CALCULUS

Chapter 13

Limits & Derivatives

13

SYLLABUS 5.1–5.4 5.6 5.7 5.12 5.13 5.15

A car's speedometer reads 80 km/h. If you ask what that number means, you meet a problem that took mathematicians many centuries to settle.

Speed is distance divided by time. But the speedometer shows your speed *right now*, at a single instant. In one instant no time passes, and no distance is covered. Speed at an instant therefore appears to be $0 \div 0$, which has no value. How can the needle point to 80?

The solution is to stop asking about the instant, and to ask instead about short intervals around it.

Measure your average speed over the next hour. Then over the next minute. Then over the next second, and then over the next hundredth of a second. Each interval is shorter than the one before, and each average lies closer to one particular number. That number is the speed at the instant. The averages approach it, even though no calculation made at the instant itself can reach it.

This idea is called the limit. When a value cannot be reached directly, you find it by asking what nearby values approach. The rest of this chapter, and most of calculus, is built on it.

The derivative follows from the limit. It measures the exact rate at which a quantity changes: how steep a curve is at one point, or how fast an apple is falling at the moment it lands.

By the end of this chapter you will evaluate limits, decide when a function is continuous and differentiable, and differentiate from first principles. You will apply the product, quotient and chain rules, differentiate the standard functions of the course, and find the equations of a tangent and a normal. You will evaluate limits of the form $\frac{0}{0}$ and $\frac{\infty}{\infty}$ with l'Hôpital's rule. You will also differentiate more than once, and meet the second derivative and the notation for every higher derivative.

The speedometer's 80 km/h looked like $0 \div 0$. By the end of the chapter you will see that it is a derivative.

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13.1 Limits

A **limit** is the value a function approaches as its input approaches some point. We write

$$\lim_{x \to a} f(x) = L$$

to mean: as x gets **arbitrarily close** (as close as you like) to a, $f(x)$ gets arbitrarily close to L. The value of f at a itself **does not affect the limit**, so a limit can exist at a point where the function is undefined.

For most functions at most points, the limit is found by direct substitution. If f is defined at a and its graph passes through that point unbroken, then the value it approaches is simply the value it takes:

$$\lim_{x \to 2} (x^2 + 3x) = 2^2 + 3(2) = 10.$$

The cases that need a method are the ones where substitution fails. When substitution gives $\frac{0}{0}$, factorise the numerator and the denominator and cancel the common factor. The cancelled function agrees with the original everywhere except at the hole, so its value there is the limit.

EXAMPLE 13.1

Find $\lim_{x \to 3} \dfrac{x^2 - 9}{x - 3}$.

SOLUTION

First try substituting $x = 3$. It gives $\frac{0}{0}$, which is undefined: the function has a hole at $x = 3$. But for every $x \neq 3$ you can factorise and cancel:

$$\frac{x^2 - 9}{x - 3} = \frac{(x - 3)(x + 3)}{x - 3} = x + 3.$$

Away from the hole the function is simply $x + 3$, so as $x \to 3$ it approaches 6:

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3} = \boxed{6}$$

At $x = 3$ the function has no value at all, yet 6 is the value it approaches. A limit describes the value the function approaches, not the value it takes.

The form $\frac{0}{0}$ is called **indeterminate** (so is $\frac{\infty}{\infty}$, §13.8). It does not mean the limit fails to exist. It means more work is needed.

One limit of this kind must be memorised, because the derivatives of the trigonometric functions depend on it:

$$\lim_{x \to 0} \frac{\sin x}{x} = 1.$$

When there is no algebraic method, you can *estimate* a limit. Calculate the function at values closer and closer to the target, from both sides. Here is the result for this one, with x in radians:

x	-0.1	-0.01	0.01	0.1
$\dfrac{\sin x}{x}$	0.99833	0.99998	0.99998	0.99833

Both sides approach 1. A table or a zoomed-in graph is a valid way to **conjecture** (guess, before proving) the value of a limit. On a calculator paper it is often the fastest way to check an algebraic answer.

EXAMPLE 13.2

EXT Prove that $\lim_{x\to 0}\dfrac{\sin x}{x}=1$.

SOLUTION

The values in the table approach 1, and a geometric argument proves it. Take a sector of a circle of radius 1 with angle x radians, where $0<x<\frac{\pi}{2}$. The triangle formed by the two radii lies inside the sector, and its area is $\frac{1}{2}\sin x$. The sector itself has area $\frac{1}{2}x$ §7.5. Now draw the tangent at the end of one radius and extend the other radius until it meets that tangent. This gives a right-angled triangle with shorter sides 1 and $\tan x$. It contains the sector, and its area is $\frac{1}{2}\tan x$.

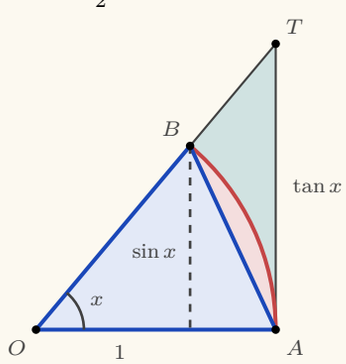


Figure 13.1 A sector OAB of angle x in a circle of radius 1. Triangle OAB lies inside the sector and the sector lies inside triangle OAT.

So

$$\tfrac{1}{2}\sin x<\tfrac{1}{2}x<\tfrac{1}{2}\tan x.$$

Divide each part by $\frac{1}{2}\sin x$, which is positive, and then take reciprocals, which reverses the inequalities:

$$1<\frac{x}{\sin x}<\frac{1}{\cos x}\qquad\Longrightarrow\qquad\cos x<\frac{\sin x}{x}<1.$$

As $x \to 0$, $\cos x$ approaches 1. The quotient $\dfrac{\sin x}{x}$ lies between $\cos x$ and 1, so it approaches 1 as well. Replacing x by $-x$ changes neither $\dfrac{\sin x}{x}$ nor $\cos x$, so the same is true as x approaches 0 from the left.

The phrase "from both sides" needs its own notation. Write $x \to a^-$ when x approaches a through values *less* than a (from the left), and $x \to a^+$ when it approaches through values *greater* than a (from the right). The value that $f(x)$ approaches from each side is a **one-sided limit**:

$$\lim_{x \to a^-} f(x) \quad \text{(left-hand limit)}, \qquad \lim_{x \to a^+} f(x) \quad \text{(right-hand limit)}.$$

DEFINITION 13.1 **When a limit exists**

The limit $\lim_{x \to a} f(x)$ exists only when the left-hand and right-hand limits both exist and are **equal**. Their common value is the limit.

The two graphs below show both cases. On the left, the graph has a jump at $x = 2$. Moving along the curve from the left, the height approaches 1; moving from the right, it approaches 3. The one-sided limits differ, so $\lim_{x \to 2} f(x)$ does not exist. On the right, the graph has only a hole at $x = 2$. Both sides approach the same height, 3, so the limit exists and equals 3, even though the function has no value at $x = 2$.

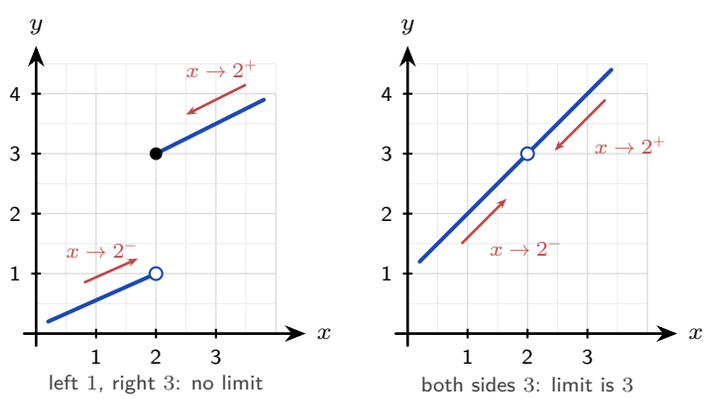


Figure 13.2 One-sided limits; an open circle is not on the graph.

EXAMPLE 13.3

Let

$$f(x) = \begin{cases} 3x - 1, & x < 2, \\ x^2 + 1, & x \geq 2. \end{cases}$$

Find $\lim_{x \to 2} f(x)$.

SOLUTION

One-sided limits can be found from the formula, with no graph. For $x < 2$ the function is $3x - 1$, so the left-hand limit is $\lim_{x \to 2^-} f(x) = 3(2) - 1 = 5$. For $x > 2$ it is $x^2 + 1$, so the right-hand limit is $\lim_{x \to 2^+} f(x) = 2^2 + 1 = 5$. The two are equal, so $\boxed{\lim_{x \to 2} f(x) = 5}$

Change the second piece to $x^2 - 1$ and the right-hand limit becomes 3. The two sides then differ, and the limit does not exist.

A modulus is handled the same way, because it is a function with two pieces. Take $g(x) = \dfrac{|2x - 6|}{x - 3}$. For $x < 3$, $|2x - 6| = 6 - 2x$, so $g(x) = \dfrac{-2(x - 3)}{x - 3} = -2$. For $x > 3$, $|2x - 6| = 2x - 6$, so $g(x) = 2$. The left-hand limit at 3 is -2 and the right-hand limit is 2, so $\lim_{x \to 3} g(x)$ does not exist.

13.1.1 Reading a limit off the graph

A graph often shows why a limit takes the value it does. The two fractions $\dfrac{\sin x}{x}$ and $\dfrac{\cos x - 1}{x}$ both give $\frac{0}{0}$ at $x = 0$, yet they approach different values. The reason is how fast each numerator approaches 0 compared with the denominator x.

Sketch $y = \sin x$ and $y = x$ on the same axes. Near the origin the two graphs are almost the same: at $x = 0.1$, $\sin x = 0.0998$. The numerator and the denominator are almost equal, so their ratio approaches 1.

Now sketch $y = \cos x - 1$. Near the origin this graph is almost flat, because $\cos x$ has a maximum at $x = 0$. At $x = 0.1$, $\cos x - 1 = -0.0050$, while $x = 0.1$. The numerator is very small compared with the denominator, so their ratio approaches 0.

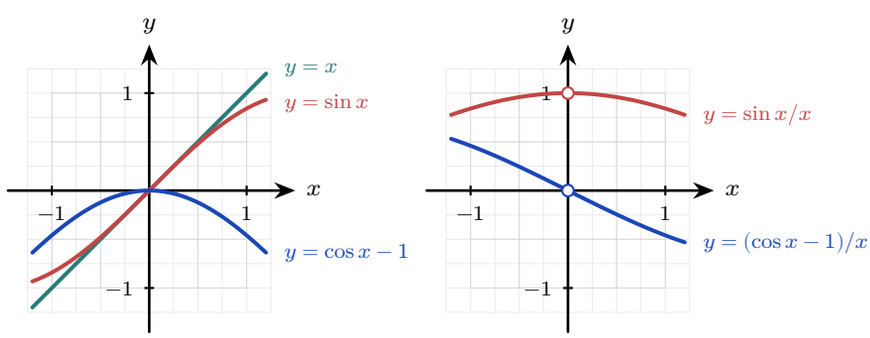


Figure 13.3 **Left:** near the origin $\sin x$ is almost x, while $\cos x - 1$ is almost flat. **Right:** each quotient has a hole at $x = 0$; $\sin x/x$ approaches 1 and $(\cos x - 1)/x$ approaches 0.

Both limits are needed later: they are the two facts from which the derivatives of $\sin x$ and $\cos x$ are built.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1, \qquad \lim_{x \to 0} \frac{\cos x - 1}{x} = 0.$$

The second limit follows from the first. Multiply the top and the bottom by $\cos x + 1$, and use $\cos^2 x - 1 = -\sin^2 x$:

$$\frac{\cos x - 1}{x} = \frac{\cos^2 x - 1}{x(\cos x + 1)} = -\frac{\sin x}{x} \cdot \frac{\sin x}{\cos x + 1}.$$

As $x \to 0$ the first fraction approaches 1 and the second approaches $\frac{0}{2} = 0$, so the product approaches 0.

Not every limit exists. HL We say that $f(x)$ converges as $x \to a$ when the limit exists, and diverges when it does not (as for series, §2.6). Three examples show ways a limit can fail to exist:

- $\frac{1}{x^2}$ increases without bound as $x \to 0$;
- $\frac{1}{x}$ increases without bound as $x \to 0^+$ but decreases without bound as $x \to 0^-$;
- $\sin x$ **oscillates** (moves up and down) between -1 and 1 as $x \to \infty$, and does not approach any single value.

