STATISTICS AND PROBABILITY

Chapter 11

Probability

11

SYLLABUS 1.10 4.5 4.6 4.11 4.13

A health service screens everyone in an age group for a rare disease. The test is a good one. Of every 100 people who have the disease, it correctly identifies 99. Of every 100 people who do not have it, it correctly gives a negative result for 95. About one person in a hundred has the disease.

Your result is positive. How likely is it that you have the disease?

A common answer is about 99%. In published studies, many doctors who were asked a question of this kind gave the same answer. The correct answer is about 17%. The test is not faulty and none of the numbers are unusual. The error is in the reasoning. That 99% describes the test *when the disease is present*. The quantity you actually want is different: how likely the disease is *now that the result is positive*. Those two numbers are not the same, and this chapter develops the rule that connects them.

Probability puts a number on how likely an event is. With that number you can judge a risk and base a decision on evidence instead of a feeling. But it is easy to give a correct answer to the wrong question. Two habits prevent that mistake, and both are practised throughout the chapter. The first is to state precisely which event you are asking about. The second is to know where a probability came from. Some are calculated from symmetry, such as the faces of a fair die. Others are measured by counting what happened in many trials. The two kinds are found in different ways, and they can be trusted in different ways.

By the end of this chapter you will be able to estimate a probability from data and find an expected number of occurrences, find probabilities from Venn diagrams, tree diagrams, tables and counting, find conditional probabilities and test events for independence, and use Bayes' theorem to reverse a conditional probability. When we return to the test in the last section, you will be able to find its answer with one formula.

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11.1 The Language of Probability

Probability measures how likely something is, on a scale from 0 (impossible) to 1 (certain). To find it, we need to be precise about what "something" means.

Five words are used precisely from here on. A die illustrates each one.

- **Experiment**: any process with an uncertain result. Rolling a die is an experiment.
- **Trial**: one repetition of the experiment. One roll is one trial.
- **Outcome**: the result of a single trial, such as a 3.
- **Sample space** U: the set of all possible outcomes. For one die, $U = \{1, 2, 3, 4, 5, 6\}$. A sample space is not a sample in the sense of Ch. 10: it lists every possible outcome, not a part of a population.
- **Event**: any subset of the sample space, that is, a collection of outcomes you are interested in. "An even number" is the event $\{2, 4, 6\}$.

When every outcome is equally likely, you can find a probability by counting: the fraction of outcomes that belong to the event.

FORMULA 11.1 **Equally likely outcomes** FORMULA BOOKLET · 4.5

$$P(A) = \frac{n(A)}{n(U)}$$

Here $n(A)$ is the number of outcomes in event A, and $n(U)$ the number in the whole sample space. Every probability lies between 0 and 1: $0 \le P(A) \le 1$.

EXAMPLE 11.1

A fair die is rolled once. Find the probability of (a) an even number, (b) a number greater than 4.

SOLUTION

The sample space has $n(U) = 6$ equally likely outcomes, so

$$P(\text{even}) = \frac{n(\{2,4,6\})}{6} = \frac{3}{6} = \boxed{\frac{1}{2}}, \qquad P(\text{greater than } 4) = \frac{n(\{5,6\})}{6} = \frac{2}{6} = \boxed{\frac{1}{3}}$$

Counting works only because the outcomes are equally likely.

11.1.1 Relative frequency

When there is no **symmetry** (a reason, such as identical faces, for every outcome to be equally likely) to count with, the probability can be estimated by experiment. Repeat the trial many

times and record how often the event occurs. The fraction of trials in which it happens is the **relative frequency** of the event, and it is an experimental estimate of the probability.

KEY 11.2 **Relative frequency**

After n trials,

$$P(A) \approx \frac{\text{number of trials in which } A \text{ occurred}}{n}$$

The estimate is rough when n is small, and it becomes more stable as n grows. There are two kinds of probability in this chapter. A *theoretical* probability comes from the structure of the experiment, such as the symmetry of a fair die. An *experimental* probability comes from data.

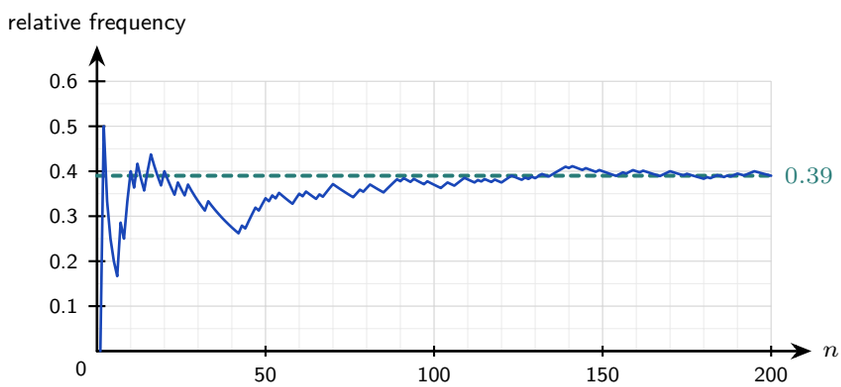


Figure 11.1 The relative frequency of "point-up" after each of 200 simulated drops of a drawing pin. The estimate changes a lot while n is small and stays close to 0.39 as n grows.

EXAMPLE 11.2

A drawing pin is dropped 200 times and lands point-up 78 times. Estimate the probability that a dropped pin lands point-up.

SOLUTION

There is no symmetry argument for a drawing pin, so use the relative frequency:

$$P(\text{point-up}) \approx \frac{78}{200} = \boxed{0.39}$$

A further 200 drops would probably not give exactly 78 again.

11.1.2 The complement

Sometimes the event you want is difficult to count, but its opposite is easy. The **complement** of A, written A', is the event that A does *not* happen. Since A either happens or it does not, their probabilities must add to 1.

FORMULA 11.3 **Complementary events** FORMULA BOOKLET · 4.5

$$P(A) + P(A') = 1$$

Rearranged: $P(A') = 1 - P(A)$.

When a question contains the phrase "at least one," the complement is usually the quicker method.

EXAMPLE 11.3

A fair coin is tossed three times. Find the probability of getting at least one head.

SOLUTION

Compare the direct count with the complement. Counting every way to get one, two, or three heads is possible but slow. The complement of "at least one head" is "no heads at all," a single outcome: TTT.

$$P(\text{at least one head}) = 1 - P(\text{TTT}) = 1 - \frac{1}{8} = \boxed{\frac{7}{8}}$$

One subtraction replaces a count of seven outcomes. The saving becomes far greater as the number of tosses increases.

11.1.3 Expected number of occurrences

A probability also gives the number of times an event should occur, on average.

KEY 11.4 **Expected number of occurrences**

If an event has probability p and the experiment is repeated n times, the **expected number of occurrences** is

$$\text{expected number} = np.$$

This is an average, not a guaranteed count. If you repeated the whole set of n trials many times and averaged the counts, that average would be close to np.

The expected number need not be a whole number. For 50 rolls of a fair die, the expected number of sixes is $50 \times \frac{1}{6} \approx 8.33$, although every actual count is a whole number.

The link to relative frequency holds when the trials are independent $(\text{the result of one does not change the probabilities for another})$ and p is the same on every trial. Then, over many trials, the fraction of trials in which the event occurs is usually close to p. The count itself may still be some distance from np, but that distance becomes small compared with n as n grows.

In Ch. 12 the same np returns as the mean of the binomial distribution, $E(X) = np$.

EXAMPLE 11.4

A fair die is rolled 60 times. Find the expected number of sixes.

SOLUTION

Each roll gives a six with probability $\frac{1}{6}$, so over 60 rolls:

$$\text{expected sixes} = 60 \times \frac{1}{6} = \boxed{10}$$

YOUR TURN 11.1 The Language of Probability

Equally likely outcomes

1.

(a) A fair die is rolled. Find the probability that the number is odd.

(b) A card is drawn from a pack of 52. Find the probability that it is a heart.

(c) A bag holds 4 red, 6 green and 5 blue counters. Find the probability that a counter drawn at random is green.

(d) A letter is chosen at random from the word STATISTICS. Find the probability that it is an S.

(e) A fair spinner with eight equal sectors, numbered 1 to 8, is spun. Find the probability that it lands on a multiple of 3.

2. List the sample space in each case.

(a) Two fair coins are tossed. List the sample space, and find the probability of exactly one head.

(b) Three fair coins are tossed. List the sample space, and find the probability of exactly two heads.

Relative frequency and expected number

3. A spinner has five sectors numbered 1 to 5, but it is not fair. In 400 spins it lands on 3 exactly 96 times.

(a) Estimate the probability of landing on 3.

(b) Using that estimate, find the expected number of 3s in 250 spins.

(c) Explain why the probability in Question 2(a) can be found without any experiment, while the one in part (a) of this question cannot.

4.

(a) A fair die is rolled 90 times. Find the expected number of 5s.
(b) Records show that a train was late on 45 of the last 300 days. Estimate the probability that it is late on a given day, and the expected number of late days in a month of 20 working days.

The complement

5.

(a) Given $P(A) = 0.35$, find $P(A')$.
(b) A letter is chosen at random from the word PROBABILITY. Find the probability that it is not a B.
(c) A fair coin is tossed four times. Find the probability of at least one tail.
(d) A fair die is rolled twice. Find the probability that at least one of the rolls is not a six.

Concept check

6. A fair coin has landed heads five times in a row. A student says: "*the next toss is more likely to be tails, because the results have to even out.*" Explain why the student is wrong.

7. A student writes: "*the probability of a six on one roll is* $\frac{1}{6}$*, so the probability of at least one six in six rolls is* $6 \times \frac{1}{6} = 1$." Identify the error, and find the correct probability.

11.2 Counting the Outcomes HL

To use the formula $P(A) = \dfrac{n(A)}{n(U)}$, you must find the two numbers in it. For a die you list the outcomes. For a committee of four chosen from twelve people you cannot: there are 495 possible committees. The counting tools of Ch. 2, §2.8 let you use the formula at that scale, and this section uses them to find probabilities.

11.2.1 Which tool, and when

Three questions tell you which tool to use: *how many objects are you choosing from, how many are you taking, and does the order matter?* Your answer to the last one tells you which of the two formulas below to use.

