that this curve cannot be rearranged to give y in terms of the functions of this course: y appears both inside an exponential and multiplied by x. The formula still gives the gradient at every point of the curve where its denominator $e^y + x$ is not zero. Here implicit differentiation is not a shortcut. It is the method this course gives for such a curve.

To find $\frac{d^2y}{dx^2}$ implicitly, differentiate the expression for $\frac{dy}{dx}$ again with respect to x, using the quotient rule if it is a fraction in x and y. Every term in y again gains a factor $\frac{dy}{dx}$. Then replace $\frac{dy}{dx}$ by its expression, so that the answer contains no derivative.

EXAMPLE 14.19

Find $\dfrac{d^2y}{dx^2}$ for the curve $y^2 = 4x$.

SOLUTION

A second derivative is found by differentiating the first one implicitly again. Differentiating with respect to x gives

$$2y\frac{dy}{dx} = 4 \implies \frac{dy}{dx} = \frac{2}{y}.$$

Now differentiate $\dfrac{2}{y} = 2y^{-1}$ with respect to x. It is a function of y, so the chain rule gives a factor $\dfrac{dy}{dx}$ again:

$$\frac{d^2y}{dx^2} = -2y^{-2}\,\frac{dy}{dx} = -\frac{2}{y^2}\cdot\frac{2}{y} = \boxed{-\frac{4}{y^3}}$$

The last step replaces $\dfrac{dy}{dx}$ by $\dfrac{2}{y}$, so that the answer contains no derivative.

EXAMPLE 14.20

Show that $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$, $\dfrac{d}{dx}\arcsin x = \dfrac{1}{\sqrt{1-x^2}}$ and $\dfrac{d}{dx}\arccos x = -\dfrac{1}{\sqrt{1-x^2}}$.

SOLUTION

With implicit differentiation you can prove the three inverse trigonometric derivatives that were stated without proof in §13.5. Let $y = \arctan x$, so that $\tan y = x$. Differentiate both sides with respect to x:

$$\sec^2 y\,\frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\sec^2 y}.$$

The answer should be in terms of x. Use the identity $\sec^2 y = 1 + \tan^2 y$. Since $\tan y = x$, $\sec^2 y = 1 + x^2$, so

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}.$$

For $y = \arcsin x$, write $\sin y = x$, so that $\cos y\,\dfrac{dy}{dx} = 1$. Now $\cos^2 y = 1 - \sin^2 y = 1 - x^2$, and $\arcsin$ takes values in $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$, where $\cos y \ge 0$,

so the positive root is the right one:

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}.$$

For $y = \arccos x$, the same steps on $\cos y = x$ give $-\sin y \dfrac{dy}{dx} = 1$. Here $0 \le y \le \pi$, where $\sin y \ge 0$, so $\sin y = \sqrt{1-x^2}$ and

$$\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}.$$

This agrees with $\arccos x = \frac{\pi}{2} - \arcsin x$ from §13.5: the two derivatives add to zero. The last two results hold for $-1 < x < 1$. At $x = \pm 1$ the tangent to each graph is vertical, so there is no derivative there.

The same method proves the derivative of $\ln x$, which was read from a graph in §13.5. Let $y = \ln x$, so that $e^y = x$. Differentiating both sides with respect to x, using the derivative of e^x from §13.5, gives $e^y \dfrac{dy}{dx} = 1$, and since $e^y = x$,

$$\frac{d}{dx}\ln x = \frac{1}{e^y} = \frac{1}{x}.$$

YOUR TURN 14.5 Implicit Differentiation HL

Finding the gradient function

34. Find $\dfrac{dy}{dx}$ for each curve.

(a) $x^2 + y^2 = 16$

(b) $xy = 12$

(c) $x^2 - y^2 = 9$

(d) $x^3 + y^3 = 6$

35. Each of these needs the product rule.

(a) Find $\dfrac{dy}{dx}$ for $x^2 + xy = 5$.

(b) Find $\dfrac{dy}{dx}$ for $x^2 y + y^2 = 4$.

36. Find $\dfrac{dy}{dx}$ for each curve in parts (a) to (c).

(a) $e^y = x^2 + y$

(b) $\sin y + x^2 = 4$

(c) $x \ln y = 4$

(d) Find $\dfrac{d^2y}{dx^2}$ for $y^2 = 9x$, in terms of y.

Gradients, tangents and normals

37.

(a) Find the gradient of $x^2 + y^2 = 169$ at the point $(5, -12)$.

(b) Find the equation of the tangent to $xy = 4$ at the point $(2, 2)$.

(c) On the curve $x^2 + 4y^2 = 16$, find the two points at which the tangent is horizontal.

(d) Explain why implicit differentiation gives a gradient in terms of both x and y, while differentiating $y = f(x)$ gives one in terms of x alone.

38.

(a) Find the gradient of $e^y + x = 3$ at the point $(2, 0)$.

(b) Show that $(1, 2)$ lies on the curve $x^3 + y^3 = 9$, and find the equation of the normal there.