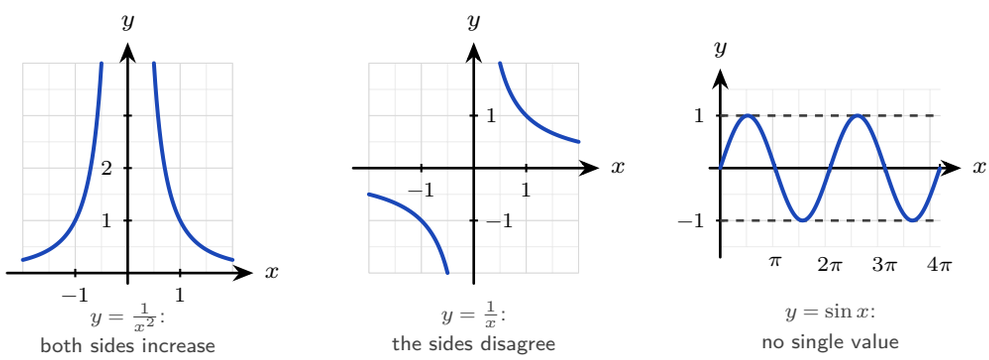


Figure 13.4 Three limits that do not exist. Left: $\frac{1}{x^2}$ as $x \to 0$. Centre: $\frac{1}{x}$ as $x \to 0$. Right: $\sin x$ as $x \to \infty$, which keeps moving between the dashed lines $y = \pm 1$.

For $\frac{1}{x}$, writing that the limit "equals infinity" hides the disagreement between the two sides, so it is not an acceptable answer.

Deciding whether a limit as $x \to \infty$ converges or diverges needs a method, because there is no value of x to substitute. For a ratio of polynomials this method works: divide every term, above and below, by the highest power of x that appears. What is left is a set of constants together with terms like $\dfrac{1}{x}$ and $\dfrac{1}{x^2}$, and each of those approaches 0.

The result depends on the degrees of the top and the bottom.

- If the bottom has the higher degree, the limit is 0.
- If the degrees are equal, the limit is the ratio of the leading coefficients.
- If the top has the higher degree, the fraction increases or decreases without bound, and the limit does not exist.

EXAMPLE 13.4

Evaluate (a) $\lim_{x\to\infty} \dfrac{5x-2}{x}$, (b) $\lim_{x\to\infty} \dfrac{4x^2-x}{3x^2+5}$.

SOLUTION

Divide above and below by the highest power of x in each.

(a) The highest power is x, so divide both terms of the numerator by it:

$$\lim_{x\to\infty} \frac{5x-2}{x} = \lim_{x\to\infty} \left(5 - \frac{2}{x}\right) = 5 - 0 = \boxed{5}$$

(b) The highest power is x^2, so divide all four terms by x^2:

$$\begin{aligned} \lim_{x\to\infty} \frac{4x^2-x}{3x^2+5} &= \lim_{x\to\infty} \frac{4 - \dfrac{1}{x}}{3 + \dfrac{5}{x^2}} \\ &= \frac{4-0}{3+0} \\ &= \boxed{\frac{4}{3}} \end{aligned}$$

Every term that approaches 0 came from a lower power of x than the one you divided by. Both parts have equal degrees, so each answer is the ratio of the leading coefficients.

YOUR TURN 13.1 Limits

Direct substitution and factorising

1. Find each limit by direct substitution.

(a) $\lim_{x\to 2}(3x-1)$

(b) $\lim_{x\to 0}(x^2+2x+5)$

(c) $\lim_{x\to 3}(x^2-9)$

(d) $\lim_{x\to 1} \dfrac{x+1}{x+3}$

2. Each of these gives $\frac{0}{0}$. Factorise, cancel, then take the limit.

(a) $\lim_{x\to 4} \dfrac{x^2-16}{x-4}$

(b) $\lim_{x\to 1} \dfrac{x^2-1}{x-1}$

(c) $\lim_{x\to 2} \dfrac{x^2-x-2}{x-2}$

(d) $\lim_{x\to -3} \dfrac{x^2+x-6}{x+3}$

One-sided limits

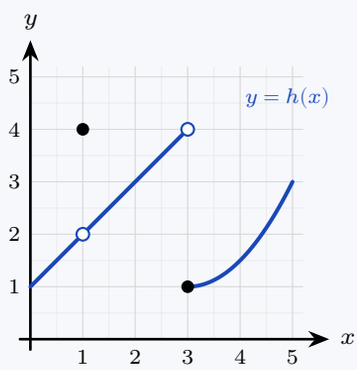
3. Explain why $\lim_{x\to 0} \dfrac{|x|}{x}$ does not exist.

4. For each function, find the left-hand and right-hand limits at the value of x given, and state whether the limit exists there.

(a) $f(x) = \begin{cases} x^2, & x < 1 \\ 2x + 1, & x \geq 1 \end{cases}$ at $x = 1$

(b) $g(x) = \begin{cases} x + 2, & x < 3 \\ 8 - x, & x > 3 \end{cases}$ with $g(3) = 1$, at $x = 3$. Compare the limit with $g(3)$.

(c) The graph of $y = h(x)$ is shown for $0 \leq x \leq 5$. An open circle marks a point that is not on the graph.



Write down $\lim_{x \to 1} h(x)$ and $h(1)$. Then write down the left-hand and right-hand limits at $x = 3$, and state whether $\lim_{x \to 3} h(x)$ exists.

5. Find $\lim_{x \to 1^-} \dfrac{x^2 - 1}{|x - 1|}$ and $\lim_{x \to 1^+} \dfrac{x^2 - 1}{|x - 1|}$, and hence state whether $\lim_{x \to 1} \dfrac{x^2 - 1}{|x - 1|}$ exists. *(Hint: $|x - 1| = 1 - x$ when $x < 1$)*

Limits at infinity

6. State whether each of the following converges or diverges as $x \to \infty$, and give the limit where one exists.

(a) $\dfrac{1}{x}$

(b) x^2

(c) e^{-x}

(d) $2 + \dfrac{5}{x}$

7. Evaluate each limit by dividing every term by the highest power of x.

(a) $\lim_{x \to \infty} \dfrac{6x - 1}{2x + 3}$

(b) $\lim_{x \to \infty} \dfrac{2x^2 + 3x}{5x^2 - 1}$

(c) $\lim_{x \to \infty} \dfrac{x + 4}{x^2 + 1}$

(d) $\lim_{x \to \infty} \dfrac{x^2 + 1}{x - 3}$

Estimating a limit

8. GDC Consider $y = \dfrac{\sin x}{x}$.

(a) Write down the value of $\lim_{x\to 0}\dfrac{\sin x}{x}$.

(b) Graph the curve for $-10 \le x \le 10$ on your GDC, and sketch what you see.

(c) Describe what the graph does at $x = 0$, and what it does as $x \to \pm\infty$.

9. GDC Work in radians.

(a) Evaluate $\dfrac{\cos x - 1}{x}$ for $x = \pm 0.1$ and $x = \pm 0.01$. State the value that its limit as $x \to 0$ appears to take.

(b) Estimate $\lim_{x\to 0}\dfrac{1-\cos x}{x^2}$ by evaluating the expression at $x = 0.1$ and $x = 0.01$. State the exact value you think the limit takes.

Concept check

10. A student is asked for $\lim_{x\to 3}\dfrac{x^2-9}{x-3}$ and writes: "*substituting $x = 3$ gives $\frac{0}{0}$, so the limit does not exist.*" Explain why this conclusion is wrong, and find the limit.

13.2 The Derivative

Calculus is the mathematics of change. You already know how to measure change: it is the gradient. Rise over run gives the change in y for each unit of change in x, and if the gradient doubles, the quantity changes twice as fast. But a gradient is defined for a *straight line* only. One number describes the whole line, because a line changes at the same rate everywhere.

Many real quantities do not behave like that. A cooling cup, a thrown ball, a growing population: in each case the rate of change is itself changing. So the question "what is the gradient of this curve?" has no single answer, because the answer is different at every point. Ask a smaller question instead: what is the gradient of the curve *at one point*? Once you can answer that for one point, you can answer it for every point, and the rate of change becomes a function of its own.

Here is the method. On the graph of $y = f(x)$, choose your point x and a second point $x + h$ close to it. Join the two points with a straight line, called the **secant**, and find its gradient. You can already calculate that:

$$\frac{f(x+h) - f(x)}{h},$$

the rise over the run. This expression is the **difference quotient**. It is the average rate of change over the interval of width h.

The secant gradient is still not the answer, because the question was about a single point.

Now make h smaller and smaller. The second point moves towards the first, and the secant turns towards one limiting line, which touches the curve at the chosen point. That line is the **tangent**. Its gradient is the rate of change at a single point, and it is called the **derivative**.

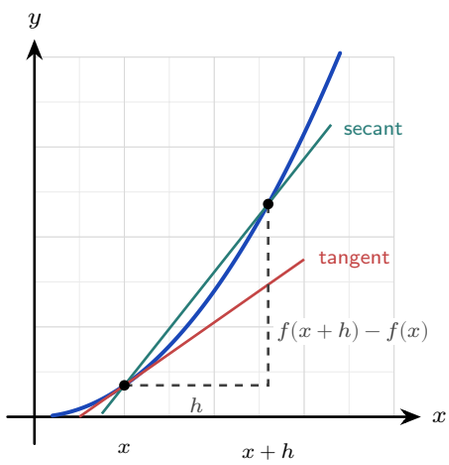


Figure 13.5 The secant joins two points on the curve; its gradient, rise over run, is an average rate of change. As $h \to 0$ the second point moves to the first and the secant approaches the tangent.

DEFINITION 13.2 **The derivative in words**

In words: the **derivative** of f at a point is the **gradient of the tangent** to the curve $y = f(x)$ at that point. It is the limit of the gradients of the secants through that point, as the second point moves towards it.

The **derivative** of f at x is written $f'(x)$, or $\dfrac{dy}{dx}$ when $y = f(x)$:

FORMULA 13.1 **Derivative from first principles** FORMULA BOOKLET · 5.12 HL

$$y = f(x) \implies \frac{dy}{dx} = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative exists when this limit exists. To **differentiate** f is to find its derivative. Computing a derivative directly from this definition is called differentiation **from first principles**.

The derivative is itself a function, called the **gradient function**. For each value of x, the number $f'(x)$ is the gradient of the curve $y = f(x)$ at the point with that x-coordinate. For example, the derivative of $f(x) = x^2$ is $f'(x) = 2x$ (§13.3). So the gradient of $y = x^2$ is 2 at $x = 1$, 0 at $x = 0$ and -4 at $x = -2$. One formula, $2x$, gives the gradient at every point of the curve.

13.2.1 Notation, and where it came from

Calculus was developed independently by Isaac Newton and Gottfried Leibniz, and each was trying to solve a different problem.

- **Isaac Newton**, in England in the 1660s, was studying *motion*: falling bodies, and later the orbits of the planets under gravity. He needed the speed of a moving object at a single instant, which is the problem of the speedometer at the start of this chapter.
- **Gottfried Leibniz**, in Germany and France in the 1670s, was studying the *geometry of curves*: how to find the tangent to any curve, and the area under it. He wanted one set of symbols and rules that would work for every curve.

Both built on earlier methods for tangents by Pierre de Fermat and Isaac Barrow. Newton's own notation, a dot above the variable, is still seen in physics, but it is not used in this course. The notation used here comes from Leibniz.

The IB course uses these forms for the first derivative:

$$\frac{dy}{dx}, \qquad f'(x), \qquad \frac{dV}{dr}, \qquad \frac{ds}{dt}.$$

Leibniz wrote $\frac{dy}{dx}$. He built it from $\frac{\Delta y}{\Delta x}$, the ratio of two finite changes, and the letter d marks the limit as those changes approach zero. The form $f'(x)$ names the function being differentiated. A **prime** (a small dash) on a single letter means the same: $y' = \frac{dy}{dx}$, $u' = \frac{du}{dx}$. The form $\frac{d}{dx}f(x)$, used in the formula cards of this chapter, means "the derivative of $f(x)$ with respect to x".

The Leibniz form is the one that carries the most information, because it names both variables. When the quantities are not called x and y, the letters change with them. A volume V changing with radius r has derivative $\frac{dV}{dr}$, and a displacement s changing with time t has derivative $\frac{ds}{dt}$. Read each as a rate: how fast the top quantity changes per unit of the bottom one. Examination questions use these forms often, and choosing the right letters is an important part of setting up the problem.

The symbol $\dfrac{dy}{dx}$ is written as a fraction, but it is not one. It is a single symbol standing for the limit of the fraction $\dfrac{\Delta y}{\Delta x}$, and dy and dx have no independent meaning at this level. The notation is still well designed, because the limit keeps some of the behaviour of a fraction. That is why the chain rule in §13.6 can be written as $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$, with the du appearing to cancel. The cancelling is a memory aid, not a proof, but it works because the underlying limits behave that way.

Leibniz's notation became the standard one, largely because it shows the variable of differentiation. So it extends directly to the chain rule, to implicit differentiation, and to integration, where $\int f(x)\,dx$ carries the same dx.

13.2.2 Where derivatives are used

A derivative is a rate of change, and rates of change appear wherever one quantity depends on another.

- **Motion.** Differentiate displacement with respect to time to get velocity, and again to get acceleration. This is the subject of §14.4.