Order matters when the positions are different from one another: first, second and third place; president, secretary and treasurer; the order of speakers at a meeting. Order does not matter when you end up with a set: a committee, a hand of cards, six lottery numbers. Choosing Ali then Ben gives the same committee as choosing Ben then Ali, and a count that treats those as different has counted the committee twice.

Both formulas use $n!$, read "n factorial", the product $n(n-1)(n-2)\cdots 2\times 1$. Choosing r objects from n distinct objects:

$$\text{order matters:}\quad {}^nP_r=\frac{n!}{(n-r)!},\qquad \text{order does not:}\quad {}^nC_r=\frac{n!}{r!\,(n-r)!}.$$

Each unordered selection of r objects can be arranged in $r!$ orders, so ${}^nP_r = r!\times {}^nC_r$. This factor $r!$ is the only difference between the two formulas.

The first formula counts the choices one position at a time. There are n choices for the first position, $n-1$ for the second, and so on down to $n-r+1$ for the rth. Their product is $n(n-1)\cdots(n-r+1)$. This equals $\frac{n!}{(n-r)!}$, because dividing $n!$ by $(n-r)!$ removes the factors from $n-r$ down to 1. Ch. 2, §2.8 derives both formulas in full.

CAUTION Read the question for a word that tells you the order matters: *arrange*, *rank*, *in a row*, *first and second*. Without one, it is usually a combination.

Order matters?	Formula	Signal words	Examples
yes	${}^nP_r$	arrange, rank, first and second	medal places, officers
no	${}^nC_r$	choose, select, committee	committee, hand of cards, lottery numbers

11.2.2 Probability by counting

Count the sample space with one method, count the event with the same method, and divide.

To count the event, split the objects into the groups that the question names. Choose the required number from each group with ${}^nC_r$. Then multiply the counts, because the event needs one choice from each group. Many probability questions that need counting have this shape. Take the rest of the selection only from the objects outside the named group. For exactly two kings in five cards, the other three come from the 48 non-kings. Choosing them from all 50 remaining cards would also count hands with more kings.

EXAMPLE 11.5

A committee of 4 is chosen at random from 7 girls and 5 boys. Find the probability that it contains exactly 3 girls.

SOLUTION

A committee is a set, so order does not matter and both counts are combinations. The sample space is every committee of 4 from the 12 students:

$$n(U) = {}^{12}C_4 = 495.$$

For the event, read the requirement as an *and*: 3 of the 7 girls *and* 1 of the 5 boys. When an event needs one choice *and* another, the counts multiply Ch. 2, §2.8:

$$n(A) = {}^{7}C_3 \times {}^{5}C_1 = 35 \times 5 = 175.$$

So

$$P(\text{exactly 3 girls}) = \frac{175}{495} = \boxed{\frac{35}{99}} \approx 0.354$$

TIP Write down $n(U)$ before anything else. This makes you decide what one outcome *is*, and both counts must then use that same kind of outcome.

11.2.3 At least one, again

The complement of §11.1 saves the most work here. "At least one" covers many cases, while "none" needs only one count. When objects are drawn one at a time without replacement and the question does not ask about the order of the draws, each result is a set of objects, so ${}^{n}C_{r}$ counts it.

EXAMPLE 11.6

A box holds 20 components, of which 3 are defective. Four are drawn at random **without replacement** (not put back). Find the probability that at least one is defective.

SOLUTION

Counting the cases directly means one defective, or two, or three: three separate counts. The complement is one count:

$$\begin{aligned} P(\text{none defective}) &= \frac{{}^{17}C_4}{{}^{20}C_4} \\ &= \frac{2380}{4845} \\ &= \frac{28}{57}. \end{aligned}$$

Therefore

$$P(\text{at least one defective}) = 1 - \frac{28}{57} = \boxed{\frac{29}{57}} \approx 0.509$$

Only three components in twenty are defective, yet a sample of four is more likely than not to contain one.

11.2.4 When order does matter

When the positions in a question are different from one another, count the sample space in order with ${}^nP_r$, and count the event in order as well. You may instead count both without order, using ${}^nC_r$. The answer is the same, because each count without order is the ordered count divided by the same $r!$. What you may not do is count one in order and the other without.

EXAMPLE 11.7

Eight athletes run a final, and all orders of finishing are equally likely. Find the probability that a particular three of them take the first three places, in some order.

SOLUTION

Here the positions are distinguishable, so the sample space is ordered:

$$n(U) = {}^8P_3 = 336$$

ordered ways to fill the three medal positions. The three named athletes can fill them in $3! = 6$ orders, so

$$P = \frac{6}{336} = \boxed{\frac{1}{56}}$$

The same answer comes from counting unordered: ${}^8C_3 = 56$ possible medal-winning sets, one of which is the named three. Either tool gives the right answer, provided the numerator and the denominator use the same one.

The binomial distribution of Ch. 12 is this section's ${}^nC_r$ applied to repeated trials.

YOUR TURN 11.2 Counting the Outcomes HL

Choosing the counting tool

8. In each part decide whether order matters, find $n(U)$ and then the probability.

(a) Three of nine books are chosen at random to take on holiday. Find the probability that one particular book is among them.

(b) Gold, silver and bronze medals are won by three of nine finalists, and all finishing orders are equally likely. Find the probability that Lena wins gold and her sister Mia wins silver.

(c) Five people sit in a row of five seats in a random order. Find the probability that Ana and Ben sit next to each other.

Probability by counting groups

9. A team of 5 is chosen at random from 6 women and 4 men.

(a) Find the number of possible teams.
(b) Find the probability that the team contains exactly 3 women.
(c) Find the probability that the team is all women.
(d) Find the probability that the team contains at least one man.

At least one, again

10. A box contains 15 light bulbs, of which 4 are faulty. Three are chosen at random without replacement.

(a) Find the probability that none is faulty.
(b) Find the probability that at least one is faulty.
(c) Find the probability that exactly two are faulty.

When order does matter

11. A four-digit code is made from four different digits chosen from $1, 2, \ldots, 9$, and every such code is equally likely.

(a) Find the probability that the digits of the code are in increasing order.
(b) Find the probability that the code starts with 1 and ends with 9.

12. Four of the 12 students in a club are chosen at random to be president, vice-president, secretary and treasurer. Find the probability that Maya and Leo are both given a post,

(a) counting ordered selections for both $n(U)$ and $n(A)$;
(b) counting unordered selections for both, and check that the answers agree.

Concept check

13. Three of nine books are chosen at random. To find the probability that one particular book is chosen, a student writes: "$n(A) = {}^8C_2 = 28$ *and* $n(U) = {}^9P_3 = 504$, *so the probability is* $\frac{28}{504} = \frac{1}{18}$." Identify the error, and find the correct probability.

11.3 Combining Events

Many questions involve more than one event. You want the probability that a student plays football *or* tennis, or that a card is a king *and* a heart. Two operations combine them: the **union** $A \cup B$ ("A or B or both") and the **intersection** $A \cap B$ ("A and B together").

The word *or* needs care, because everyday English and mathematics do not use it the same way. "Tea or coffee?" offers you one drink, not both. In mathematics $A \cup B$ always includes the case where both happen. This is called the **inclusive or**, and it is the reason the addition rule below needs a correction term. Questions that mean one but not both usually say so explicitly.

A **Venn diagram** makes the structure visible. The rectangle is the sample space, written U and called the **universal set**: everything under discussion lies inside it. Each circle is an event, and every point of the diagram lies in exactly one of four regions. Each region has a name made from intersections and complements. For example, $F \cap T'$ means "in F and not in T": the students who play football but not tennis. The diagram below is for a class of 30 students, of whom 18 play football (F), 12 play tennis (T) and 7 play both.

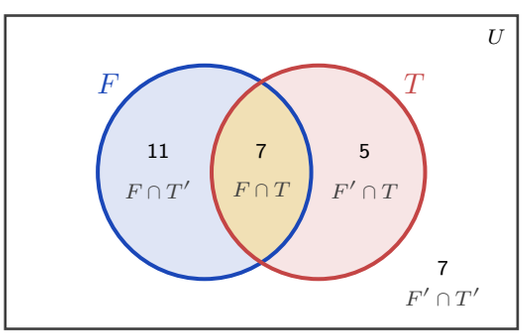


Figure 11.2 The four regions for the class of 30: football only $F \cap T'$, both $F \cap T$, tennis only $F' \cap T$, and neither $F' \cap T'$. The four counts add to 30.

Fill a Venn diagram from the middle outwards. Put the overlap in first, then subtract it from each circle total to get the "only" regions, then use the grand total to find the outside. If you write a circle total, such as the 18 football players, straight into the "only" region, the 7 students in the overlap are counted twice: once there and once in the middle. The addition rule is designed to correct the same mistake.

11.3.1 The addition rule

If you add $P(A)$ and $P(B)$ to find $P(A \cup B)$, you count the overlap twice: once in A, once in B. Subtracting it once corrects the double count.

FORMULA 11.5 **Combined events** FORMULA BOOKLET · 4.6

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

When two events cannot happen together, they are **mutually exclusive**: their intersection contains no outcomes at all, written $A \cap B = \emptyset$, where $\emptyset$ is the **empty set**. Then $P(A \cap B) = 0$ and the rule simplifies.

FORMULA 11.6 **Mutually exclusive events** FORMULA BOOKLET · 4.6

For mutually exclusive events:

$$P(A \cup B) = P(A) + P(B)$$

Events are **exhaustive** when together they cover the whole sample space, so that at least one of them must happen. Their union then has probability 1. On a die, "even" and "odd" are both mutually exclusive and exhaustive: no roll is in both, and every roll is in one of them. An event A and its complement A' always form such a pair. When events are mutually exclusive and exhaustive, their probabilities add to 1.

EXAMPLE 11.8

In the class of 30 above, 18 students play football, 12 play tennis, and 7 play both. A student is chosen at random. Find the probability that the student plays (a) football or tennis, (b) neither.

SOLUTION

(a) The addition rule gives the answer directly:

$$P(F \cup T) = \frac{18}{30} + \frac{12}{30} - \frac{7}{30} = \boxed{\frac{23}{30}}$$

(b) The probability of playing neither is the complement of "football or tennis":

$$P(\text{neither}) = 1 - \frac{23}{30} = \boxed{\frac{7}{30}}$$

Three events need three circles, and the same instruction applies: start in the middle. The centre region belongs to all three events, and every other region is found by subtracting what has already been placed.