39. Show that the point $(2, 1)$ lies on the curve $x^2 - 3xy + y^2 = -1$, and find the equations of the tangent and the normal at that point.

Concept check

40. A student differentiates $x^2 + y^2 = 25$ with respect to x and writes $2x + 2y = 0$. Identify the error, and find $\dfrac{dy}{dx}$.

14.6 Related Rates HL

When two quantities are linked by an equation and both change over time, their rates of change are linked too. In a **related rates** problem you connect those rates with the chain rule. Differentiate, with respect to time, the equation that links the quantities.

The method has three steps.

1. Write an equation relating the quantities.
2. Differentiate every term with respect to t, so that the rate of each variable appears.
3. Substitute the values known at the instant in question, and solve for the rate you need.

Each variable is a function of t, so, as in §14.5, a term in x gains a factor $\dfrac{dx}{dt}$: for example, $\dfrac{d}{dt}(x^2) = 2x\dfrac{dx}{dt}$. A negative rate means that the quantity is decreasing at that instant.

Sometimes the equation contains a variable whose rate is neither given nor asked for. Remove that variable before differentiating, using a fixed relation from the shape such as similar triangles.

TIP Substitute the given values only *after* differentiating. Putting a value such as $r = 5$ into the volume of a sphere, $V = \frac{4}{3}\pi r^3$, first turns a variable into a constant, and a constant has rate of change zero.

EXAMPLE 14.21

A 10 m ladder leans against a wall. The foot slides away from the wall at $0.6\,\text{m s}^{-1}$. How fast is the top sliding down when the foot is 6 m from the wall?

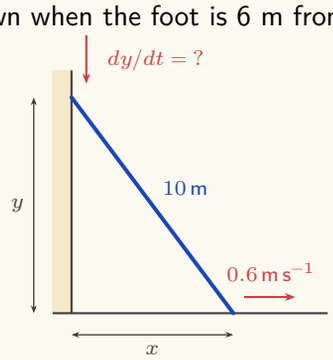


Figure 14.12 The ladder at the instant the foot is 6 m from the wall.

SOLUTION

Let x be the distance of the foot from the wall and y the height of the top. Then $x^2 + y^2 = 100$. Differentiate with respect to t:

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt}.$$

When $x = 6$, $y = \sqrt{100 - 36} = 8$. With $\dfrac{dx}{dt} = 0.6$:

$$\frac{dy}{dt} = -\frac{6}{8}(0.6) = \boxed{-0.45\ \text{m s}^{-1}}$$

The negative sign means the top moves downward, as expected.

EXAMPLE 14.22

A water tank is an inverted cone with its vertex at the bottom, as tall as it is wide: at every level, the water surface has radius $r = \frac{h}{2}$, where h is the water depth. Water flows in at $2\,\text{m}^3\,\text{min}^{-1}$. How fast is the depth rising when $h = 3$ m?

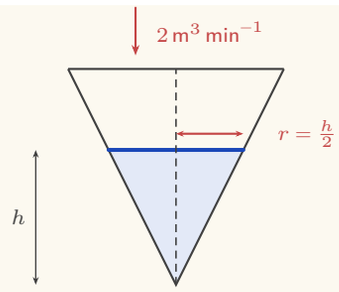


Figure 14.13 A cross-section of the tank through its axis.

SOLUTION

Write the volume in terms of h alone, using the shape constraint $r = \frac{h}{2}$:

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \left(\frac{h}{2}\right)^2 h = \frac{\pi h^3}{12}.$$

Differentiate with respect to time:

$$\frac{dV}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt} \implies 2 = \frac{9\pi}{4}\frac{dh}{dt} \implies \frac{dh}{dt} = \boxed{\frac{8}{9\pi} \approx 0.283 \text{ m min}^{-1}}$$

Remove r *before* differentiating. If r is left in, you need the product rule and the rate $\frac{dr}{dt}$ as well, and r is easy to treat by mistake as a constant. Removing r first avoids this.

The result also has a physical meaning: the same inflow raises the level more slowly as the cone widens.

The quantities linked need not be lengths and volumes. When the link is an angle, the relating equation is trigonometric, and the method is unchanged.

EXAMPLE 14.23

A balloon rises vertically at $3\,\mathrm{m\,s}^{-1}$ from a point 40 m from an observer. Find the rate at which the angle of elevation of the balloon is increasing when the balloon is 40 m high.

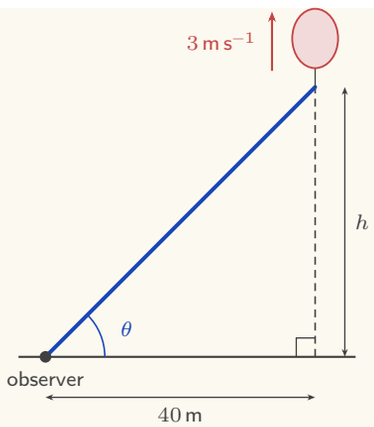


Figure 14.14 The balloon, the observer and the angle of elevation θ.