- **Optimisation.** A curve is flat where its derivative is zero, so setting $f'(x) = 0$ locates maxima and minima. Manufacturers use this to find the packaging that uses the least material for a required volume.
- **Medicine.** The rate at which a drug leaves the bloodstream determines the dosing interval, and that rate is a derivative of the concentration with respect to time.
- **Economics.** *Marginal* cost is the derivative of total cost with respect to quantity produced, and it answers what one more unit costs to make.
- **Computer graphics.** A surface is shaded according to the direction it faces, which is found from derivatives of the surface, and this shading helps computer images look solid.

13.2.3 Differentiating from first principles

To differentiate from first principles, substitute into $\dfrac{f(x+h)-f(x)}{h}$, expand, and simplify until the factor h in the denominator cancels. **Only then let** $h \to 0$, because the division by h is valid only while $h \neq 0$.

EXAMPLE 13.5

Differentiate $f(x) = x^2 + 4x$ from first principles.

SOLUTION

Form the difference quotient and expand:

$$\begin{aligned}\frac{f(x+h)-f(x)}{h} &= \frac{(x+h)^2 + 4(x+h) - x^2 - 4x}{h} \\ &= \frac{2xh + h^2 + 4h}{h} \\ &= 2x + h + 4.\end{aligned}$$

Now let $h \to 0$:

$$f'(x) = \lim_{h\to 0}(2x + h + 4) = \boxed{2x + 4}$$

EXAMPLE 13.6

EXT Differentiate $f(x) = \dfrac{1}{x}$ from first principles.

SOLUTION

Form the difference quotient:

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right) = \frac{1}{h}\cdot\frac{x-(x+h)}{x(x+h)} = \frac{-h}{h\,x(x+h)} = \frac{-1}{x(x+h)}.$$

As $h \to 0$:

$$f'(x) = \lim_{h\to 0}\frac{-1}{x(x+h)} = \boxed{-\frac{1}{x^2}}$$

TIP In examinations, first-principles differentiation questions usually involve polynomials. The $\frac{1}{x}$ example above shows how far the definition extends; it is not itself an examination task.

13.2.4 Continuity and differentiability HL

The derivative is defined as a limit, and a limit can fail to exist. So a function need not have a derivative at every point. Two ideas, continuity and differentiability, describe where it does.

A function f is **continuous** at $x = a$ when three things are true:

1. $f(a)$ is defined;
2. $\lim_{x \to a} f(x)$ exists, so the left-hand and right-hand limits are equal (§13.1);
3. this limit equals $f(a)$.

On a graph, a continuous function has no hole and no jump at that point: you can draw through it without lifting the pen.

A function f is **differentiable** at $x = a$ when the first-principles limit $f'(a)$ exists. On a graph, the curve is continuous there and also *smooth*: it has one non-vertical tangent at that point, with no corner.

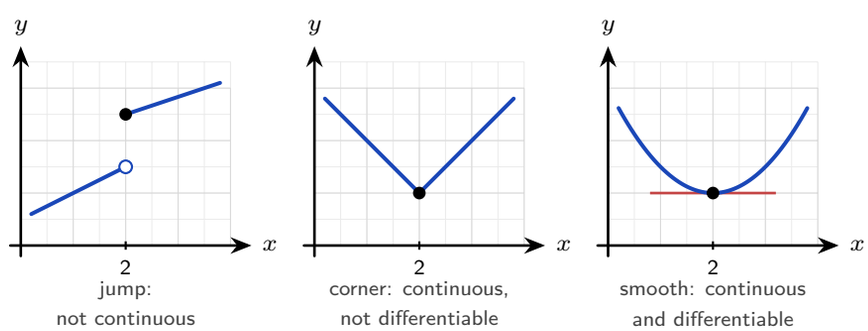


Figure 13.6 Three behaviours at $x = 2$. An open circle marks a point that is not on the graph. Only the smooth curve on the right has a single tangent (red) at the point.

- The two ideas are **linked in one direction only**: a differentiable function is continuous, but a continuous function need not be differentiable.
  - If f is differentiable at a point, then it is continuous there. The reason is that $f(a+h) - f(a) = \dfrac{f(a+h) - f(a)}{h} \times h$. As $h \to 0$ the fraction approaches $f'(a)$ and the factor h approaches 0, so the product approaches 0 and $f(a+h)$ approaches $f(a)$. A jump or a hole always prevents a derivative.
  - If f is continuous at a point, it may still fail to be differentiable there. A corner is the usual reason.

The modulus function shows the second case.

EXAMPLE 13.7

Show that $f(x) = |x - 1|$ is continuous at $x = 1$ but not differentiable there.

SOLUTION

Check the two properties in turn. For continuity: $f(1) = 0$, and as $x \to 1$ from either side, $|x - 1| \to 0$. The limit exists and equals $f(1)$, so f is continuous at $x = 1$.
For differentiability, form the difference quotient at 1:

$$\frac{f(1+h) - f(1)}{h} = \frac{|h|}{h} = \begin{cases} 1 & h > 0, \\ -1 & h < 0. \end{cases}$$

The quotient approaches 1 from the right and -1 from the left. The one-sided limits are not equal, so the limit does not exist and f has no derivative at $x = 1$. On the graph, the V has a corner there: the gradient is -1 just to the left and 1 just to the right, so there is no single tangent.

TIP Before you differentiate at a point, check that the curve has no jump, hole or corner there. If it has one, the derivative does not exist at that point.

YOUR TURN 13.2 The Derivative

Notation

11.

(a) Write down three notations for the derivative of $y = f(x)$.

(b) A volume V depends on a radius r. Write down the notation for the rate of change of V with respect to r, and state in words what it measures.

First principles

12. Differentiate each from first principles, showing the limit.

(a) $f(x) = 3x^2$ (b) $f(x) = x^2 - 3x$

(c) $f(x) = 5 - 2x^2$

13. Differentiate $f(x) = x^3$ from first principles. *(Hint: $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$)*

14. EXT Differentiate $f(x) = \dfrac{2}{x}$ from first principles.

Estimating a gradient numerically

15. GDC The definition of the derivative can also be used numerically. For $f(x) = \sin x$, evaluate $\dfrac{f(0+h)-f(0)}{h}$ for $h = 0.1$, $h = 0.01$ and $h = 0.001$, working in radians. State the value the quotient is approaching, and what this suggests about the gradient of $y = \sin x$ at $x = 0$.

16. GDC Repeat the same calculation for $f(x) = e^x$ at $x = 0$, then at $x = 1$. Explain what the two answers suggest about the derivative of e^x.

Continuity and differentiability

17. For each function, state which of the three conditions for continuity at $x = 2$ fails, if any.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

(b) $f(x) = \begin{cases} x + 1, & x < 2 \\ x^2, & x \geq 2 \end{cases}$

(c) $f(x) = \begin{cases} x^2, & x \neq 2 \\ 1, & x = 2 \end{cases}$

18. Let $g(x) = \begin{cases} x + 1, & x < 1 \\ 3 - x, & x \geq 1. \end{cases}$

(a) Show that g is continuous at $x = 1$.

(b) Use the difference quotient at $x = 1$ to show that g is not differentiable at $x = 1$.

(c) A function h has a jump at $x = 4$. Explain why h cannot be differentiable at $x = 4$.

Concept check

19. A student differentiates $f(x) = x^2$ from first principles and writes $\dfrac{(x+h)^2 - x^2}{h} = \dfrac{x^2 + h^2 - x^2}{h} = h$, which tends to 0 as $h \to 0$, so $f'(x) = 0$. Identify the error and correct the working.

13.3 Differentiating Powers

First principles works, but it is slow. The two first-principles calculations above show a pattern: the x^2 term gave $2x$, and $\frac{1}{x} = x^{-1}$ gave $-x^{-2}$.

In both cases the power decreased by one, and the original power became a multiplier in front. This is the **power rule**.

FORMULA 13.2 **Power rule** FORMULA BOOKLET · 5.3

$$\frac{d}{dx}\,x^n = nx^{n-1}$$

Valid for every rational n.

The rule holds for rational powers too. This is what allows $\sqrt{x}$ to be differentiated below, and it is needed throughout §13.5 onwards.

EXAMPLE 13.8

EXT Two examples are not a proof. Prove from first principles that $\frac{d}{dx}\,x^n = nx^{n-1}$ for every positive integer n.

SOLUTION

The binomial theorem §2.9 gives

$$(x+h)^n = x^n + nx^{n-1}h + (\text{terms containing } h^2 \text{ or a higher power of } h).$$

Subtract x^n and divide by h:

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + (\text{terms containing } h).$$

As $h \to 0$ every term after the first approaches 0, which leaves nx^{n-1}. This argument needs n to be a positive integer. The rule for every real power, for $x > 0$, is proved at the end of §13.6, from $x^n = e^{n\ln x}$ and the chain rule.

The rule applies only to a term written as x^n, so rewrite every root and every reciprocal as a power first: $\sqrt{x} = x^{1/2}$ and $\frac{1}{x^2} = x^{-2}$.

Two more rules let you differentiate any polynomial. A constant multiplier is unchanged, and a sum is differentiated one term at a time:

$$\frac{d}{dx}(c\,f(x)) = c\,f'(x), \qquad \frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x).$$

Both follow from the definition in §13.2. In the difference quotient of $c\,f(x)$ the constant c is a common factor of the numerator. The difference quotient of $f(x)+g(x)$ is the sum of the difference quotients of f and g. Letting $h \to 0$ in each case gives the rule. A constant on its own has derivative zero, because its graph is a horizontal line with gradient zero.

EXAMPLE 13.9

Differentiate $y = \sqrt{x} + \dfrac{1}{x^2}$.

SOLUTION

Rewrite both terms as powers before differentiating:

$$y = x^{1/2} + x^{-2} \implies \frac{dy}{dx} = \tfrac{1}{2}x^{-1/2} - 2x^{-3} = \boxed{\frac{1}{2\sqrt{x}} - \frac{2}{x^3}}$$

EXAMPLE 13.10

Find the gradient of $y = x^3 - 5x$ at the point where $x = 2$.

SOLUTION

Differentiate, then substitute:

$$\frac{dy}{dx} = 3x^2 - 5, \qquad \text{at } x = 2, \quad \frac{dy}{dx} = 3(4) - 5 = \boxed{7}$$

The derivative is a function. A gradient *at a point* is that function evaluated at the point.

EXAMPLE 13.11

Find the points on the curve $y = x^3 - 5x$ where the gradient is 7.

SOLUTION

This is the reverse question: the gradient is given and the point is asked for. Set the derivative equal to 7 and solve for x:

$$3x^2 - 5 = 7 \implies x^2 = 4 \implies x = 2 \text{ or } x = -2.$$

Substitute each value into y, not into $\dfrac{dy}{dx}$, to find the points: $y(2) = 8 - 10 = -2$ and $y(-2) = -8 + 10 = 2$. The curve has gradient 7 at $\boxed{(2, -2) \text{ and } (-2, 2)}$

Setting the derivative equal to 0 in the same way finds the points where the tangent is horizontal.

YOUR TURN 13.3 Differentiating Powers

Polynomials

20. Differentiate each of the following.

(a) $y = x^7$

(b) $y = 5x^3 - 2x$

(c) $y = x^4 - 3x^2 + 6$

(d) $y = 4x^5 + x$

(e) $y = 2x^3 - 7x^2 + x - 9$ (f) $y = 6$

Rewriting as a power

21. Rewrite each function as a power of x, then differentiate. In part (f), divide through first.

(a) $y = 2/x$ (b) $y = \sqrt{x}$
(c) $y = x^3 + 1/x$ (d) $y = 1/x^3$
(e) $y = \sqrt[3]{x}$ (f) $y = (x^2 + 1)/x$

22. Expand or divide to write each function as a sum of powers of x, then differentiate.

(a) $y = (2x + 1)^2$
(b) $y = \dfrac{x^3 - 4}{x^2}$
(c) $y = \dfrac{x + 1}{\sqrt{x}}$

Gradient at a point

23. Find the gradient of each curve at the point given.

(a) $y = x^2 - 5x$ at $x = 3$ (b) $y = x^3 - 4x$ at $x = -1$
(c) $y = \sqrt{x}$ at $x = 4$ (d) $y = 2x^3 + x$ at $x = 1$

24. Find the value or values of x at which each curve has the stated gradient.

(a) $y = x^2 - 6x$, gradient 0
(b) $y = x^3$, gradient 12
(c) $y = x^3 - 3x^2 + 1$, gradient 9

Concept check

25. A student differentiates $y = \dfrac{3}{x^2}$ and writes $y = 3x^{-2}$, so $\dfrac{dy}{dx} = -6x^{-1}$. Identify the error and give the correct derivative.

13.4 Tangents and Normals

The derivative gives the gradient of the tangent at any point, and a gradient plus a point determines a line. So you can now write the equation of the tangent to a curve at a given point. The **normal** is the line perpendicular to the tangent there, with gradient the negative reciprocal, because the gradients of perpendicular lines multiply to -1 §4.2.