EXAMPLE 11.9

In a year group of 40 students, 20 study French, 18 study German and 15 study Spanish. Of these, 8 study French and German, 6 study German and Spanish, 7 study

French and Spanish, and 3 study all three. A student is chosen at random. Find the probability that the student studies (a) exactly one of the three languages, (b) none of them.

SOLUTION

Start with the 3 in the centre. The 8 who study French and German include those 3, so $8-3=5$ study those two and no other. In the same way $6-3=3$ study German and Spanish only, and $7-3=4$ study French and Spanish only.

Now the "only" regions. French has 20 altogether, of which $5+3+4=12$ are already placed, leaving 8 for French only. German gives $18-(5+3+3)=7$, and Spanish gives $15-(4+3+3)=5$.

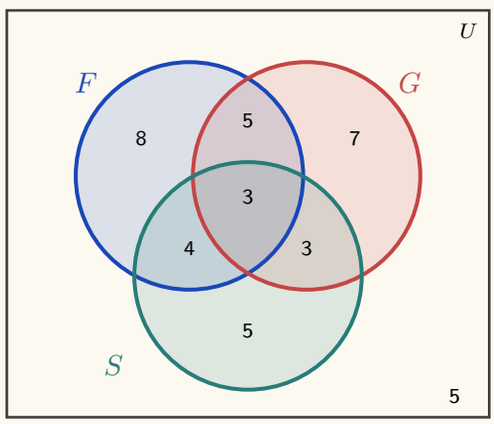


Figure 11.3 The three-circle diagram, filled from the centre outwards.

(a) Exactly one language means the three outer regions: $8+7+5=20$, so

$$P(\text{exactly one})=\frac{20}{40}=\boxed{\frac{1}{2}}$$

(b) The seven regions inside the circles total $8+7+5+5+4+3+3=35$, so 5 students study none:

$$P(\text{none})=\frac{5}{40}=\boxed{\frac{1}{8}}$$

No formula was used. Once the diagram is filled, every probability is a count divided by 40.

11.3.2 Sample space diagrams and two-way tables

When an experiment has two parts, drawing the whole sample space is often faster than any formula. A **sample space diagram** plots one part on each axis. For two dice, the 36 equally likely outcomes form a grid, and any event is just a set of cells to count.

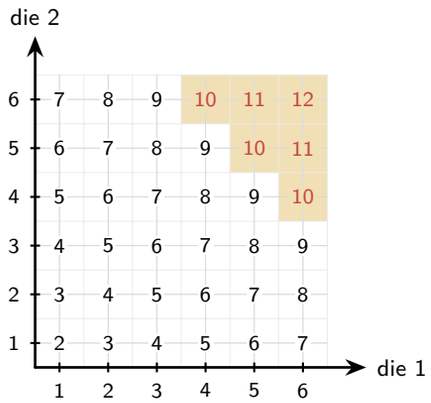


Figure 11.4 The 36 equally likely outcomes of rolling two dice, each labelled with its total. The six cells with a total of at least 10 are shaded.

Six outcomes have a total of at least 10, so

$$P(\text{total} \geq 10) = \frac{6}{36} = \frac{1}{6}.$$

A **two-way table** (a table of counts split by two characteristics) does the same for data about a group of people or objects. It splits the group by two characteristics at once. Every probability with no condition is a cell, a row total or a column total divided by the grand total. With a condition, the denominator becomes the total of the row or column that the condition names, as §11.4 shows.

EXAMPLE 11.10

The table shows the 100 students of a year group by age group and club membership.

	Club	No club	Total
Junior	14	26	40
Senior	21	39	60
Total	35	65	100

A student is chosen at random. Find the probability that the student (a) is a junior in a club, (b) is in a club or is a junior.

SOLUTION

(a) Read the single cell where the two conditions meet:

$$P(J \cap C) = \frac{14}{100} = \boxed{0.14}$$

(b) Apply the addition rule with the row and column totals:

$$P(J \cup C) = \frac{40}{100} + \frac{35}{100} - \frac{14}{100} = \frac{61}{100} = \boxed{0.61}$$

YOUR TURN 11.3 Combining Events

The addition rule

14.

(a) Given $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$, find $P(A \cup B)$.

(b) Events A and B are mutually exclusive, with $P(A) = 0.6$ and $P(B) = 0.3$. Find $P(A \cup B)$.

(c) Given $P(A \cup B) = 0.8$, $P(A) = 0.5$ and $P(B) = 0.4$, find $P(A \cap B)$.

(d) Given $P(A) = 0.3$, $P(B) = 0.4$ and $P(A \cup B) = 0.7$, determine whether A and B are mutually exclusive.

(e) A and B are mutually exclusive and exhaustive, with $P(A) = 0.35$. Find $P(B)$.

15.

(a) A card is drawn from a pack of 52. Find the probability that it is neither a heart nor a picture card (jack, queen or king).

(b) In a group of 40 people, 25 like coffee, 20 like tea and 10 like both. Find the probability that a person chosen at random likes coffee or tea.

(c) For the group in part (b), find the probability that the person likes neither.

Venn diagrams

16.

(a) Two events satisfy $n(A \text{ only}) = 8$, $n(B \text{ only}) = 5$, $n(A \cap B) = 4$ and $n(\text{neither}) = 3$. Find $P(A)$.

(b) Given $n(U) = 50$, $n(A) = 22$, $n(B) = 18$ and $n(A \cap B) = 7$, find $n(A \cap B')$ and $n(A' \cap B')$.

17. A survey of 50 households finds that 28 have a cat, 22 have a dog and 15 have a rabbit. Ten have a cat and a dog, 6 have a dog and a rabbit, 5 have a cat and a rabbit, and 3 have all three.

(a) Draw a Venn diagram showing the number of households in each region.

(b) A household is chosen at random. Find the probability that it has at least two of the three kinds of pet.

(c) Find the probability that it has none of the three.

Sample space diagrams and two-way tables

18. Two fair dice are rolled.

(a) Find the probability that the total is 7.

(b) Find the probability that the two numbers differ by exactly 1.

(c) Find the probability that the two dice show the same number.

19. The table shows 80 people classified by sex and by writing hand.

	Left	Right	Total
Male	7	33	40
Female	5	35	40
Total	12	68	80

One person is chosen at random.

(a) Find the probability that the person is left-handed.

(b) Find the probability that the person is female and right-handed.

(c) Find the probability that the person is female or left-handed.

Concept check

20. A card is drawn from a pack of 52. A student writes:
"$P(\textit{heart or king}) = \frac{13}{52} + \frac{4}{52} = \frac{17}{52}$." Identify the error, and find the correct probability.

11.4 Conditional Probability and Independence

New information can change a probability. The chance (probability) that a random person has a particular illness is small, but the chance *given* that they tested positive is different. **Conditional probability** is the probability of one event once you know another has happened.

Knowing B has occurred **reduces the sample space** to B alone. Within that smaller set, you ask what fraction also lies in A.

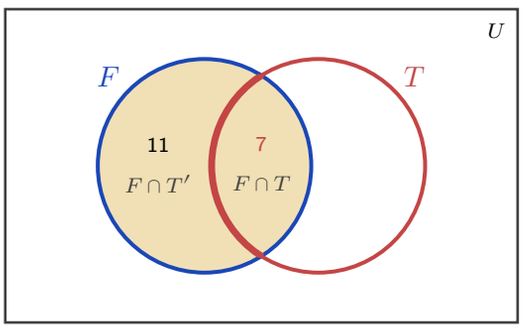


Figure 11.5 Given F, only the 18 students inside F remain. Of these, 7 are also in T, so $P(T \mid F) = \frac{7}{18}$.

FORMULA 11.7 **Conditional probability** FORMULA BOOKLET · 4.6

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

The formula needs $P(B) \neq 0$.

Read $P(A \mid B)$ as "the probability of A given B." The denominator $P(B)$ is the probability of the new, restricted sample space B; the numerator is the probability of the part of B where A also holds.

- Conditioning is not symmetric (swapping the two events changes the answer). In general $P(A \mid B)$ and $P(B \mid A)$ **are different numbers**. They have the same numerator, $P(A \cap B)$, but different denominators, $P(B)$ and $P(A)$. The event after the bar is the one you already know.

EXAMPLE 11.11

In the class of 30 from before, a student chosen at random is known to play football. Find the probability that the student also plays tennis.

SOLUTION

Restrict to the 18 football players. Of those, 7 also play tennis:

$$P(T \mid F) = \frac{P(T \cap F)}{P(F)} = \frac{7/30}{18/30} = \boxed{\frac{7}{18}}$$

This differs from $P(F \mid T) = \frac{7}{12}$, because the denominator is now the 12 tennis players.

11.4.1 The multiplication rule

Rearranging the conditional formula gives a rule for the probability that two events both happen.

KEY 11.8 **Multiplication rule**

$$P(A \cap B) = P(A)\,P(B \mid A)$$

In words: the chance of both is the chance of the first times the chance of the second given the first. This is exactly how you move along the branches of a tree diagram, as the next section shows.

EXAMPLE 11.12

A drawer holds nine pens, four of which are red. Two pens are taken at random, one after the other, without replacement. Find the probability that both are red.

SOLUTION

Let A be "the first pen is red" and B "the second pen is red". Four of the nine pens are red, so $P(A) = \frac{4}{9}$.

For the second factor, assume A has happened. Eight pens remain and three of them are red, so $P(B \mid A) = \frac{3}{8}$. The multiplication rule combines the two:

$$\begin{aligned} P(A \cap B) &= P(A)\,P(B \mid A) \\ &= \frac{4}{9} \times \frac{3}{8} \\ &= \frac{12}{72} \\ &= \boxed{\frac{1}{6}} \end{aligned}$$

The second factor is $\frac{3}{8}$ and not $\frac{4}{9}$ because the first pen is not put back. Removing a red pen reduces both the number of red pens and the total, so both parts of the fraction fall by one.

Four words appear in many probability questions, and each one tells you which operation to use.