SOLUTION

Let h be the height of the balloon and θ the angle of elevation. The observer's distance is fixed at 40 m, so

$$\tan\theta = \frac{h}{40}.$$

Differentiate both sides with respect to t. The left side needs the chain rule. The angle must be in radians, because $\dfrac{d}{d\theta}\tan\theta = \sec^2\theta$ holds only in radians §13.5:

$$\sec^2\theta\,\frac{d\theta}{dt} = \frac{1}{40}\,\frac{dh}{dt}.$$

When $h = 40$ the triangle is isosceles, so $\theta = \frac{\pi}{4}$ and $\sec^2\theta = 2$. Substituting this and $\dfrac{dh}{dt} = 3$:

$$2\,\frac{d\theta}{dt} = \frac{3}{40} \implies \frac{d\theta}{dt} = \boxed{\frac{3}{80} = 0.0375 \text{ rad s}^{-1}}$$

The balloon rises at a constant rate, but the angle does not increase at a constant rate. As the balloon climbs, $\sec^2\theta$ grows, so $\dfrac{d\theta}{dt}$ falls: the observer's head turns more and more slowly, even though the balloon never slows down.

YOUR TURN 14.6 Related Rates HL

One formula, one rate

41.

(a) A square's side grows at $2\,\text{cm s}^{-1}$. Find the rate of change of its area when the side is 5 cm.

(b) A circle's radius grows at $3\,\text{cm s}^{-1}$. Find the rate of change of its area when $r = 4$ cm.

(c) A cube's side grows at $1\,\text{cm}\,\text{s}^{-1}$. Find the rate of change of its volume when the side is 10 cm.

(d) A circle's radius grows at $0.4\ \text{cm}\,\text{s}^{-1}$. Find the rate of increase of its circumference.

42. A spherical snowball melts so that its volume decreases at $20\,\text{cm}^3\,\text{min}^{-1}$. Find the rate of change of its radius when $r = 8$ cm, and explain what the sign of your answer means.

Pythagoras and similar triangles

43. A ladder 13 m long leans against a vertical wall. Its top slides down the wall at $0.5\,\text{m}\,\text{s}^{-1}$. Find the rate at which the foot moves away from the wall when the top is 5 m above the ground.

44. Water is poured at $20\,\text{cm}^3\text{s}^{-1}$ into a cone whose vertex points downwards. The water depth h is always twice the radius r of the water surface.

(a) Show that the volume of water is $V = \frac{2}{3}\pi r^3$.

(b) Find the rate at which r increases when $r = 5$ cm.

45. A conical tank, vertex down, is 6 m deep and has a top radius of 2 m. Water drains out at $0.5\,\text{m}^3\,\text{min}^{-1}$. Find the rate of change of the water depth when the depth is 3 m, and interpret its sign. *(Hint: use similar triangles to write r in terms of h)*

Angles

46. An aircraft flies horizontally at a height of 3 km, at $0.2\,\text{km}\,\text{s}^{-1}$, directly towards an observer on the ground. Find the rate of change of the angle of elevation of the aircraft when its horizontal distance from the observer is 4 km.

47. A searchlight rotates at $0.2\,\text{rad}\,\text{s}^{-1}$ and its beam sweeps along a straight wall 50 m away. Find the speed of the spot of light along the wall when the beam makes an angle of $\frac{\pi}{6}$ with the perpendicular to the wall.

Concept check

48. The radius of a circle grows at $0.5\ \text{cm}\,\text{s}^{-1}$. To find the rate of change of its area when $r = 6$ cm, a student writes $A = \pi(6)^2 = 36\pi$, so $\dfrac{dA}{dt} = 0$. Identify the error and find the correct rate.

Chapter 14 · Derivative Applications

Reflections

TOK This chapter is full of conditions that are necessary but not sufficient (they must hold, but they are not enough on their own): $f'(x) = 0$ holds at every smooth maximum, and, for a twice-differentiable function, $f''(x) = 0$ at every point of inflexion, yet neither condition guarantees one. What kind of knowledge is a necessary condition? Does knowing where something must be count as knowing where it is?

TOK Calculus makes exact claims about idealised objects (simplified, perfect versions of real things). The can that uses least metal has its height equal to its diameter. Real cans are taller and narrower, because the model ignores printing, shipping and the size of a hand. When a model is exact and its conclusion still differs from the world, which one is wrong: the model, or our reading of it?

IM Many laws of nature can be written as optimisation problems. In the 1600s Pierre de Fermat proposed that light travels along the path that takes the least time, and with that one idea you can explain why a straw looks bent in water. Euler and Lagrange later extended it into the *principle of least action*, which restates much of classical mechanics as finding where a quantity is stationary.

RW Many machine-learning models are trained by *gradient descent*, which applies this chapter's search for a minimum to a very large function. A model has millions of adjustable numbers, and the size of its errors is a function of those numbers; training means making that function as small as possible. The computer measures the gradient at its current point, takes a small step 'downhill' (in the direction that makes the function smaller), and repeats. As the gradient approaches zero the steps shrink. A small gradient may mean that a minimum has been reached, but it can also occur near a saddle point (a flat point that is neither a maximum nor a minimum), so $f' = 0$ is once more necessary and not sufficient.