The tangent and the normal are both straight lines, so both are written with the same equation, the point-gradient form from Ch. 4:

$$y - y_1 = m(x - x_1),$$

where (x_1, y_1) is a known point on the line and m is its gradient. The point (x_1, y_1) is the same for both lines, because both lines pass through the point of contact (x_1, y_1). Only the gradient m is different.

KEY 13.3 **Tangent and normal**

At the point (x_1, y_1) on the curve $y = f(x)$, both lines have the form $y - y_1 = m(x - x_1)$, with

$$m_{\text{tangent}} = f'(x_1), \qquad m_{\text{normal}} = -\frac{1}{f'(x_1)}$$

(for $f'(x_1) \neq 0$; if $f'(x_1) = 0$ the normal is the vertical line $x = x_1$).

So a question of this type follows three steps.

1. Substitute x_1 into f to find y_1.
2. Differentiate, then substitute x_1 to find the tangent gradient $f'(x_1)$. For the normal, use $-1/f'(x_1)$.
3. Substitute the point and the gradient into $y - y_1 = m(x - x_1)$.

On a calculator paper a graphic display calculator (GDC) can do part of this work. It finds the value of the derivative at a point, and if your calculator has a tangent feature, it also draws the tangent and gives its equation. Two cautions apply. A GDC that draws tangents may not draw normals, so find the normal gradient $-1/f'(x_1)$ by hand. A line copied from the screen shows no method, so write the point and the gradient you used, then the equation. Use the GDC to check your algebra, not to replace it.

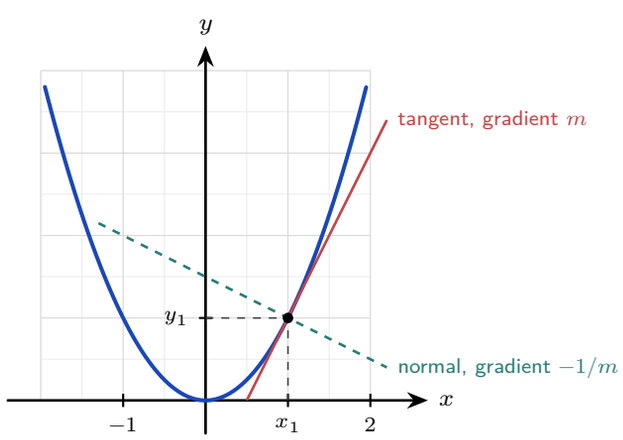


Figure 13.7 The tangent and the normal at the point (x_1, y_1). Their gradients m and $-1/m$ multiply to -1.

EXAMPLE 13.12

Find the equations of the tangent and the normal to $y = x^2 - 3x + 2$ at the point where $x = 2$.

SOLUTION

First find the point: $y(2) = 4 - 6 + 2 = 0$, so the point is $(2, 0)$. Then find the gradient: $\frac{dy}{dx} = 2x - 3$, so at $x = 2$ the tangent gradient is $2(2) - 3 = 1$. The tangent passes through $(2, 0)$ with gradient 1:

$$y - 0 = 1(x - 2) \implies \boxed{y = x - 2}$$

The normal is perpendicular, so its gradient is -1:

$$y - 0 = -1(x - 2) \implies \boxed{y = -x + 2}$$

Always find the y-coordinate first. A common error is to write the equation of the line using the x-value alone.

13.4.1 Tangents through a point off the curve

A harder question names a point the tangent must pass *through*, and that point is not on the curve. The point where the tangent touches the curve, the **point of contact**, is not given, so make it the unknown.

1. Let the point of contact be $(a, f(a))$.
2. Write the equation of the tangent there: $y - f(a) = f'(a)(x - a)$.
3. Substitute the coordinates of the given point, because the tangent passes through it. This gives an equation in a alone.
4. Solve for a, and substitute each value back into the tangent equation.

For a quadratic curve the equation in a is quadratic, so it usually has two roots, and each root gives a separate tangent. If the given point lies inside the curve, for example above a parabola that opens upwards, the equation has no real roots and no tangent passes through the point.

EXAMPLE 13.13

Find the equations of both tangents to $y = x^2 - 2x + 3$ that pass through the point $(2, -1)$.

SOLUTION

First check where the point sits: at $x = 2$ the curve is at $4 - 4 + 3 = 3$, not -1, so $(2, -1)$ is off the curve and is not the point of contact.

Let the contact point be $(a,\ a^2-2a+3)$. Since $\dfrac{dy}{dx}=2x-2$, the tangent there has gradient $2a-2$, so

$$y-(a^2-2a+3)=(2a-2)(x-a)\implies y=(2a-2)x-a^2+3.$$

Now substitute the point $(2,-1)$, because the tangent passes through it:

$$-1=2(2a-2)-a^2+3\implies a^2-4a=0\implies a(a-4)=0,$$

so $a=0$ or $a=4$. Substituting each value back into $y=(2a-2)x-a^2+3$:

$$a=0:\quad \boxed{y=-2x+3},\qquad\qquad a=4:\quad \boxed{y=6x-13}$$

Two roots give two tangents, and both pass the check at the given point: $-2(2)+3=-1$ and $6(2)-13=-1$.

YOUR TURN 13.4 Tangents and Normals

Tangents

26. Find the equation of the tangent to each curve at the point given.

(a) $y=x^2$ at $(2,4)$

(b) $y=x^2-4x+3$ at $x=1$

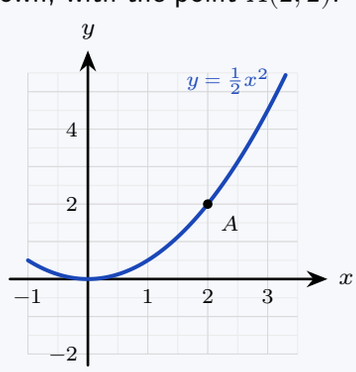
(c) $y=1/x$ at $x=1$

(d) $y=\sqrt{x}$ at $x=9$

27.

(a) Find the coordinates of the point on $y=x^2-6x$ at which the tangent is horizontal.

(b) The graph of $y=\frac{1}{2}x^2$ is shown, with the point $A(2,2)$.



Find the gradient of the curve at A. On the diagram, sketch the tangent at A, and write down the coordinates of the point where it meets the x-axis.

Normals

28. Find the equation of the normal to each curve at the point given.

(a) $y = x^3$ at $(1, 1)$

(b) $y = \sqrt{x}$ at $x = 4$

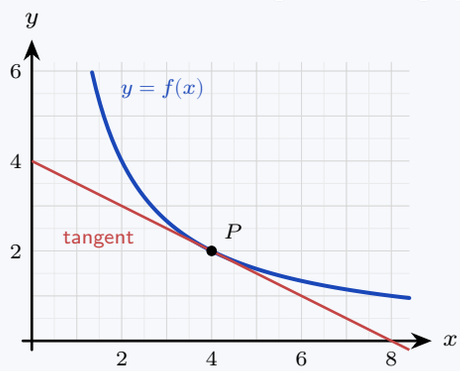
(c) $y = \dfrac{2}{x}$ at $x = 1$

29.

(a) A tangent and a normal are drawn at the same point of a curve. Explain the relationship between their gradients.

(b) Find the equation of the normal to $y = x^2 - 4x + 7$ at the point where $x = 2$.

(c) The graph of $y = f(x)$ is shown, with the tangent to the graph at the point $P(4, 2)$.



Write down $f'(4)$. Hence find the equation of the normal at P.

Tangents through a point off the curve

30. The tangent to $y = x^2$ at the point where $x = a$ passes through $(0, -4)$.

(a) Find the two possible values of a.

(b) Write down the two tangents, and check that each one passes through $(0, -4)$.

(c) Show that no tangent to $y = x^2$ passes through the point $(0, 1)$.

31. Find the equations of the two tangents to $y = x^2 + 3$ that pass through the point $(1, 0)$.

Concept check

32. A student finds the tangent to $y = x^2$ at the point $(3, 9)$ and writes $y - 9 = 2x(x - 3)$. Identify the error and find the correct equation of the tangent.

13.5 The Standard Derivatives

The power rule covers algebraic functions. The trigonometric, exponential and logarithmic functions have derivatives of their own. You can find each one by reading gradients from a

graph. Most are also proved: the trigonometric ones from first principles in this section, and the logarithm in §14.5. The derivative of e^x is the one this course does not prove. Read each graph once before you memorise the table.

The idea is to move along the graph of f from left to right, and at each point plot its gradient as a height. The curve you plot is f'. In practice, sketch it in this order.

1. **Start with the local maximum and minimum points of f.** There the tangent is horizontal, so $f'(x) = 0$. Mark these x-values on the axis: the graph of f' meets the x-axis there.
2. **Where f is increasing** (rising from left to right), its gradient is positive, so draw f' *above* the x-axis.
3. **Where f is decreasing** (falling from left to right), its gradient is negative, so draw f' *below* the x-axis.
4. **Where f is steepest**, f' is furthest from the axis.

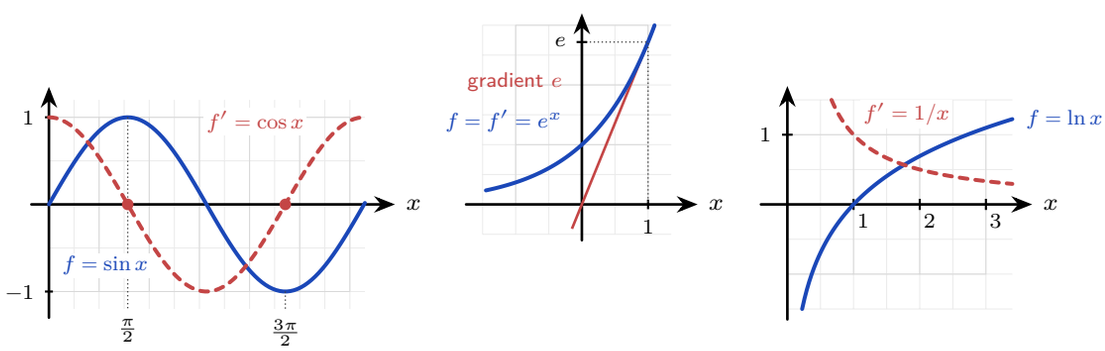


Figure 13.8 Derivatives read from gradients: $\sin x$ (left), e^x (centre), $\ln x$ (right).

Read each panel separately.

- **Sine.** At $x = 0$ the sine graph is increasing most steeply, so f' starts at its maximum. At the top of the curve, $x = \frac{\pi}{2}$, the tangent is horizontal, so f' is zero. After the top the graph decreases, so f' is negative. A curve that starts at 1, decreases through zero at $\frac{\pi}{2}$ and reaches -1 at π agrees with $\cos x$. A graph can only suggest this; the proof rests on the limit of §13.1.
- **The exponential.** Measure the gradient of $y = e^x$ at any point and you get the height of the curve at that point. So here $f' = f$. This is a property of the number e of §1.1, read here from the graph. This course does not prove it.
- **The logarithm.** Just to the right of 0 the graph of $\ln x$ is almost vertical, so f' is very large. As x increases the curve becomes flatter, so f' decreases towards zero. It stays positive everywhere. Those features agree with $\dfrac{1}{x}$.

Those, with the derivative of $\cos x$, are the four standard derivatives to learn.

FORMULA 13.4 **Standard derivatives** FORMULA BOOKLET · 5.6

$$\frac{d}{dx}\sin x = \cos x, \quad \frac{d}{dx}\cos x = -\sin x, \quad \frac{d}{dx}e^x = e^x, \quad \frac{d}{dx}\ln x = \frac{1}{x}$$

The results for $\sin x$ and $\cos x$ hold only for x in radians.

Note the exponential in particular: e^x is its own derivative. More precisely, the functions equal to their own derivative are exactly those of the form $y = Ce^x$, and e^x is the one passing through $(0, 1)$.

The card says radians because the derivative of $\sin x$ is built on the limit $\lim_{x\to 0}\frac{\sin x}{x} = 1$ from §13.1, and that limit equals 1 only in radians. An angle of x degrees is $u = \frac{\pi x}{180}$ radians, so the quotient is $\frac{\sin u}{x} = \frac{\pi}{180}\cdot\frac{\sin u}{u}$, which approaches $\frac{\pi}{180}$. The derivative would then be $\frac{\pi}{180}\cos x$. Radians are the unit that makes this constant equal to 1, which is why calculus uses them.

EXAMPLE 13.14

EXT Three of these derivatives were read from the graphs. Prove from first principles that $\frac{d}{dx}\sin x = \cos x$ and $\frac{d}{dx}\cos x = -\sin x$.