Word	What it tells you to do	In symbols
given	restrict the sample space to that event	$P(A \mid B)$
and	both happen: the intersection	$P(A \cap B)$
or	one or the other or both: the union	$P(A \cup B)$
at least one	everything except "none": use the complement	$1 - P(\text{none})$

To find the probability that both A and B happen, multiply $P(A)$ by the probability of B *given* A: $P(A \cap B) = P(A)\,P(B \mid A)$. You may use $P(B)$ itself only when the events are independent, as the next subsection shows.

11.4.2 Independence

Two events are **independent** when knowing that one has happened does not change the probability of the other, so $P(A \mid B) = P(A)$. Rearranging $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ gives the multiplication rule with the letters the other way round, $P(A \cap B) = P(B)\,P(A \mid B)$. Substituting $P(A \mid B) = P(A)$ gives $P(A \cap B) = P(A)\,P(B)$. This test treats A and B in the same way, so it does not matter which event you call A.

FORMULA 11.9 **Independent events** FORMULA BOOKLET · 4.6

A and B are **independent** if and only if

$$P(A \cap B) = P(A)\,P(B).$$

Provided $0 < P(B) < 1$, the test is equivalent to

$$P(A \mid B) = P(A) = P(A \mid B').$$

This conditional form is not printed in the booklet, but it says directly what independence means: whether or not B happened, the probability of A is the same. To test independence, compute both sides of either form and compare them.

The B' part follows by subtraction. Since A happens either with B or with B', $P(A \cap B') = P(A) - P(A \cap B)$. If $P(A \cap B) = P(A)\,P(B)$, this gives $P(A \cap B') = P(A) - P(A)\,P(B) = P(A)\,P(B')$. Dividing by $P(B')$ gives $P(A \mid B') = P(A)$.

EXAMPLE 11.13

A fair coin is tossed and a fair die is rolled at the same time, so that the 12 combined outcomes are equally likely. Let H be "heads" and S be "rolling a six." Are H and S independent?

SOLUTION

Here $P(H) = \frac{1}{2}$ and $P(S) = \frac{1}{6}$. The outcome "heads and a six" is one of the 12 equally likely combined outcomes:

$$P(H \cap S) = \frac{1}{12} = \frac{1}{2} \times \frac{1}{6} = P(H)\,P(S).$$

The two sides match, so the events are independent, which is what you would expect: the coin cannot affect the die.

The two-way table of §11.3 passes the same test. There, $P(C \mid J) = \frac{14}{40} = 0.35$ and $P(C) = \frac{35}{100} = 0.35$: knowing a student is a junior does not change the chance they are in a club, so, in this year group, age group and club membership are independent.

Exact independence like this is unusual in real data. Two characteristics recorded on the same group of people are often linked at least a little, and then the two sides of the test are not equal.

EXAMPLE 11.14

The table classifies the 200 students of a year group by whether they take music and whether they take drama.

	Drama	No drama	Total
Music	30	50	80
No music	20	100	120
Total	50	150	200

A student is chosen at random. Determine whether the events M (takes music) and D (takes drama) are independent.

SOLUTION

Read the two single-event probabilities from the totals, and the intersection from the cell where the two conditions meet:

$$P(M) = \frac{80}{200} = 0.4, \qquad P(D) = \frac{50}{200} = 0.25, \qquad P(M \cap D) = \frac{30}{200} = 0.15.$$

Compare the product with the intersection:

$$P(M)\,P(D) = 0.4 \times 0.25 = 0.10 \neq 0.15 = P(M \cap D),$$

so M and D are **not** independent.

The conditional form gives the same verdict and also shows the direction of the link. Restricting to the 80 music students gives $P(D \mid M) = \frac{30}{80} = 0.375$. For the year group as a whole, $P(D) = 0.25$. So knowing that a student takes music raises the probability of drama from a quarter to three eighths.

TIP Independence is an exact condition, so any difference between the two numbers, however small, means the events are not independent. Compare exact values, not rounded ones. Write both numbers down and then the verdict; a bare "not independent" with only one side computed earns little credit.

The verdict is about the group in the table only. A difference between proportions in a sample does not, by itself, establish dependence in a wider population.

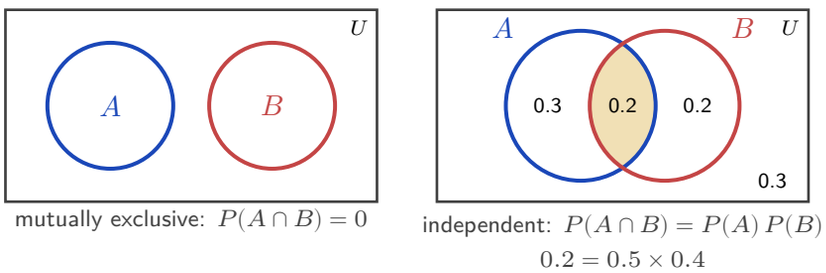


Figure 11.6 Separate circles mean the events cannot happen together. Independence needs an overlap of exactly the right size.

CAUTION Mutually exclusive and independent are different ideas. If two events with non-zero probability are mutually exclusive, then one happening prevents the other completely, so $P(A \mid B) = 0 \neq P(A)$ and the events cannot be independent. Test independence with $P(A \cap B) = P(A)\,P(B)$; do not assume it.

YOUR TURN 11.4 Conditional Probability and Independence

Conditional probability and the multiplication rule

21.

(a) Given $P(A \cap B) = 0.2$ and $P(B) = 0.5$, find $P(A \mid B)$.

(b) Given $P(A) = 0.6$ and $P(B \mid A) = 0.3$, find $P(A \cap B)$.

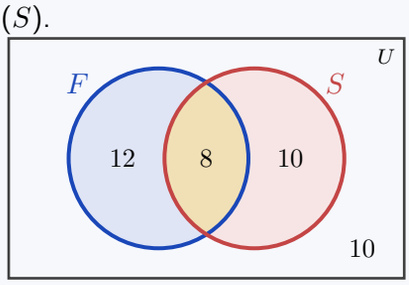
(c) Given $P(A) = 0.5$, $P(B) = 0.4$ and $P(A \cup B) = 0.7$, find $P(A \mid B)$.

(d) Two cards are drawn from a pack of 52 without replacement. Find the probability that both are kings.

(e) A card is drawn from a pack of 52. Given that it is a picture card, find the probability that it is a king.

(f) Two fair dice are rolled. Given that the first die shows an even number, find the probability that the total is 8.

(g) The Venn diagram shows the numbers of students in a group of 40 who study French (F) and Spanish (S).



A student is chosen at random. Find $P(F \cup S)$, $P(F \mid S)$ and $P(S \mid F')$.

(h) In another group of 30 students, 16 study French, 11 study Spanish and 5 study neither language. Draw a Venn diagram showing the number of students in each region, and find $P(F \mid S)$ for this group.

22. The table classifies 120 patients by the treatment they received and whether their condition improved.

	Improved	**Not improved**	**Total**
New treatment	36	24	60
Standard treatment	30	30	60
Total	66	54	120

One patient is chosen at random. Let I be "the condition improved" and N be "received the new treatment".

(a) Find $P(I \mid N)$.
(b) Find $P(N \mid I)$.
(c) Find $P(I \mid N')$, and use it to determine whether I and N are independent.

Independence

23.

(a) Events A and B are independent, with $P(A) = 0.5$ and $P(B) = 0.4$. Find $P(A \cap B)$.
(b) Given $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \cap B) = 0.2$, determine whether A and B are independent.
(c) Events A and B have $P(A) = 0.3$, $P(B) = 0.2$ and $P(A \cup B) = 0.5$. Show that A and B are mutually exclusive, and determine whether they are independent.

24. In a class of 25 students, 15 study Spanish, 10 study art and 6 study both.

(a) Given that a student studies Spanish, find the probability that they also study art.
(b) Given that a student studies art, find the probability that they also study Spanish.
(c) Determine whether studying Spanish and studying art are independent.

25. Events A and B have $P(A) = 0.4$, $P(B) = 0.5$ and $P(A \mid B) = 0.4$. Find $P(A \mid B')$, and state whether A and B are independent.

Concept check

26. Events A and B have $P(A) = 0.3$ and $P(B) = 0.4$, and are mutually exclusive. A student says: "*mutually exclusive events do not affect each other, so A and B are independent.*" Explain the difference between *mutually exclusive* and *independent*, and show that A and B are not independent.

11.5 Tree Diagrams

When events happen in sequence, a **tree diagram** shows every possible sequence of outcomes. A **path** is one route from the start of the tree to an end, with one branch for each stage. Each branch carries the probability of that step, given the steps before it. You then need only two rules. **Multiply along the branches** of a single path (the multiplication rule). **Add across paths** that satisfy the condition (the paths are mutually exclusive).

When each draw is made **with replacement**, the object goes back and the contents are mixed before the next draw. Every stage then has the same probabilities, and the stages are independent. Without replacement, the first outcome changes the probabilities on the second branches, so those branches need care.

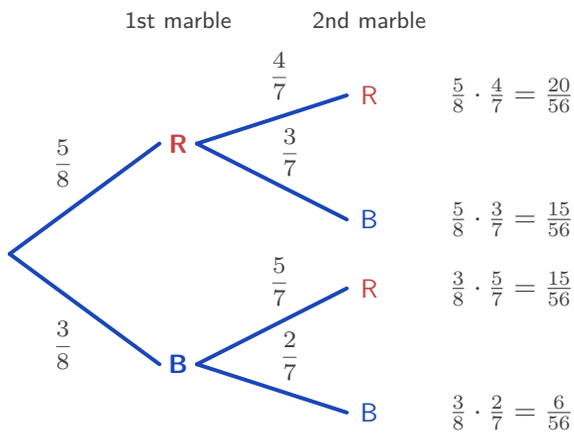


Figure 11.7 Two marbles drawn from 5 red and 3 blue without replacement.

EXAMPLE 11.15

A bag holds 5 red and 3 blue marbles, and two are drawn without replacement, as in the tree above. Find the probability that (a) both are red, (b) one of each colour is drawn, (c) at least one is blue.

SOLUTION

The second branches change because after drawing one red, only 4 reds remain out of 7 marbles.

