GEOMETRY & TRIGONOMETRY

CHAPTER 9

Vectors

9

SYLLABUS 3.12–3.18 HL ONLY

A camera tripod has three legs, not four. The choice is deliberate. Part of the reason is geometry.

Three points that are not in a line lie in exactly one plane. The three feet of a tripod therefore always fix a single plane, so all three can rest on the ground together. A fourth foot need not lie in that plane, which is why a four-legged stool can rock.

Ordinary arithmetic cannot express that fact. The fact is not about how large anything is. It is about position, direction and flatness. To state it, and to prove it, you need a language in which a direction is something you can calculate with.

A **vector** is a quantity that has both a size and a direction. Speed is a number, but velocity is a vector. Distance is a number, but displacement is a vector. A pilot who flies due north through a wind from the west does not travel due north, and does not travel at the aircraft's airspeed (its speed through the air). The motion and the wind point in different directions, and both must be counted.

Once a quantity has a direction, addition changes. Add two vectors of length three. If they point the same way, the sum has length six. If they point in opposite directions, the sum has length zero. Any length between those two is possible, and it depends on the angle between them.

Vectors model position and movement in two and three dimensions. They also connect three ways of solving a problem: drawing it, writing equations, and calculating with vectors. By the end of this chapter you will add and scale vectors and use them to prove geometric facts. You will use the scalar product to find angles and test for right angles, and the vector product to find areas and normals. You will write equations of lines and planes, model straight-line motion, decide how two lines or three planes are arranged, and find where they meet, the angles between them and the shortest distances.

By then the tripod will have its explanation in a few lines of algebra: three points that are not in a line always fix exactly one plane.

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9.1 Vectors: Magnitude and Direction

A vector has exactly two properties: a **magnitude**, which is its length, and a **direction**. Where it sits in space does not matter. Two arrows of the same length pointing the same way are the same vector.

An arrow drawn to represent a vector is called a **directed line segment**. It is called "directed" because it has an arrowhead, and a "line segment" because it is a piece of a line with two ends. Examination questions use this phrase; it means an arrow and nothing more.

Vectors must be kept apart from ordinary numbers, which in this context are called **scalars**. Three notations are in common use, and they all mean the same thing. Printed books, including this one, use a bold letter $\mathbf{a}$. By hand you cannot write bold type, so you write either a letter with an arrow above it, $\vec{a}$, or an underlined letter, $\underline{a}$. An arrow is also used for a vector named by its two endpoints: $\overrightarrow{AB}$ means the vector from the point A to the point B.

The usual representation is the **column vector**. It lists how far to travel along each axis to get from the tail to the head.

$$\mathbf{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}.$$

Here $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are the **base vectors**: arrows of length 1 along the x-, y- and z-axes. The IB writes the base vectors without a hat, and so do these notes. Some books print them as $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$, $\hat{\mathbf{k}}$. The hat marks a vector of length 1 and is introduced in §9.1.2. The formula booklet uses only the column form; use whichever is convenient.

9.1.1 The algebra of arrows

Vectors are added **tip to tail**. Draw the second vector starting where the first one ends. The sum is the arrow from the start of the first to the end of the second. In components (the entries of the column, one for each axis) it is simple: add the matching entries, one coordinate at a time.

KEY 9.1 **Adding and scaling vectors**

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} + \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} a_1 + b_1 \\ a_2 + b_2 \\ a_3 + b_3 \end{pmatrix}, \qquad k\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} ka_1 \\ ka_2 \\ ka_3 \end{pmatrix}.$$

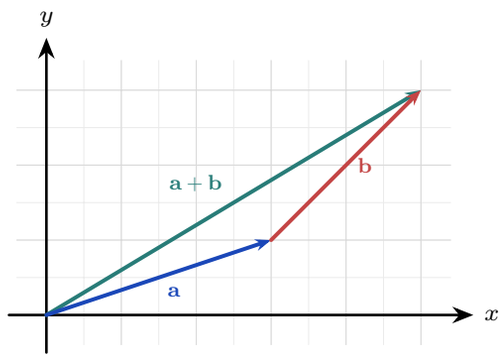


Figure 9.1 Adding tip to tail.