SOLUTION

Expand $\sin(x+h)$ with the compound-angle identity $\sin(x+h) = \sin x\cos h + \cos x\sin h$ §8.2, then collect the terms in $\sin x$:

$$\frac{\sin(x+h)-\sin x}{h} = \sin x\cdot\frac{\cos h - 1}{h} + \cos x\cdot\frac{\sin h}{h}.$$

As $h \to 0$ the two fractions approach 0 and 1, the limits of §13.1, so the quotient approaches $\cos x$. The same steps with $\cos(x+h) = \cos x\cos h - \sin x\sin h$ give

$$\frac{\cos(x+h)-\cos x}{h} = \cos x\cdot\frac{\cos h - 1}{h} - \sin x\cdot\frac{\sin h}{h},$$

which approaches $-\sin x$. The derivative of $\ln x$ needs implicit differentiation, and it is proved in §14.5. The claim about $y = Ce^x$ needs the product rule, and it is proved at the end of §13.6.

At Higher Level the formula booklet gives a wider set of standard derivatives. HL They are listed here for reference and used freely in the rules that follow. The reciprocal and inverse trigonometric functions in the table are defined in Ch. 8; only their derivatives are needed here.

FORMULA 13.5 **Further standard derivatives** FORMULA BOOKLET · 5.15 HL

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x \qquad \frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}a^x = a^x \ln a \qquad \frac{d}{dx}\log_a x = \frac{1}{x \ln a}$$

$$\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\arctan x = \frac{1}{1+x^2}$$

Angles in radians. For a^x and $\log_a x$, $a > 0$ and $a \neq 1$, and $\log_a x$ needs $x > 0$. The $\arcsin$ and $\arccos$ results hold for $|x| < 1$.

The derivative of $\arccos$ is the negative of the derivative of $\arcsin$, because $\arccos x = \frac{\pi}{2} - \arcsin x$. Let $y = \arcsin x$, so that $\sin y = x$ and $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$. Then $\cos\left(\frac{\pi}{2} - y\right) = \sin y = x$, and $\frac{\pi}{2} - y$ lies between 0 and π, the range of $\arccos$. So $\arccos x = \frac{\pi}{2} - \arcsin x$. The two functions add to a constant, so their derivatives add to zero.

Each result in the table can be derived once the right tool is available. The first six ($\tan$, $\sec$, cosec, $\cot$, a^x and $\log_a x$) need the quotient and chain rules, and are derived at the end of §13.6. The three inverse trigonometric results need implicit differentiation, and are derived at the end of §14.5.

The tables are used term by term, and a constant multiplier is unchanged, exactly as with powers.

EXAMPLE 13.15

Differentiate (a) $y = 3\tan x - 2\cos x$, (b) $y = 5^x + \ln x$, (c) $y = \arccos x + 3\arctan x$.

SOLUTION

(a)

$$\frac{dy}{dx} = \boxed{3\sec^2 x + 2\sin x}$$

The derivative of $\cos x$ is $-\sin x$, so subtracting a cosine adds a sine.

(b)

$$\frac{dy}{dx} = \boxed{5^x \ln 5 + \frac{1}{x}}$$

The factor $\ln 5$ is a constant, and it appears because $5^x = e^{x \ln 5}$ (derived at the end of §13.6). For e^x the factor is $\ln e = 1$, which is why e^x has no extra factor.

(c)

$$\frac{dy}{dx} = \boxed{-\frac{1}{\sqrt{1-x^2}} + \frac{3}{1+x^2}}$$

The leading minus sign is the only thing separating the $\arccos$ entry of the table from the $\arcsin$ entry. Neither inverse trigonometric derivative contains a trigonometric function; both are algebraic, so they appear again as the results of integrals in Ch. 15.

YOUR TURN 13.5 The Standard Derivatives

The four standard derivatives

33. Differentiate each of the following.

(a) $y = \sin x + \cos x$

(b) $y = e^x - 3\ln x$

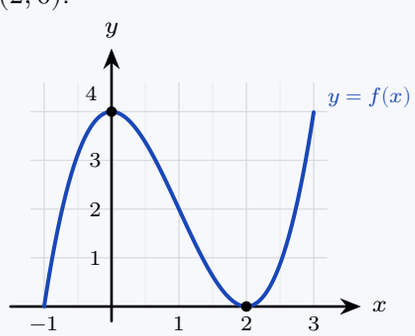
(c) $y = 4\cos x$

(d) $y = 2\sin x - 5e^x$

34.

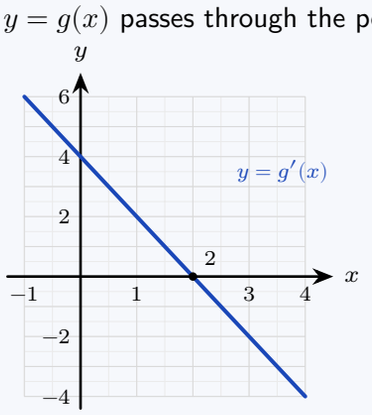
(a) The results $\dfrac{d}{dx}\sin x = \cos x$ and $\dfrac{d}{dx}\cos x = -\sin x$ hold only when x is measured in radians. Explain why the choice of unit matters.

(b) The graph of $y = f(x)$ is shown for $-1 \le x \le 3$. It has a local maximum at $(0, 4)$ and a local minimum at $(2, 0)$.



Sketch the graph of $y = f'(x)$ for $-1 \le x \le 3$. Show where it meets the x-axis.

(c) The graph of $y = g'(x)$, the derivative of a function g, is the straight line shown for $-1 \le x \le 4$. The graph of $y = g(x)$ passes through the point $(2, 5)$.



Sketch the graph of $y = g(x)$ for $-1 \le x \le 4$.

The Higher Level table

35. Differentiate each of the following.

(a) $y = \tan x$
(b) $y = \arctan x$
(c) $y = 3^x$
(d) $y = \ln x + \tan x$
(e) $y = \sec x$
(f) $y = \arcsin x$

36. Differentiate each of the following.

(a) $y = \operatorname{cosec} x$
(b) $y = 2\cot x$
(c) $y = \log_2 x$
(d) $y = 3\arccos x - 4^x$
(e) Show that $\dfrac{d}{dx}(\arcsin x + \arccos x) = 0$, and explain why this result is expected.

Gradients and tangents

37. Find the gradient of $y = \sin x$ at $x = \dfrac{\pi}{3}$, giving an exact answer.

38. Find the point on the curve $y = e^x$ at which the gradient is 1, and write down the equation of the tangent there.

39. Find the equation of the tangent to $y = \ln x$ at the point where $x = e$.

40. Find the values of x, for $0 \le x \le 2\pi$, at which the curve $y = x + 2\cos x$ has gradient zero.

Concept check

41. A student writes $\dfrac{d}{dx}\,2^x = x \cdot 2^{x-1}$.

(a) Explain why the power rule does not apply here, and give the correct derivative.
(b) By writing $5^x = e^{x\ln 5}$, find $\dfrac{d}{dx}\,5^x$.

13.6 The Rules of Differentiation

Most functions are built from simpler ones, by composing, multiplying or dividing them. Three rules cover these three cases. With the standard derivatives, they let you differentiate almost any function in the course.

13.6.1 The chain rule

A composite function §5.1 is a function inside a function, such as $(2x^2 + 3)^5$ or $\sin(2x)$. The **chain rule** differentiates it in two steps: differentiate the outer function, then multiply by the derivative of the inner function.

FORMULA 13.6 **Chain rule** FORMULA BOOKLET · 5.6

If $y = g(u)$ where $u = f(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

The meaning of the rule is that **rates multiply**. Suppose a car's fuel use y depends on its speed u, and its speed depends on time x. If fuel use rises 3 units for every unit of speed, and speed rises 2 units for every unit of time, then fuel use rises $3 \times 2 = 6$ units for every unit of time. That is exactly $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$. A chain of dependencies multiplies its rates, in the same way that gears in a machine multiply their turning ratios.

The chain rule contains the power rule as a special case. When the inner function is just x, the extra factor is 1: for $y = x^7$ the chain rule gives

$$\frac{dy}{dx} = 7x^6 \times \frac{d}{dx}(x) = 7x^6 \times 1 = 7x^6.$$

The factor 1 is the derivative of the inner function x, and it is not usually written. Replace the inner function x by $2x + 1$ and the same rule gives

$$\frac{d}{dx}(2x+1)^7 = 7(2x+1)^6 \times 2 = 14(2x+1)^6.$$

Now the derivative of the inner function is 2, not 1, so it must be written.

Two habits help you apply the chain rule correctly:

1. **Identify the inner function first**, before differentiating anything. Ask which function is applied to x first: that one is the inner function.
2. **Multiply by the derivative of the inner function.** Differentiating the outer function alone and stopping there is a common error, so check for it at the end of every chain rule question.

The outer function need not be a power. Any of the standard functions of §13.5, such as e^u, $\ln u$ or $\sin u$, can be the outer function, with the inner function kept unchanged inside it.

EXAMPLE 13.16

Differentiate (a) $y = \ln(2 + \cos x)$ and (b) $y = \sqrt{x^3 + 2}$.

SOLUTION

(a) Outer is $\ln u$, inner is $u = 2 + \cos x$. The outer derivative is $\frac{1}{u}$, which must be written at $u = 2 + \cos x$:

$$\frac{dy}{dx} = \frac{1}{2+\cos x} \cdot (-\sin x) = \boxed{-\frac{\sin x}{2+\cos x}}$$

(b) Rewrite the root as a power first, $y = (x^3+2)^{1/2}$, so the outer is $u^{1/2}$ and the inner is $u = x^3+2$:

$$\frac{dy}{dx} = \tfrac{1}{2}(x^3+2)^{-1/2} \cdot 3x^2 = \boxed{\frac{3x^2}{2\sqrt{x^3+2}}}$$

The root itself needs $x^3+2 \geq 0$, and the derivative puts that root in a denominator, so this expression holds for $x^3+2 > 0$, that is $x > -\sqrt[3]{2}$.

Part (a) shows a common error to avoid: the outer derivative of a logarithm is $\frac{1}{\text{inner}}$, so it is $\frac{1}{2+\cos x}$ here, never $\frac{1}{x}$.

13.6.2 The product rule

To differentiate a product of two functions, you cannot simply multiply their derivatives. The mistake is a natural one to make. Seeing why it gives a wrong answer leads to the correct rule.

Take $u = x$ and $v = x$, so that $y = uv = x^2$. The derivative of x^2 is $2x$. But $u' = 1$ and $v' = 1$, and $u'v' = 1$. The two answers do not agree, so multiplying the derivatives gives a wrong answer.

The reason is that a product uv changes for two reasons at the same time: u changes, and v changes.

Suppose both u and v increase. Part of the increase in uv comes from the change in u, multiplied by the whole of v. The rest comes from the change in v, multiplied by the whole of u. The correct rule adds these two parts, which is why each term keeps one factor un-differentiated.

A rectangle makes this concrete. Let u and v be the two side lengths, so that uv is the area. Now stretch both sides slightly, u by a small amount Δu and v by Δv. The extra area has three parts. There is a thin strip along the top, of area (change in v) $\times\, u$, and a thin strip along the side, of area (change in u) $\times\, v$. There is also a tiny corner rectangle where the two strips meet. That corner is a product of two small quantities, so its area approaches zero faster than the strips and adds nothing in the limit. The two strips are the two terms of the product rule.

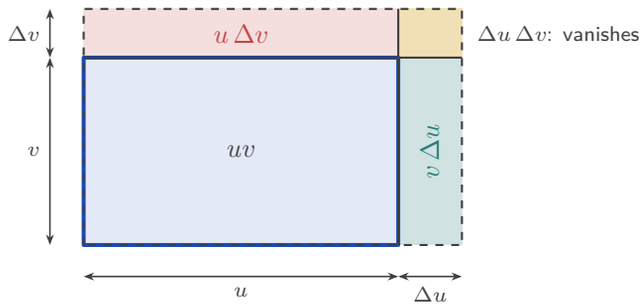


Figure 13.9 The product rule as an area. Growing both sides adds strips of area $u\,\Delta v$ and $v\,\Delta u$, the two terms of the rule; the corner $\Delta u\,\Delta v$ vanishes in the limit.

FORMULA 13.7 **Product rule** FORMULA BOOKLET · 5.6

If $y = uv$, then

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

EXAMPLE 13.17

Differentiate $y = (x^2 + 1)e^x$.

SOLUTION

Let $u = x^2 + 1$ and $v = e^x$, so $u' = 2x$ and $v' = e^x$:

$$\frac{dy}{dx} = (x^2 + 1)e^x + e^x(2x) = e^x(x^2 + 2x + 1) = \boxed{(x + 1)^2 e^x}$$

Factorise the answer where you can. A factorised form makes later work much easier, for example finding where $\frac{dy}{dx} = 0$.

The rules combine, and many examination questions require it. When one factor of a product is itself a composite, the product rule gives the structure of the answer and the chain rule gives that factor's derivative. Find each of u' and v' completely before assembling anything.

EXAMPLE 13.18

Differentiate $y = x^3 e^{2x}$.