(a) For both red, multiply along the top path:

$$P(\text{both red}) = \frac{5}{8} \times \frac{4}{7} = \frac{20}{56} = \boxed{\frac{5}{14}}$$

(b) "One of each" covers two paths, red-then-blue and blue-then-red; add them:

$$\begin{aligned} P(\text{one of each}) &= \frac{5}{8}\cdot\frac{3}{7} + \frac{3}{8}\cdot\frac{5}{7} \\ &= \frac{15+15}{56} \\ &= \frac{30}{56} \\ &= \boxed{\frac{15}{28}} \end{aligned}$$

(c) For "at least one blue," use the complement, which is "both red":

$$P(\text{at least one blue}) = 1 - \frac{5}{14} = \boxed{\frac{9}{14}}$$

All three rules appear here: multiply along a path, add across alternatives, and take the complement for "at least one."

EXAMPLE 11.16

The same bag holds 5 red and 3 blue marbles, but now the first marble is replaced before the second is drawn. Find the probability that both are red, and compare with the answer without replacement.

SOLUTION

The second draw has the same probabilities as the first, so

$$P(\text{both red}) = \frac{5}{8}\times\frac{5}{8} = \boxed{\frac{25}{64}} \approx 0.391$$

Without replacement the answer was $\frac{5}{14} \approx 0.357$, which is smaller. Removing a red marble leaves fewer reds for the second draw, so the second red becomes less likely. Replacing it removes that effect.

A tree diagram and a two-way table can show the same data. The first-stage branches are the row totals divided by the grand total. The second-stage branches are conditional probabilities: each cell divided by its own row total. Multiplying along a path gives one cell divided by the grand total.

EXAMPLE 11.17

Take the year group of 100 from §11.3: 40 juniors, of whom 14 are in a club, and 60 seniors, of whom 21 are. Show the data as a tree diagram, and show that each path product is one cell of the two-way table.

SOLUTION

Divide each row total by 100 for the first stage, and each cell by its row total for the second.

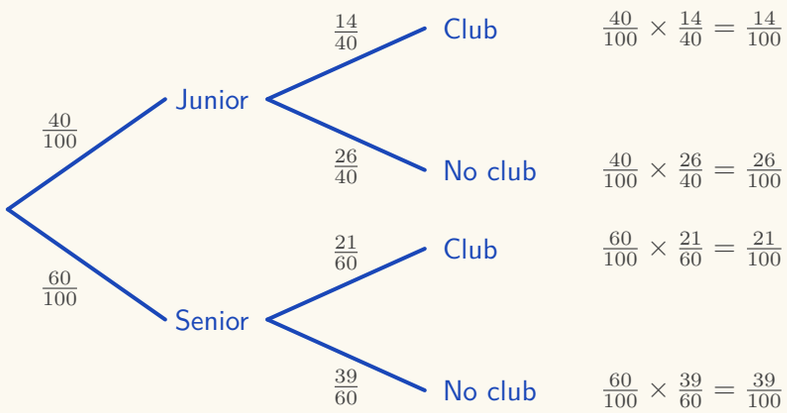


Figure 11.8 The year group of 100 as a tree. Each path product is one cell of the two-way table.

When you multiply along a path, the row total cancels and the result is the cell. The table shows the cells and leaves the conditional probabilities to be worked out; the tree shows the conditional probabilities and leaves the cells to be worked out.

When the same trial is repeated n times independently, as in draws with replacement, "at least one success" has a complement that is a single product, because "no successes at all" means failure every time.

KEY 11.10 **At least one**

For n independent trials in which the event has probability p each time,

$$P(\text{at least one}) = 1 - (1-p)^n.$$

EXAMPLE 11.18

A machine part fails on any given day with probability 0.02, independently of other days, and a failed part is replaced before the next day. Find the probability that it fails at least once during a 30-day month.

SOLUTION

The complement is a single product: the part survives all 30 days, each day with probability 0.98.

$$P(\text{at least one failure}) = 1 - (0.98)^{30} \approx 1 - 0.545 = \boxed{0.455}$$

A part that is safe on 98 days out of 100 still has a probability close to one half of failing within a month. A small daily probability, repeated over many days, gives a much larger probability of at least one failure.

11.5.1 Three stages without replacement

A third draw adds a third stage to the tree. Each branch still carries the probability of that step given everything before it, and without replacement the denominator falls by one at every draw.

The counting needs more care. With two draws, "one of each" had two paths. With three draws, an event such as "exactly two of the three" has three paths, because the one different draw can come first, second or last. Identify those paths before computing anything. Each of these paths uses the same denominators, and the same numerators in a different order, so all of them have the same probability. Evaluate one path and multiply by the number of orders.

EXAMPLE 11.19

A bag contains 7 green and 3 yellow counters. Three are drawn at random without replacement. Find the probability that (a) all three are green, (b) exactly two are green.

SOLUTION

(a) Follow one path. The bag has one counter fewer at each draw:

$$\begin{aligned} P(\text{GGG}) &= \frac{7}{10} \times \frac{6}{9} \times \frac{5}{8} \\ &= \frac{210}{720} \\ &= \boxed{\frac{7}{24}} \end{aligned}$$

(b) Exactly two green means one yellow, and the yellow may be drawn first, second or third. The three paths are YGG, GYG and GGY, with the yellow draw in red:

Path	1st draw	2nd draw	3rd draw	Product
YGG	$\frac{3}{10}$	$\frac{7}{9}$	$\frac{6}{8}$	$\frac{126}{720}$
GYG	$\frac{7}{10}$	$\frac{3}{9}$	$\frac{6}{8}$	$\frac{126}{720}$
GGY	$\frac{7}{10}$	$\frac{6}{9}$	$\frac{3}{8}$	$\frac{126}{720}$

Each numerator is the product of 3, 7 and 6 in some order, so each is 126. Each denominator is $10 \times 9 \times 8 = 720$. So every path has probability $\frac{126}{720}$, and adding across

the three paths gives

$$\begin{aligned} P(\text{exactly two green}) &= 3 \times \frac{126}{720} \\ &= \frac{378}{720} \\ &= \boxed{\frac{21}{40}} = 0.525 \end{aligned}$$

Count the orders first, because a missed path gives an answer that is too small.
The number of orders is ${}^3C_1 = 3$: the number of ways to choose which draw is yellow.
Counting gives the same answer: $\dfrac{{}^7C_2 \times {}^3C_1}{{}^{10}C_3} = \dfrac{63}{120} = \dfrac{21}{40}$.

YOUR TURN 11.5 Tree Diagrams

Two draws, with and without replacement

27.

(a) A bag holds 3 white and 2 black balls. Two are drawn with replacement. Find the probability that both are white.

(b) Repeat part (a) for two balls drawn without replacement.

(c) A box holds 6 working and 4 defective bulbs. Two are drawn without replacement. Find the probability that exactly one is defective.

Different probabilities on the second branches

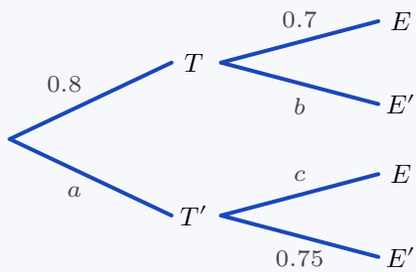
28. The probability that it rains on a given morning is 0.3. If it rains, the probability that Sam is late for school is 0.6; if it does not rain, the probability is 0.1.

(a) Find the probability that Sam is late.

(b) Find the probability that it rains and Sam is not late.

(c) Show the information for 100 mornings in a two-way table of rain against lateness. Check that the path product for "it rains and Sam is late" equals the matching cell divided by 100.

(d) The tree diagram below is partly completed. It shows whether Kenji's bus is on time (T) and whether he arrives early for work (E).



Write down the values of a, b and c, and find $P(E)$.

(e) On each of two days, Kenji's bus is on time with probability 0.8, independently of the other day. Draw a tree diagram for the two days, and find the probability that the bus is on time on exactly one of the two days.

Repeated independent trials

29.

(a) A machine works on any given day with probability 0.9, independently of other days. Find the probability that it works on two consecutive days.

(b) Ali and Bea take turns to roll a fair die, Ali first, and the first to roll a six wins. Find the probability that Ali wins with his second roll.

(c) Using the machine in part (a), find the probability that it fails on at least one of five consecutive days.

30. A component works with probability 0.95, independently of other components. A circuit contains four of them.

(a) Find the probability that all four components work.

(b) Find the probability that exactly one of the four components fails.

Three stages without replacement

31. A bag holds 4 red and 6 blue counters. Three are drawn without replacement.

(a) Find the probability that all three are blue.

(b) Find the probability that exactly two are red.

Concept check

32. A bag holds 3 red and 2 blue counters. Two are drawn without replacement. A student writes: "$P(\textit{one of each colour}) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$." Identify the error, and find the correct probability.

11.6 Bayes' Theorem HL

We can now return to the screening test from the start of the chapter. About one person in a hundred has the disease. The test correctly identifies 99% of those who have it, and wrongly reports a positive for 5% of those who do not. Suppose your result is positive.

The difficulty is that the test gives you $P(\text{positive} \mid \text{disease})$, but what you actually want is the reverse, $P(\text{disease} \mid \text{positive})$. Conditional probability is not symmetric, so these are different numbers. **Bayes' theorem** finds $P(B \mid A)$ when you know $P(A \mid B)$. In this sense it reverses the condition.

Read the letters the way the booklet uses them. B is the thing you want to know about, and A is the evidence you have observed. In the screening test, B is having the disease and A is the positive result.

The theorem is built from the conditional formula and the multiplication rule. The conditional formula, with the letters in this order, gives $P(B \mid A) = \frac{P(A \cap B)}{P(A)}$. The multiplication rule rewrites the numerator as $P(A \cap B) = P(B)\,P(A \mid B)$. For the denominator, A happens either together with B or together with B', and these two cases are mutually exclusive. So $P(A) = P(A \cap B) + P(A \cap B') = P(B)P(A \mid B) + P(B')P(A \mid B')$. Putting these together:

FORMULA 11.11 **Bayes' theorem** FORMULA BOOKLET · 4.13 HL

$$P(B \mid A) = \frac{P(B)\,P(A \mid B)}{P(B)\,P(A \mid B) + P(B')\,P(A \mid B')}$$

The formula needs $P(A) > 0$, so that the evidence is something that can happen, and $0 < P(B) < 1$, so that both branches of the first stage exist.