Scalar multiplication by k scales the length by $|k|$ and reverses the direction when k is negative. This gives a simple test: two non-zero vectors are **parallel** exactly when one is a scalar multiple of the other, $\mathbf{a} = k\mathbf{b}$, with the same k in every coordinate. Check all three coordinates: two can agree on k while the third does not.

Two particular vectors have names of their own: the negative of a vector and the zero vector.

The **negative** of $\mathbf{v}$, written $-\mathbf{v}$, is the vector of the same length pointing the opposite way. It is the case $k = -1$ of scalar multiplication. Subtraction is then just addition of the negative:

$$\mathbf{a} - \mathbf{b} = \mathbf{a} + (-\mathbf{b}).$$

The **zero vector** $\mathbf{0}$ has magnitude 0 and no direction at all. It is what you get from $\mathbf{v} + (-\mathbf{v})$, or from $0\mathbf{v}$. It is the only vector whose direction is undefined, and it is written in bold to keep it apart from the number 0.

EXAMPLE 9.1

Take $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix}$. Find $2\mathbf{a} - \mathbf{b}$, and decide whether it is parallel to $\mathbf{c} = \begin{pmatrix} -8 \\ -12 \\ 10 \end{pmatrix}$ or to $\mathbf{d} = \begin{pmatrix} 12 \\ 18 \\ -14 \end{pmatrix}$.

SOLUTION

Scale first, then subtract, working one coordinate at a time:

$$2\mathbf{a} - \mathbf{b} = \begin{pmatrix} 6 \\ 2 \\ -4 \end{pmatrix} - \begin{pmatrix} 2 \\ -4 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 6 \\ -5 \end{pmatrix}}$$

Parallel means one vector is a scalar multiple of the other, so look for a single k that works in all three coordinates. For $\mathbf{c}$, the first coordinate forces $k = -2$, and that value holds throughout:

$$-2\begin{pmatrix}4\\6\\-5\end{pmatrix} = \begin{pmatrix}-8\\-12\\10\end{pmatrix} = \mathbf{c},$$

so $\mathbf{c}$ is parallel to $2\mathbf{a} - \mathbf{b}$, pointing the opposite way because k is negative. For $\mathbf{d}$, the first two coordinates both give $k = 3$, but $3 \times (-5) = -15$, and the third coordinate of $\mathbf{d}$ is -14. No single k works, so $\mathbf{d}$ is not parallel to $2\mathbf{a} - \mathbf{b}$.

9.1.2 Magnitude and unit vectors

The size of the arrow has three names: the **magnitude** of the vector, its **length** and its **modulus**. All three are written $|\mathbf{v}|$. These notes use whichever reads more naturally in the sentence. The value is found with Pythagoras' theorem, used in three dimensions. Apply the theorem twice. The vector with components v_1, v_2 and 0 lies in the xy-plane and has length $\sqrt{v_1^2 + v_2^2}$. That length and the height v_3 are the two shorter sides of a second right-angled triangle whose hypotenuse is $\mathbf{v}$, so $|\mathbf{v}|^2 = (v_1^2 + v_2^2) + v_3^2$.

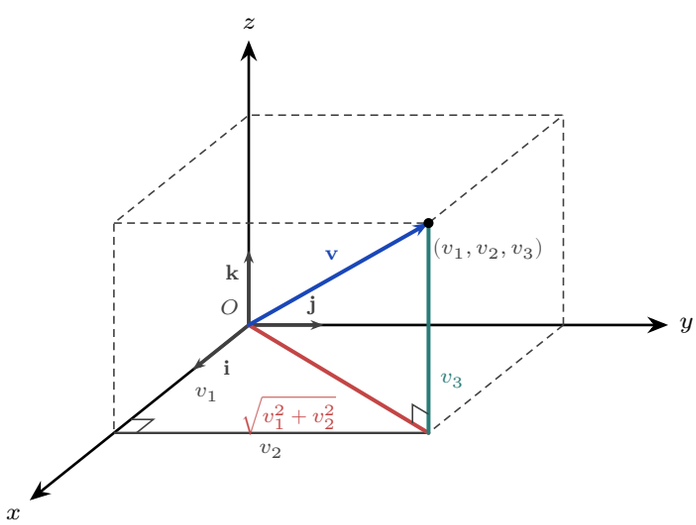


Figure 9.2 Two right-angled triangles: the floor diagonal has length $\sqrt{v_1^2 + v_2^2}$, and with the height v_3 it gives $|\mathbf{v}|^2 = v_1^2 + v_2^2 + v_3^2$.