SOLUTION

Let $u = x^3$ and $v = e^{2x}$, so $u' = 3x^2$. The second factor is a composite, with outer e^w and inner $w = 2x$, so the chain rule gives

$$v' = e^{2x} \cdot 2 = 2e^{2x}.$$

Now assemble the product rule:

$$\frac{dy}{dx} = x^3\left(2e^{2x}\right) + e^{2x}\left(3x^2\right) = e^{2x}\left(2x^3 + 3x^2\right) = \boxed{x^2(2x+3)e^{2x}}$$

The exponential factor is never zero, so the factorised form shows at once that the gradient is zero only at $x = 0$ and $x = -\frac{3}{2}$.

13.6.3 The quotient rule

The quotient rule is not a new idea. A quotient $\frac{u}{v}$ is the product $u \times v^{-1}$, so the product and chain rules together produce the quotient rule. Write $y = uv^{-1}$. The product rule, with the chain rule for v^{-1}, gives

$$\frac{dy}{dx} = u\left(-v^{-2}\frac{dv}{dx}\right) + v^{-1}\frac{du}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

The minus sign is there because a larger bottom makes the fraction smaller. When the top and the bottom are both positive, increasing the top of a fraction increases its value, so the u term is positive. Increasing the bottom *decreases* the value, so the v term is negative. The v^2 in the denominator comes from differentiating v^{-1}.

FORMULA 13.8 **Quotient rule** FORMULA BOOKLET · 5.6

If $y = \frac{u}{v}$, then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

EXAMPLE 13.19

Differentiate $y = \dfrac{x}{x^2+1}$.

SOLUTION

Let $u = x$ and $v = x^2 + 1$, so $u' = 1$ and $v' = 2x$:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - x(2x)}{(x^2+1)^2} = \boxed{\frac{1-x^2}{(x^2+1)^2}}$$

The order in the numerator is: bottom times derivative of top, minus top times derivative of bottom. Reversing the two terms changes the sign of the answer, and it is a common error.

These rules also explain the Higher Level table of §13.5. Six of its nine results follow in a line or two each.

EXAMPLE 13.20

Show that $\frac{d}{dx}\tan x = \sec^2 x$, $\frac{d}{dx}\cot x = -\operatorname{cosec}^2 x$, $\frac{d}{dx}\sec x = \sec x \tan x$ and $\frac{d}{dx}\operatorname{cosec} x = -\operatorname{cosec} x \cot x$.

SOLUTION

Write $\tan x = \frac{\sin x}{\cos x}$ and use the quotient rule:

$$\frac{d}{dx}\tan x = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The same steps on $\cot x = \frac{\cos x}{\sin x}$ give $\frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\operatorname{cosec}^2 x$.
For $\sec x = (\cos x)^{-1}$ the chain rule gives

$$\frac{d}{dx}(\cos x)^{-1} = -(\cos x)^{-2}(-\sin x) = \frac{1}{\cos x}\cdot\frac{\sin x}{\cos x} = \sec x \tan x,$$

and $\operatorname{cosec} x = (\sin x)^{-1}$ gives $-(\sin x)^{-2}\cos x = -\operatorname{cosec} x \cot x$ in the same way.

For a^x, write $a = e^{\ln a}$, so that $a^x = e^{x \ln a}$. The inner function $x \ln a$ has derivative $\ln a$, a constant, so

$$\frac{d}{dx}a^x = e^{x \ln a}\cdot \ln a = a^x \ln a.$$

For $\log_a x$, the change of base rule Ch. 1 gives $\log_a x = \frac{\ln x}{\ln a}$. That is a constant multiple of $\ln x$, so its derivative is $\frac{1}{x \ln a}$.

The same rewriting completes the power rule of §13.3. For $x > 0$ and any real power n, write $x^n = e^{n \ln x}$. The inner function $n \ln x$ has derivative $\frac{n}{x}$, so the chain rule gives

$$\frac{d}{dx}x^n = e^{n \ln x}\cdot\frac{n}{x} = x^n \cdot \frac{n}{x} = nx^{n-1}.$$

For $x > 0$ this covers the negative and fractional powers that the binomial argument of §13.3 does not cover.

With the product rule you can prove one more claim from §13.5: the functions equal to their own derivative are exactly those of the form $y = Ce^x$. Each of these is its own derivative, because $\frac{d}{dx}(Ce^x) = C\frac{d}{dx}e^x = Ce^x$. Now suppose $\frac{dy}{dx} = y$. Then

$$\frac{d}{dx}(ye^{-x}) = \frac{dy}{dx}e^{-x} - ye^{-x} = ye^{-x} - ye^{-x} = 0.$$

A function whose gradient is zero everywhere has a horizontal graph, so ye^{-x} is a constant C, and $y = Ce^x$.

Before you differentiate, decide how the function is built. The table sets out which rule to use.

What you see	Rule	Example
A function of a function	chain rule	$(2x+1)^7$, $\ln(2+\cos x)$
A product of two functions of x	product rule	$(x^2+1)e^x$
A quotient	quotient rule	$\dfrac{x}{x^2+1}$
A product with one composite factor	product rule, then chain rule for that factor	x^3e^{2x}

YOUR TURN 13.6 The Rules of Differentiation

The chain rule

42. Use the chain rule to differentiate each of the following.

(a) $y=(2x+1)^4$ (b) $y=\sin(x^2)$

(c) $y=e^{3x}$ (d) $y=\sqrt{3x+1}$

(e) $y=\ln(x^2+1)$ (f) $y=\ln(3x-2)$

43. Find the gradient of $y=(x^2-1)^3$ at the point where $x=2$.

44. A student writes $\dfrac{d}{dx}\sin(3x)=\cos(3x)$. Identify the error and give the correct derivative.

45. Use the chain rule to differentiate each of the following.

(a) $y=(2x^2+3)^5$ (b) $y=e^{x^2}$

The product rule

46. Use the product rule to differentiate each of the following.

(a) $y=x\cos x$ (b) $y=x^2e^x$

(c) $y=x^3\ln x$ (d) $y=x\sin x$

47. Explain, with a counterexample, why the derivative of a product is not the product of the derivatives.

The quotient rule

48. Use the quotient rule to differentiate each of the following.

(a) $y=\dfrac{x}{x+1}$ (b) $y=\dfrac{x+1}{x-1}$

(c) $y=\dfrac{\sin x}{x}$ (d) $y=\dfrac{e^x}{x}$

Combining the rules

49. Each of these is a product in which one factor needs the chain rule. Differentiate, and factorise your answer where possible.

(a) $y = x^2 e^{3x}$

(b) $y = x \sin 2x$

(c) $y = x\sqrt{1+x^2}$

(d) $y = x(x^2+1)^{-1}$. Then check your answer by differentiating $\dfrac{x}{x^2+1}$ with the quotient rule.

Concept check

50. A student differentiates $y = \dfrac{x^2}{x+1}$ with the quotient rule and writes $\dfrac{dy}{dx} = \dfrac{x^2 \cdot 1 - (x+1)\cdot 2x}{(x+1)^2}$. Identify the error and give the correct derivative.

13.7 Higher Derivatives

A car that is speeding up has a velocity that is changing. To describe how fast that velocity changes, you need the rate of change of a rate of change. The derivative $f'(x)$ is itself a function, so it can be differentiated again.

Differentiating $f'(x)$ gives the **second derivative**. It is written $f''(x)$, or $\dfrac{d^2y}{dx^2}$ when $y = f(x)$.

DEFINITION 13.3 **Second derivative**

The **second derivative** of f is the derivative of f':

$$f''(x) = \frac{d}{dx}(f'(x)), \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right).$$

The notation $\dfrac{d^2y}{dx^2}$ records the two steps. The operation $\dfrac{d}{dx}$ is applied twice, so the d on top carries a 2 and the dx below is written dx^2. Read it as "d two y by d x two". The 2 counts differentiations. It is not a power of y.

The second derivative measures how fast the gradient changes. For motion along a line this has a familiar name. If s is displacement at time t, then $\dfrac{ds}{dt}$ is velocity and $\dfrac{d^2s}{dt^2}$ is acceleration. See §14.4.

To find a second derivative, differentiate once, simplify, then differentiate the result. Every rule of §13.6 applies at each step, because each step is an ordinary differentiation.

EXAMPLE 13.21

For $f(x) = x^3 - 9x^2 + 5$, find $f'(x)$, $f''(x)$ and $f''(2)$.

SOLUTION

Differentiate once, then differentiate the result:

$$f'(x) = \boxed{3x^2 - 18x}, \qquad f''(x) = \boxed{6x - 18}$$

So $f''(2) = 12 - 18 = \boxed{-6}$

Each differentiation lowers the degree of a polynomial by one, so a cubic has a linear second derivative.

EXAMPLE 13.22

Find $\dfrac{d^2y}{dx^2}$ for $y = xe^{2x}$.

SOLUTION

The product rule gives the first derivative:

$$\frac{dy}{dx} = e^{2x} + 2xe^{2x} = (1 + 2x)e^{2x}.$$

This is again a product, so apply the product rule a second time:

$$\frac{d^2y}{dx^2} = 2e^{2x} + (1 + 2x) \cdot 2e^{2x} = \boxed{4(1 + x)e^{2x}}$$

Taking out the common factor e^{2x} after each step keeps the expression short enough to differentiate again.

You can continue past the second derivative. HL Differentiating n times gives the **nth derivative**, written $\dfrac{d^ny}{dx^n}$ or $f^{(n)}(x)$. The brackets matter: $f^{(3)}(x)$ is the third derivative, while $f^3(x)$ usually means $\big(f(x)\big)^3$. For the first three derivatives, primes are used instead: f', f'', f'''.

For some functions the nth derivative has a single formula, as the last column of the table shows for e^{2x}, $\sin x$ and x^{-1}.

$f(x)$	$f'(x)$	$f''(x)$	$f'''(x)$	$f^{(4)}(x)$	$f^{(n)}(x)$
e^{2x}	$2e^{2x}$	$4e^{2x}$	$8e^{2x}$	$16e^{2x}$	$2^n e^{2x}$
$\sin x$	$\cos x$	$-\sin x$	$-\cos x$	$\sin x$	repeats every 4
x^{-1}	$-x^{-2}$	$2x^{-3}$	$-6x^{-4}$	$24x^{-5}$	$(-1)^n\, n!\, x^{-(n+1)}$

Each pattern in the last column is a claim about every n, and proof by induction is how such a claim is proved: §3.4.6 proves $\dfrac{d^n}{dx^n}(xe^x) = (x+n)e^x$ in this way. Higher derivatives return in Ch. 16, where the derivatives of a function at one point are used to rebuild the function as a Maclaurin series. What the sign of f'' says about the shape of a graph is developed in Ch. 14.

YOUR TURN 13.7 Higher Derivatives

Second derivatives

51. Find the second derivative of each of the following.

(a) $f(x) = x^3 - 6x^2$ (b) $y = \sqrt{x}$

(c) $y = \cos 2x$ (d) $y = e^{-3x}$

52. Find $\dfrac{d^2y}{dx^2}$ for $y = \ln(2x)$.

53. A particle has displacement $s = t^3 - 6t^2$ metres at time t seconds. Find $\dfrac{d^2s}{dt^2}$ when $t = 3$.

54. In each case the first derivative is again a product, so the product rule is needed twice. Find $\dfrac{d^2y}{dx^2}$.

(a) $y = x^2e^x$ (b) $y = x\sin x$

Third and higher derivatives

55. Find $f'''(x)$ for $f(x) = x^5 - 2x^3$.

56. For $f(x) = x^4$, find $f^{(3)}(x)$ and $\big(f(x)\big)^3$, and evaluate each at $x = 1$.

57. For $f(x) = e^{-x}$, find $f'(x)$, $f''(x)$ and $f'''(x)$, and hence write down $f^{(n)}(x)$.

58. Let $f(x) = \sin x$.

(a) Find $f'(x)$, $f''(x)$, $f'''(x)$ and $f^{(4)}(x)$.

(b) Hence write down $f^{(10)}(x)$ and $f^{(23)}(x)$.

Concept check

59. Let $f(x) = x^3 - 2x^2$. A student writes $f'(2) = 12 - 8 = 4$, then differentiates 4 to obtain $f''(2) = 0$. Identify the error and find the correct value of $f''(2)$.

13.8 L'Hôpital's Rule HL

This chapter began with the indeterminate form $\frac{0}{0}$, and resolved it by factorising. Factorising usually does not help when the functions are trigonometric, exponential or logarithmic.

For those cases there is a second method, l'Hôpital's rule: if a limit gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, you can find the limit from the derivatives of the top and the bottom. The rule is named after the French mathematician Guillaume de l'Hôpital, and the name is said *lo - pee - TAL*, with a silent h.

KEY 13.9 L'Hôpital's rule HL

If $\lim_{x\to a}\dfrac{f(x)}{g(x)}$ gives $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)},$$

provided the limit on the right of this equation exists.

Needs f and g differentiable near a (not necessarily at a), with $g'(x)\neq 0$ there.

Differentiate the top and the bottom *separately*. This is not the quotient rule. Then try the limit again, and if it is still indeterminate, apply the rule a second time.

The rule works because near $x=a$ each function is close to its tangent line. The argument needs both functions to be zero at a, which is why the form must be $\frac{0}{0}$.