EXT A lifeguard spots a swimmer in trouble further along the shore. She runs faster than she swims, so the straight line is not the quickest route: she should run further and swim less. If she runs at speed v_1, swims at speed v_2, and her path meets the waterline at angles θ_1 and θ_2 to the perpendicular, the time is least when $\sin\theta_1/v_1 = \sin\theta_2/v_2$. This is **Snell's law** of refraction (the bending of light as it passes into another material), and it is Fermat's principle from the International box. *Try it:* she stands 30 m from the water; the swimmer is 40 m out and 60 m along the beach; she runs at $7\,\text{m}\,\text{s}^{-1}$ and swims at $2\,\text{m}\,\text{s}^{-1}$. Write the time $T(x)$ if she enters the water x m along the beach, and find the best x on your GDC.

End-of-Chapter Problems
Chapter 14: Derivative Applications

1. Consider $g(x) = x + \frac{4}{x}$ on the interval $1 \leq x \leq 3$.

(a) Find the stationary point of g in the interval and classify it. [4]

(b) Find the maximum value of g on the interval, and state where it occurs. [2]

2. Consider $f(x) = x + 2\cos x$ on the closed interval $0 \leq x \leq 2\pi$.

(a) Find the stationary points in the interval and classify them. [4]

(b) Find the greatest and least values of f on the interval. [3]

3. Consider $f(x) = e^x - 2x$.

(a) Find $f'(x)$ and the coordinates of the stationary point. [3]

(b) Determine the nature of the stationary point. [2]

(c) Explain why f has no point of inflexion. [2]

4. The curve $y = x^3 + ax^2 + bx$ has stationary points at $x = 1$ and $x = -2$.

(a) Find the values of a and b. [4]

(b) Classify each stationary point. [3]

5. For $f(x) = x^4 - 4x^3 + 6x^2$, show that $f''(1) = 0$ but that the point $(1, 3)$ is *not* a point of inflexion. [4]

6. Consider $f(x) = x^3 - 3x^2 - 9x + 5$.

(a) Find and classify the stationary points. [4]

(b) Find the point of inflexion. [2]

(c) State the interval on which f is decreasing. [2]

(d) Sketch the graph of $y = f(x)$ for $-2.5 \leq x \leq 5$. Show the stationary points, the point of inflexion and the y-intercept. [3]

7. Consider $f(x) = e^{-x^2/2}$.

(a) Find $f'(x)$, and show that $f''(x) = (x^2 - 1)\, e^{-x^2/2}$. [5]

(b) Find the exact coordinates of the points of inflexion, and state the interval on which the curve is concave down. [3]

8. Consider $f(x) = x^2 e^{-x}$.

(a) Find $f'(x)$ and the coordinates of the two stationary points. [4]

(b) Use the second derivative to determine the nature of each stationary point. [3]

(c) Find the exact x-coordinates of the points of inflexion. [2]

9. Consider $f(x) = \dfrac{\ln x}{x^2}$ for $x > 0$.

(a) Show that $f'(x) = \dfrac{1 - 2\ln x}{x^3}$. [3]

(b) Find the exact coordinates of the stationary point and determine its nature. [4]

(c) Show that $f''(x) = \dfrac{6\ln x - 5}{x^4}$, and hence find the exact x-coordinate of the point of inflexion. [4]

10. Consider $f(x) = x^2 \ln x$ for $x > 0$.

(a) Find $f'(x)$, and factorise it. [2]

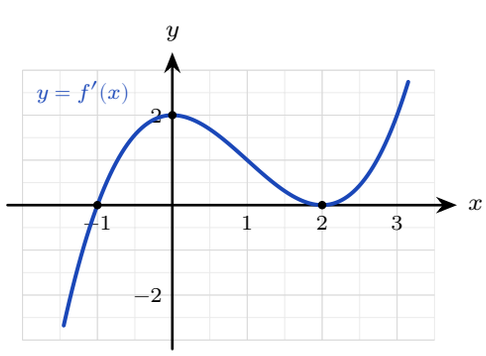
(b) Find the exact coordinates of the stationary point. [3]

(c) Find $f''(x)$, and use it to classify that point. [3]

(d) Find the exact coordinates of the point of inflexion, and show that the concavity changes there. [4]

(e) Describe the behaviour of $f(x)$ as $x \to 0^+$ and as $x \to \infty$. [3]

11. The graph of $y = f'(x)$, the derivative of a function f, is shown for $-1.5 \leq x \leq 3$. It meets the x-axis only at $x = -1$ and at $x = 2$, and its turning points are at $x = 0$ and $x = 2$.