FORMULA 9.2 **Magnitude of a vector** FORMULA BOOKLET · 3.12

The magnitude of $\mathbf{v} = \begin{pmatrix}v_1\\v_2\\v_3\end{pmatrix}$ is $|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

Dividing a vector by its own length leaves the direction untouched but sets the length to 1. The result is a **unit vector**, written $\hat{\mathbf{v}}$. It keeps only the direction of $\mathbf{v}$. To get a vector of length m in that direction, multiply it by m: the answer is $m\hat{\mathbf{v}}$.

KEY 9.3 **Unit vector**

For $\mathbf{v} \neq \mathbf{0}$, $\hat{\mathbf{v}} = \dfrac{\mathbf{v}}{|\mathbf{v}|}$ is the unit vector in the direction of $\mathbf{v}$.

The base vectors are unit vectors: $|\mathbf{i}| = |\mathbf{j}| = |\mathbf{k}| = 1$, one unit along each axis. So writing $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ says that $\mathbf{a}$ is built by taking a_1 steps of length 1 along x, then a_2 along y, then a_3 along z.

EXAMPLE 9.2

Find a vector of magnitude 14 in the direction of $\mathbf{v} = \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}$.

SOLUTION

First find the length: $|\mathbf{v}| = \sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$. The unit vector is $\hat{\mathbf{v}} = \frac{1}{7}\mathbf{v}$, and scaling it to length 14 means multiplying by 14:

$$14\hat{\mathbf{v}} = \frac{14}{7}\begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix} = \boxed{\begin{pmatrix} 4 \\ 6 \\ 12 \end{pmatrix}}$$

Reduce to a unit vector first, then rescale. These two steps give a vector of any magnitude m in a given direction.

YOUR TURN 9.1 Vectors: Magnitude and Direction

Magnitude and combining vectors

1. Find the magnitude of each vector.

(a) $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 6 \\ -2 \\ 3 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

2. Given $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ -1 \\ 4 \end{pmatrix}$, evaluate each of the following.

(a) $\mathbf{a} + \mathbf{b}$ (b) $\mathbf{a} - \mathbf{b}$

(c) $\mathbf{a} + 2\mathbf{b}$ (d) $3\mathbf{a} - \mathbf{b}$

Base vectors

3. Work with the base vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(a) Write $3\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$ as a column vector.

(b) Write $\begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$.

(c) Given $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{b} = \mathbf{i} + 3\mathbf{k}$, find $2\mathbf{a} - \mathbf{b}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, and find its magnitude.

Unit vectors and scaling

4. Unit vectors and scaling.

(a) Find the unit vector in the direction of $\begin{pmatrix} 0 \\ 3 \\ 4 \end{pmatrix}$.

(b) Find the unit vector in the direction of $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(c) Find a vector of magnitude 6 in the direction of $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

(d) Explain why $\dfrac{\mathbf{v}}{|\mathbf{v}|}$ has magnitude 1 for every non-zero vector $\mathbf{v}$.

Parallel vectors

5. Decide whether each pair of vectors is parallel, and justify your answer.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

(c) $\begin{pmatrix} 3 \\ -6 \\ 9 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 2 \\ -3 \end{pmatrix}$

(d) Find p and q so that $\begin{pmatrix} 2 \\ p \\ 6 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 6 \\ q \end{pmatrix}$ are parallel.

Concept check

6. A student says: "$|\mathbf{a} + \mathbf{b}| = |\mathbf{a}| + |\mathbf{b}|$."

(a) Give a counter-example.

(b) State when the claim is true.

9.2 Position Vectors and Geometry

Fix the origin O. The **position vector** of a point A is the arrow $\overrightarrow{OA}$ from the origin to A; its components are simply the coordinates of A. Position vectors connect the algebra of vectors to the fixed points of a geometry problem.

The arrow from one point to another is found by subtraction.

KEY 9.4 **Vector between two points**

$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, the position vector of the head minus that of the tail.