Suppose $f(a)=0$ and $g(a)=0$. Then near $x=a$ each function is close to its own tangent line at a, so

$$f(x)\approx f'(a)(x-a),\qquad g(x)\approx g'(a)(x-a).$$

Divide one by the other. The factor $(x-a)$ appears in both, so it cancels:

$$\frac{f(x)}{g(x)}\approx\frac{f'(a)(x-a)}{g'(a)(x-a)}=\frac{f'(a)}{g'(a)}.$$

So the quotient of the functions approaches the quotient of their gradients. This argument covers the $\frac{0}{0}$ case when $g'(a)\neq 0$. Only when both functions equal zero at a does each one reduce to a multiple of $(x-a)$, so only then does the cancelling work. The rule also holds for the form $\frac{\infty}{\infty}$, and for limits as $x\to\infty$. The proof of those cases is beyond this course.

The limit $\lim_{x\to 0}\dfrac{\cos x}{x}$ shows what happens when the form is not $\frac{0}{0}$ or $\frac{\infty}{\infty}$. Here $\cos 0=1$, not 0, so the form is $\frac{1}{0}$, which is not indeterminate and not one the rule covers. Differentiating top and bottom anyway would give

$$\lim_{x\to 0}\frac{-\sin x}{1}=0,$$

That answer is wrong. The limit does not exist, because the fraction increases without bound on one side of 0 and decreases without bound on the other. Check the form before you differentiate.

EXAMPLE 13.23

Use l'Hôpital's rule to evaluate (a) $\lim_{x\to 0}\dfrac{\sin x}{x}$, (b) $\lim_{x\to 0}\dfrac{\ln(1+x)-x}{x^2}$.

SOLUTION

(a) Substituting gives $\frac{0}{0}$, so differentiate top and bottom:

$$\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}=\cos 0=\boxed{1}$$

This agrees with the limit proved in §13.1. It is a consistency check and not a second proof, because the derivative of $\sin x$ was itself found from that limit.

(b) Substituting again gives $\frac{0}{0}$. One application gives $\dfrac{1/(1+x)-1}{2x}$, still $\frac{0}{0}$, so apply the rule again:

$$\begin{aligned}\lim_{x\to 0}\frac{\ln(1+x)-x}{x^2}&=\lim_{x\to 0}\frac{\dfrac{1}{1+x}-1}{2x}\\&=\lim_{x\to 0}\frac{-\dfrac{1}{(1+x)^2}}{2}\\&=\boxed{-\frac{1}{2}}\end{aligned}$$

EXAMPLE 13.24

Evaluate $\lim_{x\to\infty}\dfrac{\ln x}{x}$.

SOLUTION

As $x\to\infty$ this is $\frac{\infty}{\infty}$. Differentiate top and bottom:

$$\begin{aligned}\lim_{x\to\infty}\frac{\ln x}{x}&=\lim_{x\to\infty}\frac{1/x}{1}\\&=\lim_{x\to\infty}\frac{1}{x}\\&=\boxed{0}\end{aligned}$$

The denominator increases without bound while the numerator increases very slowly, so the ratio approaches zero. With l'Hôpital's rule you can make that comparison exact.

YOUR TURN 13.8 L'Hôpital's Rule HL

Applying the rule once

60. Evaluate each limit, checking first that it is indeterminate.

(a) $\lim_{x\to 0}\dfrac{e^x-1}{x}$ (b) $\lim_{x\to 0}\dfrac{\sin 2x}{x}$ (c) $\lim_{x\to 0}\dfrac{\tan x}{x}$

(d) $\lim_{x\to \infty}\dfrac{\ln x}{x^2}$ (e) $\lim_{x\to 1}\dfrac{\ln x}{x-1}$

61. Evaluate $\lim_{x\to\infty}\dfrac{3x^2+x}{2x^2-5}$ in two ways: by dividing by the highest power, and by l'Hôpital's rule. State which you would use in an examination, and why.

Applying the rule twice

62. Each of these needs the rule applied twice. Evaluate them.

(a) $\lim_{x\to 0}\dfrac{1-\cos 2x}{x^2}$ (b) $\lim_{x\to 0}\dfrac{e^x-1-x}{x^2}$ (c) $\lim_{x\to\infty}\dfrac{x^2}{e^x}$

Checking the form first

63. A student evaluates $\lim_{x\to 0}\dfrac{x+2}{x+1}$ by differentiating top and bottom, obtaining 1. Explain the error and give the correct value.

64. State the two indeterminate forms to which l'Hôpital's rule applies, and explain why the rule must not be used on a form such as $\frac{2}{0}$.

Concept check

65. Explain why l'Hôpital's rule requires the derivatives of the top and the bottom taken **separately**, and not the derivative of the quotient.

Chapter 13 · Limits & Derivatives

Reflections

TOK The early calculus was built on "infinitesimals," quantities treated as nonzero when dividing and as zero when convenient. In the 1730s the philosopher George Berkeley laughed at them as "ghosts of departed quantities," and his criticism was fair: the logic was not secure. The method gave correct answers for about 150 years before the limit gave it a secure foundation. Is a mathematical technique justified by its results, or only by its proof?

IM Calculus was created twice, independently. Isaac Newton developed his "method of fluxions" in England, and Gottfried Leibniz developed his version, with the $\frac{dy}{dx}$ notation we still use, some years later in Paris. A bitter dispute over who was first divided European mathematics for about a century. Britain kept Newton's dots while the rest of Europe used Leibniz's symbols, and historians often link that choice to the slower progress of British mathematics in the period. Notation is not neutral: a good symbol can make a whole subject easier to think in.

RW Some roads have average-speed cameras at both ends of a long stretch. They do not measure your speed at one instant; they time you over the whole stretch. If a car covers 10 km in 6 minutes, its average speed is 100 km/h. A theorem of calculus then shows that at some instant the car was travelling at exactly 100 km/h, even though no camera saw it. (This is the **mean value theorem**: for a smooth journey, the derivative equals the average rate at least once.) Economics uses derivatives in the same way. When a report says prices are "rising, but more slowly", it is making a claim about a second derivative. *Try it:* two cameras 4.5 km apart record a car 2 minutes 30 seconds apart. What speed must it have reached?

EXT L'Hôpital's rule is related to the Maclaurin series, a polynomial that approximates a function well near $x = 0$: $\sin x \approx x - \frac{x^3}{6}$, $e^x \approx 1 + x + \frac{x^2}{2}$. Substituting these into an indeterminate limit resolves it without differentiating repeatedly. The same problem that l'Hôpital's rule solves leads to representing functions as endless sums of powers of x Ch. 16.

TOK The speedometer says 80 km/h, but no time passes at an instant. Is instantaneous speed measured, or calculated?

End-of-Chapter Problems
Chapter 13: Limits & Derivatives

1. Consider $f(x) = |x - 2|$.

(a) Sketch the graph of f. [1]

(b) Explain briefly why f is continuous at $x = 2$. [2]

(c) Explain why f is not differentiable at $x = 2$. [2]

2. Find the equation of the tangent to $y = x^2$ that is parallel to the line $y = 6x - 1$. [4]

3. Consider $f(x) = x^3 - 3x$.

(a) Find $f'(x)$. [1]

(b) Find the coordinates of the points where the tangent is horizontal. [4]

4. Find the coordinates of the points on the curve $y = x^3 - 3x^2$ where the tangent is parallel to the line $y = 9x + 1$. [4]

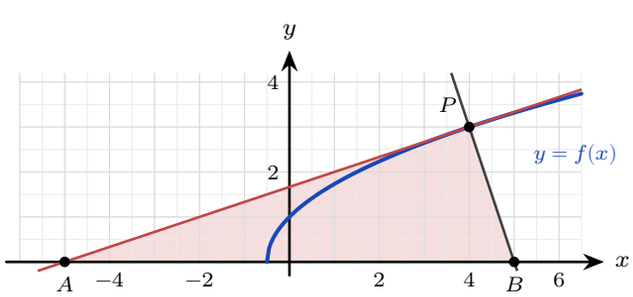
5. Consider the curve $y = e^x$ at the point where $x = 1$.

(a) Show that the tangent there passes through the origin. [4]

(b) Find the equation of the normal at the same point. [2]

6. Find the equations of *both* tangents to the curve $y = x^2$ that pass through the point $(0, -1)$. [6]

7. Let $f(x) = \sqrt{2x + 1}$ for $x \geq -\frac{1}{2}$. The point $P(4, 3)$ lies on the graph of f. The tangent to the graph at P meets the x-axis at A, and the normal to the graph at P meets the x-axis at B. Triangle APB is shaded in the diagram.



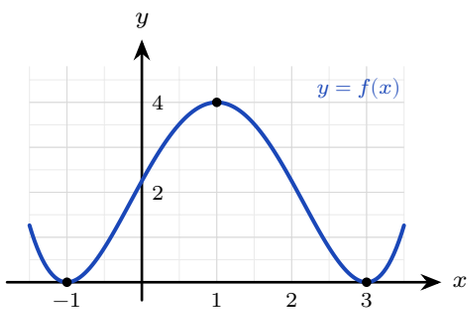
(a) Find $f'(x)$. [2]

(b) Find the equation of the tangent to the graph at P. [2]

(c) Find the equation of the normal to the graph at P. [2]

(d) Find the area of triangle APB. [4]

8. The graph of $y = f(x)$ is shown for $-1.5 \leq x \leq 3.5$. It has local minimum points at $(-1, 0)$ and $(3, 0)$, and a local maximum point at $(1, 4)$.



(a) Write down the values of x for which $f'(x) = 0$. [2]
(b) Write down the values of x, for $-1.5 < x < 3.5$, for which $f'(x) > 0$. [2]
(c) Sketch the graph of $y = f'(x)$ for $-1.5 \leq x \leq 3.5$. Show where it meets the x-axis. [3]

9. Differentiate each of the following, simplifying your answers.

(a) $y = (2x^3 - 1)^4$ [2]
(b) $y = x^2 \ln x$ [2]
(c) $y = \dfrac{x^2}{\cos x}$ [2]
(d) $y = e^{3x} \cos x$ [3]
(e) $y = \sqrt{4x^2 + 1}$ [2]
(f) $y = \dfrac{e^x}{x^2 + 1}$ [3]

10. Differentiate each of the following, simplifying where possible.

(a) $y = (3x^2 + 1)^5$ [2]
(b) $y = x^3 e^{-x}$ [2]
(c) $y = \dfrac{x^2}{x - 2}$ [3]
(d) $y = \ln(3x^2 + 2)$ [2]
(e) $y = \dfrac{\cos x}{x^2}$ [3]
(f) $y = (x + 1)^3 (2x - 5)^4$ [3]

11. Differentiate each of the following.

(a) $y = \arctan(3x)$ [2]
(b) $y = \sec(2x)$ [2]
(c) $y = \log_3(x^2 + 1)$ [2]
(d) $y = \arcsin(x^2)$ [2]
(e) $y = \operatorname{cosec}(x^2)$ [2]
(f) $y = \cot(3x)$ [2]

12. Differentiate each of the following.

(a) $y = \sec^2 x$ [2]
(b) $y = \ln(\cos x)$ [3]
(c) $y = x \arctan x$ [3]
(d) $y = \arccos(2x)$ [2]
(e) $y = e^{\sin x}$ [2]
(f) $y = \dfrac{\arctan x}{x}$ [3]

13. Differentiate $y = e^x \sin x$, and hence find the gradient of the curve at $x = 0$. [3]

14. Using the quotient rule on $\tan x = \dfrac{\sin x}{\cos x}$, prove that $\dfrac{d}{dx}\tan x = \sec^2 x$. [4]

15. A curve has equation $y = \dfrac{\ln x}{x}$ for $x > 0$.

(a) Find $\dfrac{dy}{dx}$. [3]
(b) Show that the curve has a horizontal tangent at $x = e$. [2]

16. A curve has equation $y = \dfrac{e^{2x}}{1 + x^2}$.

(a) Find $\dfrac{dy}{dx}$. [4]
(b) Find the gradient of the curve at $x = 0$. [1]

17. The displacement of a particle is $s(t) = t^3 - 6t^2 + 9t$ metres at time t seconds, for $t \geq 0$.

(a) Find the velocity $v(t) = s'(t)$. [2]
(b) Find the times at which the particle is momentarily at rest. [3]
(c) Find the acceleration at $t = 3$. [2]

18. Evaluate $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1}{x}$. [2]

19. Evaluate $\lim\limits_{x \to 0} \dfrac{e^{2x} - 1 - 2x}{x^2}$. [3]

20. Consider $f(x) = x^2 - 2x$.

(a) Find $f'(x)$ from first principles. [4]
(b) Find the equation of the tangent to the curve $y = f(x)$ at the point where $x = 3$. [3]
(c) Find the equation of the normal at the same point. [2]

21. GDC The function $f(x) = \dfrac{2^x - 1}{x}$ is not defined at $x = 0$.

(a) Copy and complete the table of values of $f(x)$, giving 5 significant figures:
$x = -0.1,\ -0.01,\ 0.01,\ 0.1$. [2]
(b) Suggest the value of $\lim\limits_{x \to 0} f(x)$, correct to 3 significant figures. [1]
(c) Use l'Hôpital's rule to find the exact value of the limit, and verify that it agrees with your answer to part (b). [2]

22. GDC A cup of coffee is left to cool. Its temperature t minutes after it is poured is modelled by $T(t) = 20 + 70e^{-0.05t}$ degrees Celsius, for $t \geq 0$.