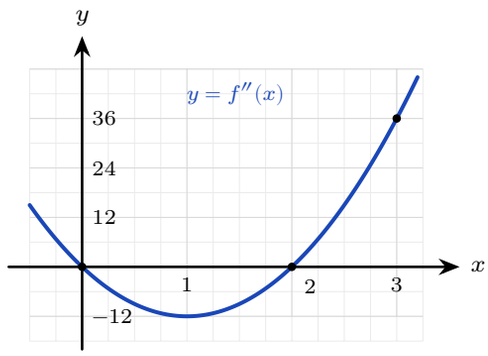
(a) Find the x-coordinate of the local minimum point of the graph of f. Justify your answer. [2]

(b) Explain why the graph of f has a stationary point at $x = 2$ that is not a turning point. [2]

(c) Write down the x-coordinates of the points of inflexion of the graph of f, and the interval on which the graph of f is concave down. [2]

(d) Given that $f(0) = 0$, sketch a possible graph of $y = f(x)$ for $-1.5 \leq x \leq 3$. Show the features found in parts (a) to (c). [3]

12. The graph of $y = f''(x)$, the second derivative of a function f, is shown for $-0.5 \leq x \leq 3.2$. It meets the x-axis only at $x = 0$ and $x = 2$. The graph of f has stationary points at $x = 0$ and $x = 3$ only.



(a) Write down the interval on which the graph of f is concave down. Give a reason. [2]
(b) Determine the nature of the stationary point at $x = 3$. [2]
(c) Explain why the second derivative test cannot be used at $x = 0$. Use the graph to determine the nature of the stationary point at $x = 0$. [3]
(d) Write down the x-coordinate of the non-stationary point of inflexion of the graph of f. [1]

13. A company's profit from selling x units is $P(x) = -2x^2 + 120x - 1000$ dollars. Find the value of x that maximises the profit, justifying that it is a maximum, and state the maximum profit. [5]

14. GDC Let D be the distance from the origin to a point (x, e^x) on the curve $y = e^x$.

(a) Write down an expression for D^2 in terms of x. [1]
(b) Show that D^2 is stationary where $x + e^{2x} = 0$. [2]
(c) Find the coordinates of the point on the curve closest to the origin, and the least distance, giving your answers to 3 significant figures. [3]

15. GDC A closed cylindrical can must hold $500\,\text{cm}^3$.

(a) Show that the surface area is $S = 2\pi r^2 + \dfrac{1000}{r}$. [3]
(b) Find the radius that minimises S, and justify it is a minimum. [4]

16. An open rectangular tank with a square base of side x must hold $32\,\text{m}^3$.

(a) Show that the surface area (base + four sides) is $A = x^2 + \dfrac{128}{x}$. [3]
(b) Find the value of x that minimises the material used. [4]

17. GDC The population of a colony, in thousands, is modelled by $P(t) = 40t\,e^{-t/5}$ for $t \geq 0$, where t is measured in years.

(a) Show that $P'(t) = 8e^{-t/5}(5 - t)$. [3]

(b) Find the time at which the population is largest, and the largest population, giving the population to three significant figures. [4]

18. A gutter is made by bending a long strip of metal of width 30 cm into three equal sections of 10 cm. The two outer sections are turned up at an angle θ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. The cross-sectional area is $A = 100\sin\theta\,(1 + \cos\theta)$.

(a) Find $\dfrac{dA}{d\theta}$. [3]

(b) Show that A is stationary when $2\cos^2\theta + \cos\theta - 1 = 0$, and find the value of θ that maximises A. [4]

(c) Find the maximum cross-sectional area. [2]

19. An open-topped box is made from a 16×10 cm rectangle of card by cutting a square of side x cm from each corner and folding up the sides.

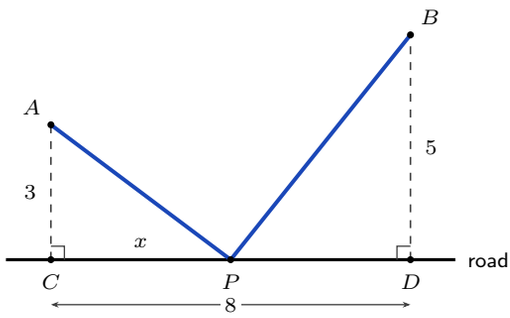
(a) Show that the volume is $V = x(16 - 2x)(10 - 2x)$, and state the possible values of x. [3]

(b) Find $\dfrac{dV}{dx}$ and show that it is zero at $x = 2$ and $x = \frac{20}{3}$. [3]

(c) Explain why $x = \frac{20}{3}$ must be rejected. [2]

(d) Find the maximum volume, justifying that it is a maximum. [4]

20. Two houses, A and B, are on the same side of a straight road. The points C and D are the points of the road nearest to A and to B, with $AC = 3$ km, $BD = 5$ km and $CD = 8$ km. A cable is laid in a straight line from A to a point P on the road between C and D, and then in a straight line from P to B. Let $CP = x$ km.