The distance between A and B is then the magnitude of that **displacement** (the vector from one point to another), $|\overrightarrow{AB}| = |\mathbf{b} - \mathbf{a}|$, and the midpoint of AB has position vector $\frac{1}{2}(\mathbf{a} + \mathbf{b})$. To reach the midpoint, travel from O to A and then along half of $\overrightarrow{AB}$, which gives $\mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b})$.

Keep three things apart. A point is a location. Its position vector is the arrow from the origin to it. A displacement is the arrow from one point to another, and it does not depend on where the origin is.

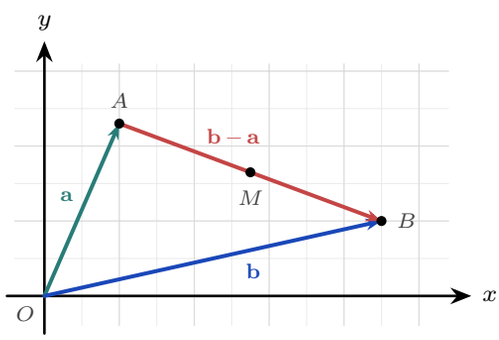


Figure 9.3 The arrow $\overrightarrow{AB}$ closes the triangle OAB; M is the midpoint.

EXAMPLE 9.3

For the points $A(2, -3, 1)$ and $B(5, 1, 6)$, write down their position vectors, then find $\overrightarrow{AB}$ and the distance AB.

SOLUTION

The position vectors have the coordinates as components: $\mathbf{a} = \overrightarrow{OA} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix}$. The displacement $\overrightarrow{AB}$ is head minus tail:

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a} = \begin{pmatrix} 5 \\ 1 \\ 6 \end{pmatrix} - \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}}, \qquad AB = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} = \boxed{5\sqrt{2}}$$

The magnitude of the displacement is the distance. To find the distance between two points, subtract their position vectors, then take the magnitude.

Three points A, B, C are **collinear** (on one straight line) when $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are parallel, that is, one is a scalar multiple of the other. This is the vector version of "same direction from a shared point", and the equation of a line in §9.4 is built from it.

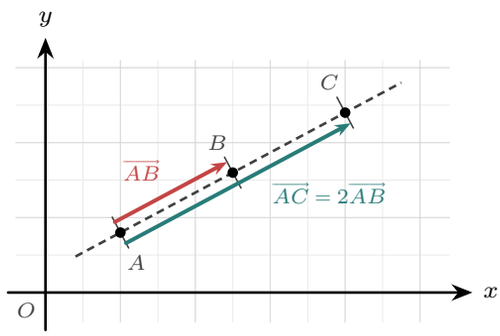


Figure 9.4 Collinear points. Both displacements start at A and $\overrightarrow{AC} = 2\overrightarrow{AB}$, so A, B and C lie on one line.

9.2.1 Proving geometry with vectors

Vectors give a way of proving geometric facts by algebra, without long arguments read from a diagram, and without coordinates. The method has three steps.

KEY 9.5 **Proving a geometric property with vectors**

1. Choose one point as the origin, and name the position vectors of the others with single letters.
2. Write every point in the figure in terms of those letters.
3. Show the required relation by algebra.

Choosing the origin well is what keeps the algebra short. Put it at a vertex where several edges meet.

EXAMPLE 9.4

Prove that the diagonals of a parallelogram bisect each other.

SOLUTION

Let the parallelogram be $OABC$, with the origin at O. Write $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. Because $OABC$ is a parallelogram, $\overrightarrow{CB} = \overrightarrow{OA} = \mathbf{a}$, so the fourth vertex is

$$\overrightarrow{OB} = \mathbf{a} + \mathbf{c}.$$

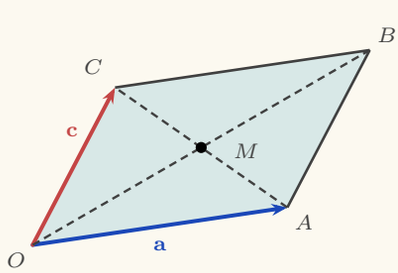


Figure 9.5 The parallelogram $OABC$. The two dashed diagonals meet at the point M, and the proof below shows that M is the midpoint of both.