(a) Find the rate of change of T at $t = 10$, giving your answer to 3 significant figures, and state what its sign tells you. [2]

(b) Find the time at which the coffee is cooling at a rate of 1 degree Celsius per minute. [2]

(c) The tangent to the graph of T at $t = 10$ is used to estimate the temperature at $t = 15$. Find this estimate and the value given by the model, each to 3 significant figures, and explain why the estimate is lower. [3]

23. GDC The profile of a hillside is modelled by $y = 4xe^{-0.5x}$ for $0 \leq x \leq 12$, where x and y are measured in metres.

(a) Find $\dfrac{dy}{dx}$. [2]

(b) Find the equation of the tangent to the profile at the point where $x = 1$, and show that it passes through the point $(-1, 0)$. [3]

(c) Find the two values of x at which the gradient of the hillside is -0.5, giving your answers to 3 significant figures. [3]

24. GDC Let $f(x) = x\cos x$ for $0 \leq x \leq \pi$.

(a) Find $f'(x)$. [2]

(b) Show that the tangent to the graph of f is horizontal where $x\tan x = 1$. [2]

(c) Find the coordinates of the point on the graph of f where the tangent is horizontal, giving your answers to 3 significant figures. [2]

(d) Find the equation of the normal to the graph of f at the point where $x = 2$. Give your answer in the form $y = mx + c$, with m and c correct to 3 significant figures. [3]

25. Let $g(x) = \dfrac{2x-1}{x+2}$, $x \neq -2$.

(a) Write down the equations of the vertical and horizontal asymptotes. [2]

(b) Use the quotient rule to show that $g'(x) = \dfrac{5}{(x+2)^2}$. [3]

(c) Hence explain why g is increasing on every interval of its domain. [2]

(d) Find the equation of the tangent to the curve at the point where $x = 0$. [2]

(e) Find the equation of the normal at the same point. [2]

(f) Sketch the graph of $y = g(x)$ for $-7 \leq x \leq 4$. Show the asymptotes, the points where the graph meets the axes, and the tangent found in part (d). [3]

26. EXT Differentiate each function from first principles, showing every step of the limit.

(a) $f(x) = \sqrt{x}$, for $x > 0$. *Multiply the difference quotient by the conjugate.* [4]

(b) $f(x) = e^x$. You may use $\lim\limits_{h\to 0} \dfrac{e^h - 1}{h} = 1$. [3]

(c) $f(x) = \sin x$. You may use the compound-angle formula for $\sin(x+h)$, together with $\lim_{h\to 0} \frac{\sin h}{h} = 1$ and $\lim_{h\to 0} \frac{\cos h - 1}{h} = 0$. [5]

27. Evaluate $\lim_{x\to 0} \frac{x - \sin x}{x^3}$, justifying each application of l'Hôpital's rule. [5]

28. Let $L = \lim_{x\to 0} \frac{\tan x - x}{x^3}$.

(a) Show that one application of l'Hôpital's rule gives $L = \lim_{x\to 0} \frac{\tan^2 x}{3x^2}$. [2]

(b) Hence find the value of L. [3]

29. Let $f(x) = xe^{2x}$.

(a) Show that $f'(x) = (2x+1)e^{2x}$ and $f''(x) = 4(x+1)e^{2x}$. [3]

(b) Find $f'''(x)$. [2]

(c) Suggest an expression for $f^{(n)}(x)$, the nth derivative of f, for every positive integer n. [2]

(d) Prove your conjecture by mathematical induction. [5]

(e) Show that $y = f(x)$ satisfies $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. [2]

30. The function f is defined by $f(x) = \frac{1 - \cos x}{x}$ for $x \neq 0$, and $f(0) = 0$.

(a) Use l'Hôpital's rule to show that f is continuous at $x = 0$. [2]

(b) Hence, using the definition of the derivative, show that $f'(0) = \frac{1}{2}$. [4]

(c) Find $f'(x)$ for $x \neq 0$. [3]

(d) Find the equation of the tangent to the graph of f at the point where $x = \pi$, giving exact coefficients. [3]

(e) Show that this tangent meets the x-axis at $x = 2\pi$. [1]

31. Evaluate each limit. Check the form before every use of l'Hôpital's rule.

(a) $\lim_{x\to 0} \frac{e^x - \cos x}{x + \sin x}$ [3]

(b) $\lim_{x\to 0} \frac{x^2 + 1}{\cos x}$ [2]

32. Let $f(x) = \frac{x^2 - 9}{|x - 3|}$ for $x \neq 3$.

(a) Find $\lim_{x\to 3^-} f(x)$ and $\lim_{x\to 3^+} f(x)$, and hence state whether $\lim_{x\to 3} f(x)$ exists. [4]

(b) Explain why no choice of the value $f(3)$ can make f continuous at $x = 3$. [2]

33. A curve has equation $y = x^2e^{2x}$.

(a) Find the coordinates of every point on the curve at which the tangent is horizontal. [5]

(b) Write down the equation of the normal at each of these points. [2]

Challenge Questions

C1. When l'Hôpital's rule says nothing. The rule in §13.8 carries a condition: $\lim \frac{f}{g} = \lim \frac{f'}{g'}$ *provided the limit on the right exists*. This investigation finds limits where that condition fails, and then uses the rule, correctly, to compare how fast functions grow.

(a) Consider $\lim_{x\to\infty} \frac{x+\sin x}{x}$. Show that it is of the form $\frac{\infty}{\infty}$, and that applying the rule once gives a limit that does not exist. Find the limit by another method, and explain why this does not contradict the rule. [5]

(b) Consider $\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$. Show that applying the rule once gives $\frac{x}{\sqrt{1+x^2}}$, and that applying it again returns the original fraction. Find the limit by another method. [4]

(c) Use the rule to evaluate $\lim_{x\to\infty} \frac{x^n}{e^x}$ for $n = 1$, 2 and 3. State a conjecture for every positive integer n, and prove it by induction on n. [5]

(d) Let $a > 0$. Show that $\lim_{x\to\infty} \frac{\ln x}{x^a} = 0$. [2]

(e) Parts (c) and (d) place $\ln x$, then every positive power x^a, then e^x, in increasing order of growth as $x \to \infty$. Investigate where the functions x^{100}, 1.01^x, $x\ln x$, $e^{\sqrt{x}}$ and x^x fit into this order, and give a precise meaning to the statement "f grows faster than g". [3]

C2. How smooth is a join? A curve built from two pieces can meet itself without a jump and still have a corner. This investigation measures how smooth a join is by counting how many derivatives exist there. For each positive integer n define

$$f_n(x) = \begin{cases} x^n & x \geq 0, \\ 0 & x < 0. \end{cases}$$

(a) By finding the one-sided limits of $\frac{f_2(h) - f_2(0)}{h}$ as $h \to 0$, show that f_2 is differentiable at $x = 0$, and state $f_2'(0)$. [3]

(b) Write down $f_2'(x)$ for every real x. Show that f_2' is continuous at 0 but not differentiable there. [3]

(c) Investigate f_3 in the same way, and find how many times f_3 can be differentiated at $x = 0$. [3]

(d) Conjecture how many times f_n can be differentiated at $x = 0$, and prove your conjecture. [4]

(e) A straight railway track along the negative x-axis is joined at the origin to a track that follows $y = f_n(x)$ for $x \geq 0$. Where the track is nearly straight, y'' measures how sharply

it bends, and a passenger feels that bending as a sideways push. Explain why the designer would choose $n = 3$ rather than $n = 2$. [2]

(f) Every f_n has only finitely many derivatives at 0. Investigate the function $g(x) = e^{-1/x}$ for $x > 0$, $g(x) = 0$ for $x \leq 0$. Use the substitution $t = \frac{1}{h}$ and Question C1(c) to show that $g'(0)$ exists, and then investigate whether g can be differentiated as many times as you like at $x = 0$. [3]

C3. The power rule, proved. Section 13.3 proved the power rule for a positive integer n in outline, using the binomial theorem. Here you write that proof out in full, and then look at how far the binomial argument reaches beyond the positive integers, before §13.6 settles every real power, for $x > 0$, with $x^n = e^{n \ln x}$.

(a) Let $n \geq 2$ be an integer. Write down the first three terms of $(x + h)^n$ in ascending powers of h. [3]

(b) Hence show that, for every integer $n \geq 2$,

$$\frac{(x+h)^n - x^n}{h} = nx^{n-1} + {}^nC_2 x^{n-2}h + \cdots + h^{n-1},$$

and explain why every term after the first still contains h. [5]

(c) Deduce that $\frac{d}{dx}x^n = nx^{n-1}$ for every integer $n \geq 2$. Check separately that the result also holds for $n = 1$. [3]

(d) Check your working by expanding in full for $n = 4$. [3]

(e) For $n = -1$ and $x \neq 0$, the binomial series of Chapter 2 gives $(x+h)^{-1} = x^{-1}\left(1 + \frac{h}{x}\right)^{-1}$ as an infinite series, valid for $|h| < |x|$. Write down its first three terms, and hence the first two terms of $\frac{(x+h)^{-1} - x^{-1}}{h}$. Explain why the step used in part (c) is no longer enough on its own. Then confirm that $\frac{d}{dx}x^{-1} = -x^{-2}$ by differentiating $\frac{1}{x}$ from first principles without a series. [4]

(f) The rule also holds for fractional powers. For $x > 0$, prove it for $n = \frac{1}{2}$ by multiplying the numerator and denominator of $\frac{\sqrt{x+h} - \sqrt{x}}{h}$ by $\sqrt{x+h} + \sqrt{x}$. Then investigate how the factorisation

$$a^q - b^q = (a - b)\left(a^{q-1} + a^{q-2}b + \cdots + b^{q-1}\right)$$

extends the method to $n = \frac{1}{q}$ for any positive integer q, and how you might then reach every positive rational power. [3]

C4. Where the derivative of sine comes from. Section 13.1 proved $\lim_{x \to 0} \frac{\sin x}{x} = 1$ by comparing three areas, and §13.5 used it to differentiate $\sin x$. Here you rebuild both arguments in full, with every inequality justified. Angles are in radians throughout. You may

use the *squeeze principle*: if $g(x) \le f(x) \le k(x)$ for all x close to a (with $x \neq a$), and $g(x) \to L$ and $k(x) \to L$ as $x \to a$, then $f(x) \to L$ as well.

(a) Let $0 < x < \frac{\pi}{2}$. On the unit circle with centre O, let $A = (1, 0)$ and $P = (\cos x, \sin x)$, and let the line OP meet the tangent to the circle at A at the point T. Show that $T = (1, \tan x)$. By comparing the areas of triangle OAP, sector OAP and triangle OAT, show that

$$\sin x < x < \tan x.$$

[4]

(b) Hence show that $\cos x < \dfrac{\sin x}{x} < 1$ for $0 < x < \frac{\pi}{2}$, and explain why the same inequality holds for $-\frac{\pi}{2} < x < 0$. Deduce that $\lim\limits_{x \to 0} \dfrac{\sin x}{x} = 1$. [4]

(c) Show that $\dfrac{\cos x - 1}{x} = \dfrac{-\sin^2 x}{x(\cos x + 1)}$. Hence show that $\lim\limits_{x \to 0} \dfrac{\cos x - 1}{x} = 0$. *(Hint: multiply above and below by* $\cos x + 1$*)* [4]

(d) The compound-angle identity gives $\sin(x + h) = \sin x \cos h + \cos x \sin h$. Use it to show that

$$\frac{\sin(x + h) - \sin x}{h} = \sin x \left(\frac{\cos h - 1}{h} \right) + \cos x \left(\frac{\sin h}{h} \right).$$

[3]

(e) Hence prove from first principles that $\dfrac{d}{dx} \sin x = \cos x$. [3]

(f) Section 13.8 evaluates $\lim\limits_{x \to 0} \dfrac{\sin x}{x}$ in one line by l'Hôpital's rule, and calls that calculation a consistency check and not a proof. Explain, with reference to part (e), exactly where the argument would be circular. [2]

(g) Put $x = \frac{\pi}{n}$ in part (a), where $n \ge 3$ is an integer, to show that $n \sin \frac{\pi}{n} < \pi < n \tan \frac{\pi}{n}$, and interpret the two bounds as half the perimeters of regular n-sided polygons drawn inside and outside a circle of radius 1. Archimedes used these bounds with $n = 96$. Investigate how quickly the bounds close in on π as n is doubled, and how the bounds for $2n$ sides can be found from those for n sides without knowing π. [3]