(a) Show that the total length of cable is $L(x) = \sqrt{x^2 + 9} + \sqrt{x^2 - 16x + 89}$. [2]

(b) Find $L'(x)$. [2]

(c) Show that $L'(x) = 0$ when $5x = 3(8 - x)$. Hence find the value of x for which $L'(x) = 0$. [3]

(d) Find the least length of cable, and justify that it is the least. [3]

21. A particle's velocity is $v(t) = 2t^2 - 14t + 20\,\text{m s}^{-1}$, $t \geq 0$.

(a) Find the times when the particle is at rest. [2]

(b) Find the minimum velocity and the time at which it occurs. [3]

22. A particle moves with displacement $s(t) = t^3 - 9t^2 + 15t$ metres, $t \geq 0$.

(a) Find the velocity and the times when the particle is at rest. [3]

(b) Find the acceleration when $t = 1$. [2]

(c) Determine whether the particle is speeding up or slowing down at $t = 1.5$. [3]

23. A particle moves with displacement $s(t) = t^3 - 15t^2 + 63t - 10$ metres, for $0 \leq t \leq 10$ seconds.

(a) Find the times at which the particle is at rest. [3]

(b) Find its acceleration at each of those times. [2]

(c) Find the intervals of time during which the particle's *speed* is increasing. [3]

24. A particle moves in a straight line with displacement $s(t) = 3\sin 2t$ metres, $t \geq 0$ seconds.

(a) Find expressions for the velocity and the acceleration. [2]

(b) Show that $a = -4s$. [2]

(c) Find the first two times at which the particle is at rest. [3]

(d) State the maximum speed of the particle, and the displacement at which it occurs. [3]

25. GDC A particle moves in a straight line with displacement $s = 5\sin t + 0.5t^2$ metres at time t seconds, for $0 \leq t \leq 6$.

(a) Find expressions for the velocity v and the acceleration a. [2]
(b) Find the times at which the particle is at rest, giving your answers to 3 significant figures. [3]
(c) Find the greatest speed of the particle for $0 \leq t \leq 6$, giving your answer to 3 significant figures. [3]
(d) Determine whether the particle is speeding up or slowing down at $t = 3$, giving a reason. [2]

26. GDC A particle moves in a straight line with displacement $s(t) = 12\left(1 - e^{-t/3}\right)$ metres, for $t \geq 0$ seconds.

(a) Find expressions for the velocity and the acceleration. [3]
(b) Write down the initial velocity, and show that the velocity is never zero. [2]
(c) Find the displacement and the velocity at $t = 3$, each to three significant figures. [2]
(d) State what happens to s and to v as $t \to \infty$, and interpret the limiting value of s. [3]
(e) Find the exact time at which the speed has fallen to half its initial value. [3]

27. The curve $\sin y + x^2 = 1$ passes through the point $(1, 0)$. Find the gradient of the curve at that point. [4]

28. A curve has equation $x^2 + xy + y^2 = 3$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [3]
(b) Find the equation of the tangent at the point $(1, 1)$. [3]

29. For the curve $x^2 + y^2 = 25$, use implicit differentiation twice to show that $\dfrac{d^2y}{dx^2} = -\dfrac{25}{y^3}$. [6]

30. The curve C has equation $x^3 + y^3 = 9xy$.

(a) Verify that the point $(2, 4)$ lies on C. [2]
(b) Use implicit differentiation to show that $\dfrac{dy}{dx} = \dfrac{x^2 - 3y}{3x - y^2}$. [4]
(c) Find the gradient of C at $(2, 4)$. [2]
(d) Find the equation of the tangent to C at $(2, 4)$. [3]

31. A stone dropped in a pond sends out a circular ripple whose radius grows at $0.5\,\text{m}\,\text{s}^{-1}$. Find the rate at which the enclosed area is increasing when the radius is 8 m. [3]

32. A street lamp is 5 m tall. A person of height 1.8 m walks away from it at $1.2\,\text{m}\,\text{s}^{-1}$. Find the rate at which the length of the person's shadow is increasing. [6]

33. A water tank is an inverted cone in which the water surface radius is always $\frac{3}{4}$ of the depth h. Water is pumped in at $1.5\,\text{m}^3\,\text{min}^{-1}$. Find the rate at which the depth is rising when $h = 2$ m. [6]

34. A 5 m ladder slides down a wall, its foot moving away at a constant $0.5\,\text{m s}^{-1}$.

(a) Find the rate at which the top descends when the foot is 3 m from the wall. [3]

(b) Find the rate at which the area of the triangle formed by the ladder, wall, and ground is changing at that instant. [4]

35. A lighthouse stands 2 km from a straight shoreline, and its beam rotates at one revolution per minute. Let θ be the angle between the beam and the perpendicular from the lighthouse to the shore, and let x be the distance from the foot of that perpendicular to the spot of light.