The numerator is the path that goes through B and ends in A. The denominator is the sum of all the paths that end in A: one through B and one through B'. You are asking what fraction of all the ways to observe A passed through B.

That description also gives the method. Written out in full the formula is long, but a tree diagram holds the same information and is quicker to set up.

1. Draw the first stage: B and B', with their probabilities.
2. Draw the second stage: A and A' after each of them, with the conditional probabilities.
3. Multiply along every path that ends in A.
4. Divide the product for the path through B by the total of all the paths that end in A.

TIP A correctly labelled tree is a complete method. Show the path products and the final division, and follow any method the question names.

The probability of B before you have any evidence, $P(B)$, is called the **base rate**. When the base rate is small, most of the people tested do not have the condition. The few **true positives** (people with the condition who test positive) can then be outnumbered by the **false positives** (people without the condition who test positive). So $P(B \mid A)$ **can be small even for an accurate test**.

You can also replace the probabilities with counts in an imagined group, such as $10\,000$ people. The counts give the same answer, and they show directly how many of the positive results come from each group.

EXAMPLE 11.20

In the screening test, let D be "has the disease" and $+$ be "tests positive." Given that $P(D) = 0.01$, $P(+ \mid D) = 0.99$ and $P(+ \mid D') = 0.05$, find $P(D \mid +)$, the probability that a person who tests positive has the disease.

SOLUTION

Draw the tree first, then multiply along every path.

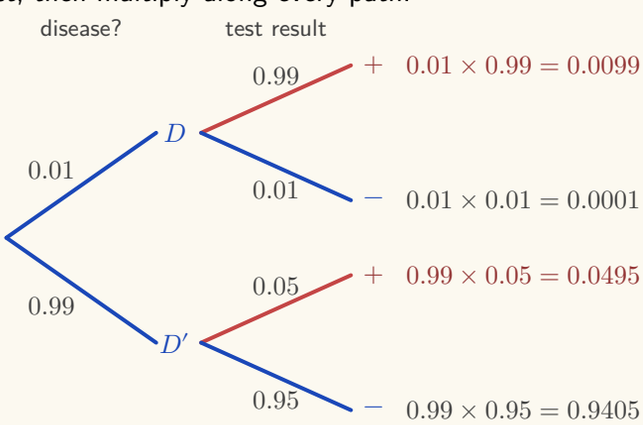


Figure 11.9 The screening test as a tree. The two second-stage branches that end in a positive result are shown in red.

Only the two paths that end in $+$ match the evidence, so they are the only ones that count. The path through D divided by the total of both gives

$$P(D \mid +) = \frac{(0.01)(0.99)}{(0.01)(0.99) + (0.99)(0.05)} = \frac{0.0099}{0.0099 + 0.0495} = \frac{0.0099}{0.0594} \approx \boxed{0.167}$$

So a person who tests positive has only about a 17% chance of having the disease. The reason is the base rate.

Imagine $10\,000$ people screened. Of the 100 who have the disease, 99 test positive. Of the 9900 who do not, 495 test positive.

	+	−	Total
Disease	99	1	100
No disease	495	9405	9900
Total	594	9406	$10\,000$

The answer is the 99 in the first column divided by that column's total, 594: $\frac{99}{594} = \frac{1}{6} \approx 17\%$. The denominator is everyone who tests positive, not everyone who has the disease.

11.6.1 Three alternatives

When the first stage has three possibilities instead of two, the denominator is the sum of the three paths that end in A. The structure is identical, and the booklet prints this version too.

FORMULA 11.12 **Bayes' theorem for three events** FORMULA BOOKLET · 4.13 HL

$$P(B_i \mid A) = \frac{P(B_i)\,P(A \mid B_i)}{P(B_1)\,P(A \mid B_1) + P(B_2)\,P(A \mid B_2) + P(B_3)\,P(A \mid B_3)}$$

B_1, B_2, B_3 must be mutually exclusive and exhaustive (exactly one of them happens), and $P(A) > 0$.

Here B_1, B_2 and B_3 are the three possibilities at the first stage, and B_i is whichever one you are asking about. Each must have a positive probability.

EXAMPLE 11.21

Three machines produce all the parts in a factory: machine 1 makes 50%, machine 2 makes 30% and machine 3 makes 20%. Their defect rates are 1%, 2% and 3% respectively. A part is found to be defective. Find the probability that it came from machine 3.

SOLUTION

Let M_1, M_2 and M_3 be the events that the part came from machine 1, 2 or 3. Let A be the event that the part is defective. These are the B_1, B_2, B_3 and A of the card.

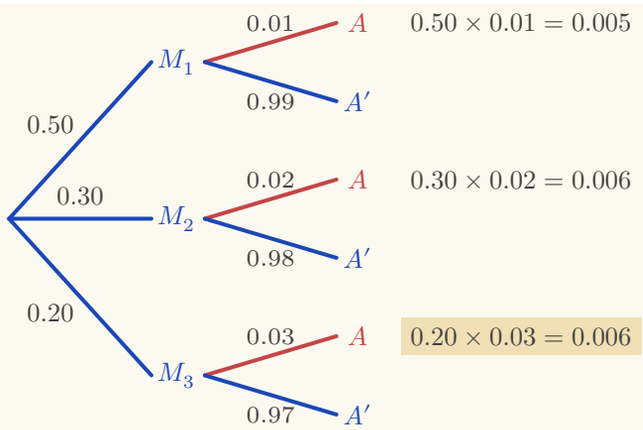


Figure 11.10 Three paths end in a defective part, with total 0.017. The path through machine 3 gives 0.006 of it.

Multiply each machine's share of production by its defect rate. Then divide the product for machine 3 by the total for all three:

$$P(M_3 \mid A) = \frac{(0.20)(0.03)}{(0.50)(0.01) + (0.30)(0.02) + (0.20)(0.03)}$$

$$= \frac{0.006}{0.005 + 0.006 + 0.006} = \frac{0.006}{0.017} \approx \boxed{0.353}$$

Machine 3 makes only a fifth of the parts but, because it is the least reliable, produces over a third of the defects.

YOUR TURN 11.6 Bayes' Theorem HL

Two alternatives

33.

(a) The probability of rain on a given morning is 0.3. If it rains, the probability of being late is 0.6; if it does not, the probability is 0.1. A person is late. Find the probability that it rained.

(b) A disease affects 2% of a population. A test is positive for 95% of those who have it and for 10% of those who do not. Given a positive result, find the probability of having the disease.

(c) Factory 1 makes 60% of all items, of which 2% are defective; factory 2 makes the other 40%, of which 5% are defective. Given that an item is defective, find the probability it came from factory 2.

34.

(a) In a mailbox, 30% of messages are spam. The word "free" appears in 90% of spam messages and in 5% of the others. A message contains the word "free". Find the probability that it is spam.

(b) A box contains two coins: one fair, and one with heads on both sides. A coin is chosen at random and tossed once, showing heads. Find the probability that it is the two-headed coin.

(c) Of the students in a year group, 70% revise for a test. Of those who revise, 80% pass; of those who do not, 30% pass. A student passes. Find the probability that the student revised.

Three alternatives

35. Three suppliers provide the components used by a factory. Supplier 1 provides 40%, supplier 2 35% and supplier 3 25%. Of their components, 1%, 3% and 4% respectively are defective.

(a) Find the probability that a component chosen at random is defective.

(b) Given that a component is defective, find the probability that it came from supplier 2.

(c) Given that a component is defective, find the probability that it came from supplier 1.

(d) Supplier 1 provides the most components but accounts for the fewest defective ones. Explain why.

Concept check

36. A disease affects 1% of a population. A test is positive for 95% of people who have the disease and for 5% of people who do not. A student says: "*the test is 95% accurate, so a person who tests positive has a 95% chance of having the disease.*"

(a) Explain the difference between $P(\text{positive} \mid \text{disease})$ and $P(\text{disease} \mid \text{positive})$.

(b) Find $P(\text{disease} \mid \text{positive})$, and explain why it is so much smaller than 0.95.

(c) Check your answer to part (b) by imagining $10\,000$ people screened.

Chapter 11 · Probability

Reflections

TOK Bayes' theorem describes how a belief should change when you receive new evidence. You start from a *prior*, the probability believed before the evidence. You take the evidence into account, and you arrive at a *posterior*, the probability believed afterwards. The screening test of §11.6 shows the lesson: evidence means little until you consider how rare the thing you are testing for is. Is Bayes' theorem a description of how people reason, or a rule for how they ought to?

TOK We say a fair die has probability $\frac{1}{6}$ because of its symmetry, and we say a drawing pin lands point-up with probability 0.39 because that is what 200 drops showed. The first number was never measured; the second was never proved. Are these two the same kind of claim? And when a measured value disagrees with a calculated one, which of the two should be revised?

IM Probability theory began with a gambler's question. In the 1650s the French writer Antoine Gombaud, known as the Chevalier de Méré, asked Blaise Pascal about several gambling problems. One was how to divide the stakes $(\text{the money bet on the game})$ fairly when a game is stopped before it ends. Pascal's letters with Pierre de Fermat on these problems described many of the methods of this chapter. Gerolamo Cardano had written about dice a century earlier.

RW Ignoring the base rate has real costs. Courts have been misled by a "one-in-a-million" DNA match probability. It was presented as if it were the probability that the defendant was innocent, and nobody asked how many other people in the population would also match by chance. In symbols, $P(\text{match} \mid \text{innocent})$ was treated as $P(\text{innocent} \mid \text{match})$. This is the same confusion as $P(+ \mid D)$ and $P(D \mid +)$. Understanding $P(D \mid +)$ versus $P(+ \mid D)$ is not only an examination topic. Mistakes about it have changed verdicts and medical decisions.

EXT How many people must be in a room for the probability of a shared birthday to be greater than one half? Fewer than many people expect. Ignore 29 February and treat all 365 days as equally likely. The complement is easier: the probability that all n birthdays are different is $\frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-n+1}{365}$. At $n = 23$ this is 0.493, so a shared birthday has probability 0.507. This is less surprising once you count pairs: 23 people make ${}^{23}C_2 = 253$ pairs, and each pair shares a birthday with probability $\frac{1}{365}$. *Try it:* show that for a class of 30 the probability is about 0.71, then check your own class.