The two diagonals are OB and AC. Find the midpoint of each.
Midpoint of OB: $\frac{1}{2}(\mathbf{0}+\mathbf{a}+\mathbf{c}) = \frac{1}{2}(\mathbf{a}+\mathbf{c})$.
Midpoint of AC: $\frac{1}{2}(\mathbf{a}+\mathbf{c})$.
The two midpoints are the same point. A single point lying halfway along both diagonals is exactly what "bisect each other" means. ■

Notice that no coordinates were used, and the result holds for every parallelogram, in a plane or in space.

EXAMPLE 9.5

Prove the **midpoint theorem**: if M and N are the midpoints of the sides OA and OB of the triangle OAB, then MN is parallel to AB and half its length.

SOLUTION

Take the origin at O, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The midpoints are then

$$\overrightarrow{OM} = \tfrac{1}{2}\mathbf{a}, \qquad \overrightarrow{ON} = \tfrac{1}{2}\mathbf{b}.$$

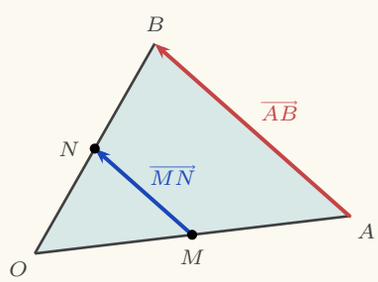


Figure 9.6 The midpoint theorem.

Now find the two displacement vectors, each as head minus tail:

$$\overrightarrow{MN} = \tfrac{1}{2}\mathbf{b} - \tfrac{1}{2}\mathbf{a} = \tfrac{1}{2}(\mathbf{b}-\mathbf{a}), \qquad \overrightarrow{AB} = \mathbf{b}-\mathbf{a}.$$

So $\overrightarrow{MN} = \frac{1}{2}\overrightarrow{AB}$.
One vector is a scalar multiple of the other, so the two segments are parallel, and the scalar is $\frac{1}{2}$, so MN is half the length of AB. ■

YOUR TURN 9.2 Position Vectors and Geometry

Vectors between points

7. Points $A(1, 2, -1)$ and $B(4, 0, 3)$ are given.

(a) Find $\overrightarrow{AB}$.

(b) Find the distance AB.

(c) Find the midpoint of AB.

(d) Find $\overrightarrow{BA}$.

8. Further position-vector work.

(a) A is $(1, 1, 1)$ and $\overrightarrow{AB} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$. Find the coordinates of B.

(b) B is the midpoint of AC, with $A(2, 1, 3)$ and $B(4, 5, 3)$. Find C.

(c) Show that $A(1, 1, 1)$, $B(2, 3, 5)$ and $C(4, 7, 13)$ are collinear.

(d) The points $A(1, 1)$, $B(4, 2)$ and $D(2, 4)$ lie in the xy-plane. Find the point C for which $ABCD$ is a parallelogram. Sketch $ABCD$ on a coordinate grid, and draw and label the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

Proving geometry with vectors

9. $OABC$ is any quadrilateral. Take the origin at O and write $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$. The points P, Q, R and S are the midpoints of OA, AB, BC and CO.

(a) Write the position vectors of P, Q, R and S in terms of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(b) Find $\overrightarrow{PQ}$ and $\overrightarrow{SR}$.

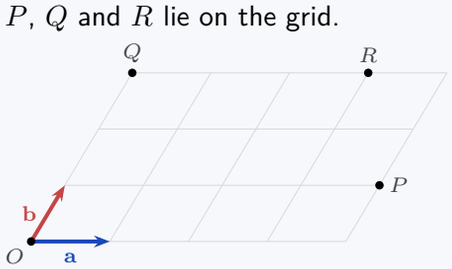
(c) Hence prove that $PQRS$ is a parallelogram.

10. In triangle OAB, take the origin at O and write $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$. M is the midpoint of AB, N is the midpoint of OA, and G is the point on OM with $OG = \frac{2}{3}OM$.

(a) Show that $\overrightarrow{OG} = \frac{1}{3}(\mathbf{a} + \mathbf{b})$.

(b) Find $\overrightarrow{BG}$ and $\overrightarrow{BN}$, and hence prove that B, G and N are collinear.

(c) The figure shows a grid of identical parallelograms. The vectors $\mathbf{a}$ and $\mathbf{b}$ are drawn from O, and the points P, Q and R lie on the grid.