(a) Write x in terms of θ, and state the value of $\dfrac{d\theta}{dt}$ in radians per minute. [2]

(b) Show that $\dfrac{dx}{dt} = 4\pi \sec^2\theta$. [3]

(c) Find the speed of the spot of light along the shore when $\theta = 30^\circ$. [3]

(d) Describe what happens to this speed as $\theta \to 90^\circ$, and explain why. [2]

36. A particle moves along the curve $y = x^2$ so that $\dfrac{dx}{dt} = 2$ at all times. Find the rate at which its distance from the origin is increasing as it passes through the point $(1, 1)$. [7]

37. A cone is inscribed in a sphere of radius R. Show that the cone of maximum volume has height $h = \frac{4R}{3}$. [7]

38. Let $f(x) = x^4 + 4x^3 + 6x^2 + 4x$.

(a) Show that $f'(x) = 4(x+1)^3$, and find the coordinates of the stationary point of f. [3]

(b) Show that the second derivative test gives no conclusion at this point, and determine its nature. [3]

(c) Determine whether the graph of f has a point of inflexion. Justify your answer. [2]

39. A particle moves in a straight line with velocity $v(t) = t^3 - 9t^2 + 24t - 20$ m s^{-1}, for $0 \le t \le 6$ seconds.

(a) Show that $v(t) = (t-2)^2(t-5)$, and find the times at which the particle is at rest. [3]

(b) State, with a reason, whether the particle changes direction at each of these times. [2]

(c) Find the greatest speed of the particle for $0 \le t \le 6$, and the time at which it occurs. [4]

(d) Determine whether the particle is speeding up or slowing down at $t = 3$, giving a reason. [2]

40. Let $f(x) = |x^2 - 4|$, for $-1 \le x \le 3$.

(a) Write $f(x)$ without the modulus sign on each of the intervals $-1 \le x \le 2$ and $2 \le x \le 3$. [2]

(b) Find the greatest and least values of f on the interval, and the values of x at which they occur. [4]

(c) Explain why the least value cannot be found by solving $f'(x) = 0$. [1]

Challenge Questions

C1. GDC **Modelling the best can.** The notes minimise the surface area of a closed can of fixed volume. At that minimum the height of the can equals its diameter. Real drink cans are taller than that. This investigation rebuilds the model around cost instead of area, and asks how far the improved model explains the cans on a shop shelf. Throughout, a closed can of radius r and height h holds a fixed volume V.

(a) Suppose the metal for the two ends costs k times as much per unit area as the metal for the curved side, where $k > 0$. Show that the cost of the can is proportional to $C = 2k\pi r^2 + \dfrac{2V}{r}$. Show that C has a minimum when $h = 2kr$, and hence that the cheapest can has $\dfrac{h}{d} = k$, where d is the diameter. [5]

(b) Now let the metal cost the same everywhere, but suppose each end is cut from its own square of side $2r$, and the corners are scrap that must still be paid for. Show that the cheapest can has $\dfrac{h}{d} = \dfrac{4}{\pi}$. [4]

(c) If instead the ends are cut from one sheet with their centres on a triangular grid, each disc uses a regular hexagon whose inscribed circle is the disc. Show that such a hexagon has area $2\sqrt{3}\,r^2$, and that the cheapest can then has $\dfrac{h}{d} = \dfrac{2\sqrt{3}}{\pi}$. [3]

(d) A typical 330 ml drink can is about 11.5 cm tall and 6.6 cm in diameter. Find the value of k in the model of part (a) that would make this shape the cheapest, and compare it with the values implied by parts (b) and (c). [2]

(e) Take $k = 1$ and write $h = xd$ with $x > 0$. Show that the surface area is

$$S = (2\pi)^{1/3}\, V^{2/3}\, \frac{1 + 2x}{x^{2/3}}.$$

Find the percentage of extra metal used by the drink-can shape of part (d) compared with the shape $x = 1$. Use the derivative $\dfrac{dS}{dx}$ at $x = 1$ to explain why a shape some way from the optimum costs only a little more. [4]

(f) List other features of a real can that a better model would include, and describe how you would investigate one of them. [3]

C2. Filling a vessel. Water is poured at a constant rate Q into an empty vessel whose horizontal cross-section at height h above the lowest point has area $A(h)$. This investigation finds how the shape decides the way the level rises, then asks the reverse question.

(a) The water between heights h and $h+\delta h$ forms a thin slice. Explain why its volume is approximately $A(h)\,\delta h$, and hence why $\dfrac{dV}{dh} = A(h)$. Deduce that $\dfrac{dh}{dt} = \dfrac{Q}{A(h)}$. [3]

(b) For a cylinder of radius R, show that the level rises at a constant rate. For a sphere of radius R filled from its lowest point, show that $A(h) = \pi(2Rh - h^2)$ for $0 \le h \le 2R$, and find the height at which the level rises most slowly. [4]

(c) Show that $\dfrac{d^2h}{dt^2} = -\dfrac{Q^2 A'(h)}{A(h)^3}$. [3]

(d) Hence state and prove a rule linking the concavity of the graph of h against t to whether the vessel is widening or narrowing at the surface. Apply it to the sphere in part (b). [3]