End-of-Chapter Problems
Chapter 11: Probability

1. A biased four-sided spinner numbered 1 to 4 was spun 250 times. The results were:

Number	1	2	3	4
Frequency	95	60	55	40

(a) Estimate the probability that the spinner lands on 1. [2]
(b) Estimate the probability that the spinner lands on an odd number. [2]
(c) The spinner is spun a further 60 times. Find the expected number of times it lands on 4. [2]

2. Four cards are dealt at random from a standard pack of 52. Find the probability that the four cards are all from different suits. [5]

3. A committee of 4 is chosen at random from 5 teachers and 4 students.

(a) Find the number of possible committees. [2]
(b) Find the probability that the committee has exactly 2 teachers and 2 students. [3]
(c) Find the probability that the committee contains at least one teacher and at least one student. [4]
(d) Explain why part (c) is quicker through its complement than by adding the cases directly. [2]

4. GDC A lottery draws 5 numbers at random from 40. A ticket also carries 5 numbers.

(a) Find the number of possible draws. [2]
(b) Find the probability that a ticket matches all 5 numbers. [2]
(c) Find the probability that a ticket matches exactly 3 of the 5. [4]
(d) The organisers propose drawing 5 from 45 instead. Find the probability that a ticket matches all five under this rule, and compare it with your answer to part (b). [3]

5. EXACT The letters of the word PARALLEL are arranged in a row, all arrangements being equally likely.

(a) Explain why the number of distinct arrangements is $\frac{8!}{2!\,3!}$, and evaluate it. [3]
(b) Find the probability that an arrangement begins with the letter P. [4]
(c) Find the probability that the three Ls are together. [4]

6. In a class of 40 students, the numbers studying two activities A and B are: only A is $2x$, both is x, only B is $x+4$, and neither is 8.

(a) Find the value of x. [3]
(b) Find $P(A \cap B)$. [1]
(c) Find $P(A \cup B)$. [2]

7. In a survey of 100 people, 50 like tea, 60 like coffee, and 35 like juice. Also, 30 like tea and coffee, 20 like tea and juice, 25 like coffee and juice, and 10 like all three.

(a) Find the number who like at least one of the three drinks. [3]
(b) Find the number who like none. [1]
(c) Find the number who like exactly one drink. [3]

8. At a school, 60% of students study French and 25% study both French and Spanish. Find the probability that a student studies Spanish, given that they study French. [3]

9. Events A and B are independent. Given $P(A) = 0.4$ and $P(A \cup B) = 0.7$, find $P(B)$. [4]

10. Events A and B satisfy $P(A) = 0.5$ and $P(B) = 0.3$.

(a) Find $P(A \cup B)$ if A and B are independent. [3]
(b) Find $P(A \cup B)$ if A and B are mutually exclusive. [2]
(c) Explain why A and B cannot be both independent and mutually exclusive here. [2]

11. In a class of 50 students, 28 study Mathematics (M), 20 study Art (A), and 10 study neither.

(a) Find the number of students who study both Mathematics and Art. [3]
(b) Find the probability that a randomly chosen student studies only Art. [2]
(c) Find $P(A \mid M)$. [2]

12. The two-way table shows whether 100 people exercise regularly.

	Exercises	**Does not**	**Total**
Male	30	20	50
Female	24	26	50
Total	54	46	100

(a) Find the probability that a person is male and exercises. [2]
(b) Find the probability that a person exercises, given that they are female. [2]
(c) Determine whether the events "male" and "exercises" are independent. [3]

13. Events A and B satisfy $P(A) = 0.45$, $P(B) = 0.4$, and $P((A \cup B)') = 0.25$.

(a) Find $P(A \cap B)$. [2]
(b) Find the probability that A occurs but B does not. [1]
(c) Find $P(A \mid B)$. [2]
(d) Determine whether A and B are independent. [2]

14. Two fair dice are rolled and the scores noted.

(a) Find the probability that the sum is a prime number. [2]
(b) Find the probability that at least one die shows a 6. [2]
(c) Given that at least one die shows a 4, find the probability that the sum is 7. [3]

15. Events A, B, and C are mutually exclusive and exhaustive, with $P(A) = 2k$, $P(B) = 3k$, and $P(C) = k + 0.1$.

(a) Find the value of k. [2]
(b) Find $P(A \cup B)$. [2]
(c) Find $P(A \mid B')$. [3]

16. A fair four-sided die (numbered 1–4) and a fair six-sided die (numbered 1–6) are rolled together, and the product of the two scores is recorded.

(a) Represent the outcomes in a sample space diagram. [2]
(b) Find the probability that the product is 12. [2]
(c) Find the probability that the product is odd. [2]
(d) Given that the product is even, find the probability that the four-sided die shows a 4. [3]

17. The 120 members of a sports club are classified by age group and by whether they wear glasses: 18 of the 50 adults and 12 of the 70 juniors wear glasses.

(a) Display this information in a two-way table, including totals. [2]
(b) Find the probability that a randomly chosen member is a junior who does not wear glasses. [1]
(c) Given that a member wears glasses, find the probability that the member is an adult. [2]
(d) Determine, using conditional probability, whether the events "adult" and "wears glasses" are independent. [3]

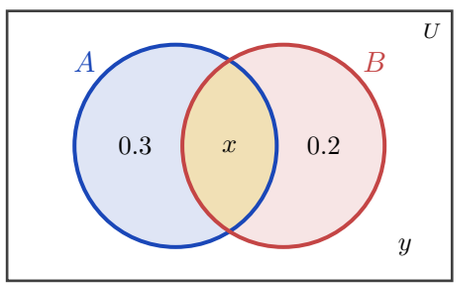
18. Events A and B satisfy $P(A) = 0.6$, $P(A \mid B) = 0.6$, and $P(A \cup B) = 0.84$.

(a) Explain why A and B are independent. [1]
(b) Find $P(B)$. [3]
(c) Write down $P(B \mid A')$, justifying your answer. [2]

19. A family has two children. Assume each child is equally likely to be a boy or a girl, independently.

(a) List the sample space. [1]
(b) Given that at least one of the children is a boy, find the probability that both are boys. [3]
(c) Given instead that the *elder* child is a boy, find the probability that both are boys. [2]
(d) Explain why the answers to (b) and (c) differ. [1]

20. The Venn diagram shows the probabilities of the events A and B and of the regions they form. It is given that $P(A) = 0.5$.



(a) Find the value of x and the value of y. [3]
(b) Find $P(A \mid B)$. [2]
(c) Show that A and B are independent. [2]
(d) Find $P(A' \mid B')$. [2]

21. A bag contains 7 red and 3 green sweets. Two are drawn without replacement.

(a) Draw a tree diagram for the two draws. [3]
(b) Find the probability that both sweets are red. [2]
(c) Find the probability that both sweets are the same colour. [3]
(d) Find the probability that at least one sweet is green. [2]

22. A box contains 4 red and 2 green counters. Three counters are drawn at random without replacement.

(a) Find the probability that all three counters are red. [2]
(b) Given that at least one counter is green, find the probability that exactly one is green. [3]
(c) Find the probability that the third counter drawn is the second green one. [2]

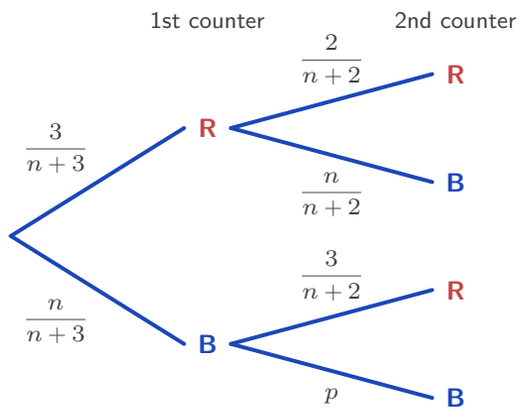
23. GDC An archer hits the target with probability 0.9 on each shot, independently of all other shots.

(a) Find the probability that her first miss is on her third shot. [2]
(b) Find the probability that she hits at least four of her first five shots. [2]
(c) Find the least number of shots for which the probability of at least one miss exceeds 0.95. [3]

24. A box contains 10 batteries, of which 3 are flat. Batteries are taken from the box one at a time, without replacement, and tested until a working one is found. Let N be the number of batteries tested.

(a) Find $P(N = 1)$. [2]
(b) Find $P(N = 2)$. [3]
(c) Find $P(N \leq 3)$. [4]
(d) Explain why N can never be greater than 4. [2]
(e) Find $P(N = 3)$ and $P(N = 4)$, and verify that the four probabilities sum to 1. [4]

25. A bag contains 3 red counters and n blue counters. Two counters are taken at random, one after the other, without replacement. The tree diagram shows the colour of each counter.



(a) Write down an expression for p in terms of n. [1]
(b) Show that the probability that the two counters are the same colour is $\dfrac{n^2 - n + 6}{(n+3)(n+2)}$. [3]
(c) The probability that the two counters are the same colour is $\frac{1}{2}$, and the bag contains more blue counters than red counters. Find the value of n. [4]
(d) Hence find the probability that the second counter is red. [2]

26. A football team's key striker is fit for a match with probability 0.85. The team wins with probability 0.7 if she is fit, and with probability 0.4 if she is not.

(a) Draw a tree diagram to represent this information. [2]
(b) Find the probability that the team wins the match. [2]
(c) Given that the team won, find the probability that the striker was fit. [3]

27. An airline routes 50% of its flights through airport 1, 30% through airport 2, and 20% through airport 3. The probabilities that a flight is delayed are 0.06, 0.03, and 0.10 respectively.

(a) Find the probability that a randomly chosen flight is delayed. [3]

(b) Given that a flight was delayed, find the probability that it was routed through airport 1. [3]

28. Three suppliers provide components: X supplies 25%, Y supplies 45%, and Z supplies 30%. The probabilities that a component from X, Y and Z is defective are 0.04, 0.02 and p respectively. Overall, 3.4% of all components are defective.