Write down $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ in terms of $\mathbf{a}$ and $\mathbf{b}$, and find $\overrightarrow{PQ}$.

(d) For the grid in part (c), M is the midpoint of PQ. Find $\overrightarrow{OM}$, and hence show that M lies on the line OR.

Concept check

11. For $A(1,2,-1)$ and $B(4,0,3)$, a student writes $\overrightarrow{AB} = \mathbf{a} - \mathbf{b} = \begin{pmatrix} -3 \\ 2 \\ -4 \end{pmatrix}$. Identify the error and find $\overrightarrow{AB}$.

9.3 The Scalar Product

This course uses two ways to multiply vectors. The first, the **scalar product** (also called the dot product), takes two vectors and returns a number, and the angle between them can be found from that number. The second, the vector product of §9.5, returns a vector. The scalar product has two forms, and they always agree.

FORMULA 9.6 **Scalar product** FORMULA BOOKLET · 3.13

For $\mathbf{v}, \mathbf{w}$ with angle θ between them,

$$\mathbf{v} \cdot \mathbf{w} = v_1 w_1 + v_2 w_2 + v_3 w_3 \qquad \text{and} \qquad \mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta.$$

The second form needs the factor $\cos\theta$ because ordinary multiplication assumes both quantities are measured along the same line: 3×4 makes sense because both numbers sit on one number line. Two vectors usually point different ways, so multiplying their lengths would measure nothing in particular.

So first bring them onto the same line. Keep $\mathbf{v}$ as it is, and use only the part of $\mathbf{w}$ that lies along $\mathbf{v}$. This part is the **projection** of $\mathbf{w}$ onto $\mathbf{v}$ (compare §7.1.1). To find it, drop a perpendicular from the tip of $\mathbf{w}$ to the line of $\mathbf{v}$. Its signed length is $|\mathbf{w}| \cos\theta$: positive when the angle is acute, and negative when it is obtuse, because the projection then points against $\mathbf{v}$. Multiplying those two gives

$$\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}| \times |\mathbf{w}| \cos\theta,$$

the length of $\mathbf{v}$ times how much of $\mathbf{w}$ goes in $\mathbf{v}$'s direction. Doing it the other way round, projecting $\mathbf{v}$ onto $\mathbf{w}$, gives $|\mathbf{w}| \times |\mathbf{v}| \cos\theta$, the same number. That is why the order does not matter.

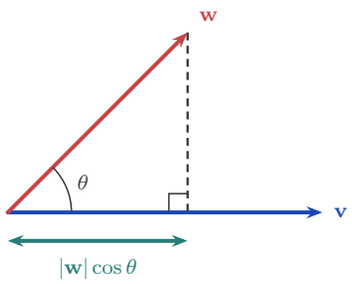


Figure 9.7 The projection of $\mathbf{w}$ onto $\mathbf{v}$ has signed length $|\mathbf{w}|\cos\theta$, shown here for an acute angle. For an obtuse angle it points the other way and the value is negative.

The scalar product then satisfies the rules you would expect of a multiplication.

- **Commutative:** $\mathbf{v}\cdot\mathbf{w} = \mathbf{w}\cdot\mathbf{v}$. The order does not matter.
- **Distributive:** $\mathbf{v}\cdot(\mathbf{w}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{u}$. Brackets expand as usual.
- **Scalars move outside:** $(k\mathbf{v})\cdot\mathbf{w} = k(\mathbf{v}\cdot\mathbf{w})$. A constant factor can be taken out.
- **Scalar product with itself:** $\mathbf{v}\cdot\mathbf{v} = |\mathbf{v}|^2$, since $\theta = 0$ and $\cos 0 = 1$. The result is the squared length.

The distributive rule comes from the projection. Draw $\mathbf{u}$ tip to tail after $\mathbf{w}$. The projection of $\mathbf{w}+\mathbf{u}$ onto $\mathbf{v}$ is then the projection of $\mathbf{w}$ followed by the projection of $\mathbf{u}$, where a projection that points against $\mathbf{v}$ counts as negative. Multiplying each projection by $|\mathbf{v}|$ gives $\mathbf{v}\cdot(\mathbf{w}+\mathbf{u}) = \mathbf{v}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{u}$. For the scalar rule, multiplying $\mathbf{v}$ by k multiplies its length by $|k|$. A negative k also reverses $\mathbf{v}$, which replaces θ by $180^\circ - \theta$ and changes the sign of $\cos\theta$. In both cases the scalar product is multiplied by k.