(e) A bowl is formed by rotating the curve $y = cx^2$ $(c > 0)$ about the y-axis. Show that $A(h) = \dfrac{\pi h}{c}$, and verify that $h = \sqrt{\dfrac{2Qct}{\pi}}$ satisfies $\dfrac{dh}{dt} = \dfrac{Q}{A(h)}$ with $h = 0$ at $t = 0$. [4]

(f) A water clock empties through a small hole in its base. At depth h the water leaves at a rate proportional to $\sqrt{h}$, so $\dfrac{dh}{dt} = -\dfrac{\lambda\sqrt{h}}{A(h)}$ for a constant $\lambda > 0$. Find how $A(h)$ must depend on h for the level to fall at a constant rate, so that equal marks show equal times, and investigate the shape of such a vessel. [3]

C3. Closest approach. Finding the point of a curve nearest to a fixed point is an optimisation problem, and its answer has a geometric description. This investigation finds that description, states exactly when it holds, and uses it to count the normals that can be drawn to a parabola from a given point.

(a) Let $(x, \sqrt{x})$ be a point on $y = \sqrt{x}$ and let D be its distance from $(4, 0)$. Show that $D^2 = x^2 - 7x + 16$. [3]

(b) Explain why minimising D^2 gives the same answer as minimising D, and why using D^2 is easier. [2]

(c) Find the point on the curve closest to $(4, 0)$, and the least distance. [4]

(d) Show that at this closest point the line joining it to $(4, 0)$ is perpendicular to the tangent to the curve. [3]

(e) Let f be differentiable on an open interval, and let (a, b) be a point not on the curve $y = f(x)$. Suppose the distance from (a, b) to $(x, f(x))$ has a local minimum at a point x_0 of the interval. Show that

$$(x_0 - a) + (f(x_0) - b)f'(x_0) = 0,$$

and deduce that the line from (a, b) to $(x_0, f(x_0))$ is perpendicular to the tangent there. (This line is the *normal* to the curve at that point.) [4]

(f) Show that the conclusion of part (e) can fail when its conditions are dropped, using: (i) the curve $y = |x|$ and the point $(0, -1)$; (ii) the ray $y = x$, $x \ge 0$, and the point $(-2, 0)$. In each case find the closest point and state which condition fails. [3]

(g) Use the condition in part (e) to show that the number of normals to $y = x^2$ that pass through the point $(0, b)$ is three if $b > \frac{1}{2}$ and one if $b \leq \frac{1}{2}$. [3]

(h) The value $b = \frac{1}{2}$ in part (g) is the radius of curvature of $y = x^2$ at its vertex (Question C4(e)). Investigate this connection, and the curve that separates the points from which three normals to $y = x^2$ can be drawn from those from which only one can. [2]

C4. Quantifying curvature. A straight line does not bend; a small circle bends sharply. This investigation builds a number that measures how sharply a curve bends at a point. It is defined by

$$\kappa = \frac{|y''|}{\left(1 + (y')^2\right)^{3/2}},$$

and κ is called the *curvature*.

(a) Show that every straight line has $\kappa = 0$ at every point, and explain why the definition must give this. [3]

(b) For $y = x^2$, find κ as a function of x. Show that the curvature is greatest at the vertex, and find its value there. [4]

(c) Show that $\kappa \to 0$ as $x \to \pm\infty$ for $y = x^2$, and explain what this says about the shape of a parabola far from its vertex. [3]

(d) Use implicit differentiation on $x^2 + y^2 = R^2$ to show that a circle of radius R has $\kappa = \dfrac{1}{R}$ at every point. Explain why a constant curvature is exactly what you should expect. [5]

(e) The *radius of curvature* is defined as $\rho = \dfrac{1}{\kappa}$. Find ρ for $y = x^2$ at the origin, and state the circle that best matches the parabola there. [3]

(f) The second derivative alone will not do as a measure of bending. For $y = x^2$, $y'' = 2$ at every point, yet the curve clearly bends more sharply at the vertex than far out along an arm. State what the denominator $\left(1 + (y')^2\right)^{3/2}$ corrects for, and use your answer to part (b) to support it. [3]

(g) Show that $\kappa = 0$ at every point of inflexion of a twice-differentiable curve. Then explain why a curve with $\kappa = 0$ at one point need not be a straight line, and give an example. [3]

(h) The curve $y = f(x)$ is stretched vertically by a factor $k > 0$, giving $y = k\,f(x)$. At a point where $f'(x) = 0$, show that the curvature of the stretched curve is k times the curvature of the original. Show also that a translation changes κ nowhere. What does this say about judging how sharply a curve bends from a graph drawn with unequal scales on the two axes? [4]

(i) Engineers join a straight railway track to a circular arc of radius R by a *transition curve*, along which the curvature increases steadily from 0 to $\frac{1}{R}$. For the cubic $y = cx^3$ with $c > 0$, show that $\kappa \approx 6cx$ when x is small and positive, and find the value of c for which this approximation reaches $\frac{1}{R}$ at $x = L$. Then investigate the curve whose curvature is exactly proportional to the distance travelled along it (the *clothoid*, or Euler spiral), and how closely the cubic approximates it. [3]