(a) Find the value of p. [3]
(b) Given that a component is defective, find the probability that it came from X. [3]

29. A condition affects 5% of a population. A screening test is positive for 90% of those who have it, and for 15% of those who do not.

(a) Find the probability that a randomly chosen person tests positive. [3]
(b) Given a positive result, find the probability the person has the condition. [3]
(c) Comment on what your answer says about screening for rare conditions. [2]

30. GDC A machine produces components, of which 6% are defective. Each component is inspected. The inspection rejects a defective component with probability 0.95, and wrongly rejects a component that is not defective with probability 0.02.

(a) Draw a tree diagram to represent this information. [3]
(b) Find the probability that a randomly chosen component is rejected. [3]
(c) Find the probability that a rejected component was in fact not defective. [3]
(d) Find the probability that a component which passes inspection is defective. [3]
(e) In a batch of 5000 components, find the expected number that are defective but pass inspection. [2]
(f) Using your answers to parts (d) and (e), comment on how well the inspection protects a customer. [2]

31. Of the 200 students at a school, 130 cycle to school (C), 90 travel by train (T), and 40 do both. The remaining students walk.

(a) Draw a Venn diagram showing the number of students in each of the four regions. [3]
(b) Find the probability that a student chosen at random walks to school. [2]
(c) Find $P(C \mid T)$. [3]
(d) Determine whether C and T are independent, justifying your answer. [3]
(e) Two students are chosen at random, without replacement. Find the probability that both of them walk. [3]
(f) State whether your answer to part (e) would be larger or smaller if the two students were chosen with replacement, and explain why. [2]

32. GDC A system contains three components A, B and C. Components A and B are connected in parallel, and this pair is connected in series with C, so the system works when C works and at least one of A and B works. Each component works with probability p, independently of the others.

(a) Show that the probability that the system works is $p(1-(1-p)^2)$. [2]
(b) Find the probability that the system works when $p = 0.9$. [1]
(c) Find the value of p for which the probability that the system works is 0.95. [2]
(d) A fourth component D, which also works with probability p independently, is connected in parallel with C. Find the probability that the system now works when $p = 0.9$. [2]

33. GDC A disease affects 0.5% of a population. A test is positive for 98% of those with the disease and for 3% of those without. A person tests positive *twice*, the two tests being independent given disease status.

(a) Find the probability the person has the disease after one positive test. [3]
(b) Find the probability the person has the disease after two positive tests. [4]
(c) Comment on the effect of the second test. [1]

34. Events A and B are independent. Prove that A' and B' are also independent, that is, show $P(A' \cap B') = P(A')\,P(B')$. *(Hint: "neither A nor B" is the complement of $A \cup B$)* [5]

35. GDC Ana and Ben take turns to roll a fair six-sided die, with Ana rolling first. The first player to roll a six wins the game.

(a) Write down the probability that Ana wins with her first roll. [1]
(b) Find the probability that Ana wins with her third roll. [2]
(c) Show that the probability that Ana wins the game is $\frac{6}{11}$. *(Hint: if both players miss on their first rolls, the game starts again, so Ana's probability p of winning appears on both sides of an equation)* [3]
(d) Find the probability that neither player has rolled a six after each has rolled ten times. [2]

36. The table classifies a sample of 200 students from a large school by whether they take music (M) and whether they take drama (D).

	Drama	**No drama**	**Total**
Music	20	61	81
No music	30	89	119
Total	50	150	200

(a) Find $P(D \mid M)$, giving your answer as an exact fraction. [2]

(b) Determine whether the events M and D are independent for these 200 students. [3]

(c) A student rounds $P(D \mid M)$ and $P(D)$ to two decimal places, finds 0.25 for both, and concludes that M and D are independent. Comment on this conclusion. [1]

(d) State whether your answer to part (b) shows that taking music and taking drama are linked across the whole school, and give a reason. [1]

37. In a games club, 50% of the members play chess (C) and 40% play go (G). Of the members who play chess, 30% also play go. A member is chosen at random.

(a) Find the probability that the member plays both games. [2]

(b) Given that the member plays go, find the probability that they also play chess. [2]

(c) Find the probability that the member plays neither game. [2]

(d) Another student answers part (a) with $0.5 \times 0.4 = 0.2$. State the assumption this student has made, and show that it is false here. [2]

Challenge Questions

C1. GDC **The birthday problem.** In a group of n people, assume that every birthday is equally likely among 365 days, that different people's birthdays are independent, and ignore leap years.

(a) Explain why, for $n = 3$, the probability that all three birthdays differ is $\dfrac{364}{365} \times \dfrac{363}{365}$, and write down an expression for the probability that all n birthdays differ. [4]

(b) Hence find, to three decimal places, the probability that at least two people share a birthday when $n = 10$, $n = 23$ and $n = 50$. [3]

(c) Find the smallest n for which the probability of a shared birthday exceeds $\frac{1}{2}$. [2]

(d) Two tempting guesses give much larger answers. The first is 183 people, about half the number of days. The second is the number of people needed for the probability that someone shares *your* birthday to exceed $\frac{1}{2}$. Show that the second guess gives 254 people (you and 253 others). Explain why each guess answers a different question from part (c), and why the count that matters is the number of *pairs*, ${}^{n}C_{2}$. Evaluate ${}^{n}C_{2}$ for $n = 23$. [5]

(e) Each pair matches with probability $\frac{1}{365}$. Find the expected number of matching pairs in a group of 23, and comment on your answer. [2]

(f) A school issues every student a random four-digit code, so there are $10\,000$ equally likely codes. Find the smallest number of students for which two students sharing a code is more likely than not. [3]

(g) Now let there be N equally likely values. Using $1 - x \approx e^{-x}$ for small x, show that the probability that all n values differ is approximately $e^{-n(n-1)/(2N)}$. Hence show that, for

large N, the smallest n for which a match is more likely than not is about $\sqrt{2N\ln 2}$. Evaluate this for $N = 365$ and $N = 10\,000$, and compare with parts (c) and (f). [4]

(h) Real birthdays are not spread evenly across the year. Investigate, using published birth data or a model of your own, whether unequal frequencies make a shared birthday more or less likely. Alternatively, investigate how large a group must be for *three* people to share a birthday with probability above $\frac{1}{2}$. [2]

C2. Three doors. A prize is behind one of three closed doors, each door equally likely. You choose a door. The host knows where the prize is, and always opens one of the other two doors that does not hold the prize; if both of those doors are empty, because you chose the prize, he chooses between them at random. You may then keep your door or switch to the last closed door.

(a) Find the probability of winning if you always switch. Treat the case where your first choice is correct separately from the case where it is not. [5]

(b) State the probability of winning if you always stay, and say which strategy is better. [1]

(c) Now use Bayes' theorem. You chose door 1 and the host opened door 3. Write down $P(\text{host opens } 3 \mid \text{prize behind } 1)$, $P(\text{host opens } 3 \mid \text{prize behind } 2)$ and $P(\text{host opens } 3 \mid \text{prize behind } 3)$, and hence find $P(\text{prize behind } 2 \mid \text{host opens } 3)$. [5]

(d) A second host knows nothing. He opens one of the two doors you did not choose at random, and it happens to be empty. Repeat the calculation of part (c) for this host, and show that switching now wins with probability $\frac{1}{2}$. [4]

(e) The two hosts performed the same visible action: they opened door 3 and it was empty. Explain what makes the answers differ, and what it is that the first host's action tells you. [3]

(f) The game is now played with $n \geq 3$ doors, one prize and each door equally likely. After your choice, the knowing host opens $n-2$ of the other doors, all of them empty, leaving your door and one other closed; if your door holds the prize, he chooses the other door to leave closed at random. Show that switching wins with probability $\dfrac{n-1}{n}$. [3]

(g) Return to three doors and a knowing host. When you have chosen the prize, this host opens door 3 with probability q and door 2 with probability $1-q$, where $0 \leq q \leq 1$. You chose door 1 and he opened door 3. Show that $P(\text{prize behind } 2 \mid \text{host opens } 3) = \dfrac{1}{1+q}$. *(Hint: the host opens door 3 with probability q if the prize is behind door 1, with probability 1 if it is behind door 2, and never if it is behind door 3)* [2]

(h) Deduce that switching is never worse than staying, whatever the value of q, and describe what the host's choice of door tells you when $q = 0$ and when $q = 1$. Suggest a further change to the host's rules that would be worth investigating. [2]

C3. Independent in pairs, but not all together. Two events A and B are independent when $P(A \cap B) = P(A)P(B)$. Three events A, B and C are called *mutually independent*

when each of the three pairs is independent and, in addition,
$P(A \cap B \cap C) = P(A)P(B)P(C)$.

Two fair coins are tossed. Let A be the event "the first coin shows heads", B the event "the second coin shows heads", and C the event "the two coins show different faces".

(a) Find $P(A)$, $P(B)$, $P(C)$, $P(A \cap B)$, $P(A \cap C)$ and $P(B \cap C)$, and show that each pair of the events A, B, C is independent. [4]

(b) Find $P(A \cap B \cap C)$, and show that A, B and C are not mutually independent. Explain in words why knowing whether A and B happened tells you whether C happened. [3]

(c) Let A, B and C be any three events that are independent in pairs, with $P(A \cap B) > 0$. Prove that $P(C \mid A \cap B) = P(C)$ if and only if $P(A \cap B \cap C) = P(A)P(B)P(C)$. *(Hint: write $P(C \mid A \cap B)$ as a fraction and replace $P(A \cap B)$ by $P(A)P(B)$; each step can be reversed)* [3]

(d) A fair eight-sided die, with faces numbered 1 to 8, is rolled once. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 5\}$ and $C = \{1, 6, 7, 8\}$. Show that $P(A \cap B \cap C) = P(A)P(B)P(C)$, but that no two of these events are independent. [3]

(e) Explain what parts (b) and (d) together show about the definition of mutual independence. Mutual independence of four events requires the product rule for every collection of two or more of them. Show that this gives 11 equations, and find a formula for the number of equations needed for n events. [3]

(f) Now toss n fair coins, and for each pair of coins i and j let E_{ij} be the event that coins i and j show the same face. Investigate which collections of the events E_{ij} are mutually independent, starting with E_{12}, E_{13} and E_{23} for three coins, and state a conjecture. [2]