The base vectors give the two simplest values. Each has length 1 and lies along its own axis, so taking the scalar product of one with itself gives $1\times 1\times\cos 0 = 1$. Different axes are at right angles, so taking the scalar product of two different base vectors gives 0.

$$\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1, \qquad \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$$

This is exactly why the component form works. Expanding $(v_1\mathbf{i}+v_2\mathbf{j}+v_3\mathbf{k})\cdot(w_1\mathbf{i}+w_2\mathbf{j}+w_3\mathbf{k})$ produces nine terms, and every term pairing two *different* base vectors is zero, because of that zero scalar product. Only the three matching terms remain, leaving $v_1w_1 + v_2w_2 + v_3w_3$.

The component form is easy to compute, and the cosine form contains the angle. Set the two forms equal and solve for $\cos\theta$ to find the angle between any two directions.

FORMULA 9.7 **Angle between two vectors** FORMULA BOOKLET · 3.13

$$\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}|\,|\mathbf{w}|}.$$

Both vectors must be non-zero.

EXAMPLE 9.6

Find the angle between $\mathbf{v} = \begin{pmatrix} 3 \\ 2 \\ 6 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 6 \\ 3 \end{pmatrix}$.

SOLUTION

Compute the three ingredients: $\mathbf{v} \cdot \mathbf{w} = 6 + 12 + 18 = 36$, $|\mathbf{v}| = \sqrt{9 + 4 + 36} = 7$, $|\mathbf{w}| = 7$. Then

$$\cos\theta = \frac{36}{49} \implies \boxed{\theta = \arccos\frac{36}{49} \approx 42.7^\circ}$$

Divide the scalar product by the product of the lengths. The method is the same whether the vectors come from a triangle, a pair of lines or a pair of planes.

A special case used often is a right angle. Since $\cos 90^\circ = 0$, the scalar product of perpendicular vectors is zero, and the converse (the statement the other way round) holds too: a zero scalar product means a right angle. This gives a test for perpendicularity that needs no angles at all, only multiplication and addition.

KEY 9.8 **Perpendicular vectors**

$\mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$ (for non-zero $\mathbf{v}, \mathbf{w}$).

Because $|\mathbf{v}|\,|\mathbf{w}|$ is positive, the sign of $\mathbf{v} \cdot \mathbf{w}$ is the sign of $\cos\theta$, so it tells you the kind of angle before you find it.

Angle θ	acute	right angle	obtuse
$\mathbf{v} \cdot \mathbf{w}$	positive	zero	negative

Parallel vectors sit at the opposite extreme. There θ is 0° or 180°, so $\cos\theta$ is $+1$ or -1 and the scalar product reaches the largest size it can. Taking the modulus covers both directions at once.

KEY 9.9 **Parallel vectors**

For non-zero $\mathbf{v}, \mathbf{w}$: $\mathbf{v} \parallel \mathbf{w} \iff |\mathbf{v} \cdot \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}|$.

EXAMPLE 9.7

Find the angle between $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$.

SOLUTION

The scalar product is $\mathbf{v} \cdot \mathbf{w} = (1)(2) + (2)(0) + (2)(-1) = 0$. Because it is zero, the vectors are perpendicular, so $\theta = 90°$ and neither magnitude was needed: a zero scalar product gives the angle immediately.

YOUR TURN 9.3 The Scalar Product

Calculating the scalar product

12. Evaluate each scalar product.

(a) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ (b) $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

13. Use the definition $\mathbf{v} \cdot \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \cos\theta$.

(a) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 4$ and an angle of $60°$ between them, find $\mathbf{a} \cdot \mathbf{b}$.

(b) Given $|\mathbf{a}| = 2$, $|\mathbf{b}| = 5$ and an angle of $120°$ between them, find $\mathbf{a} \cdot \mathbf{b}$.

(c) Explain why $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$ for every vector $\mathbf{a}$.

(d) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and $\mathbf{a} \cdot \mathbf{b} = -15$, state what this tells you about the directions of $\mathbf{a}$ and $\mathbf{b}$, and justify your answer.

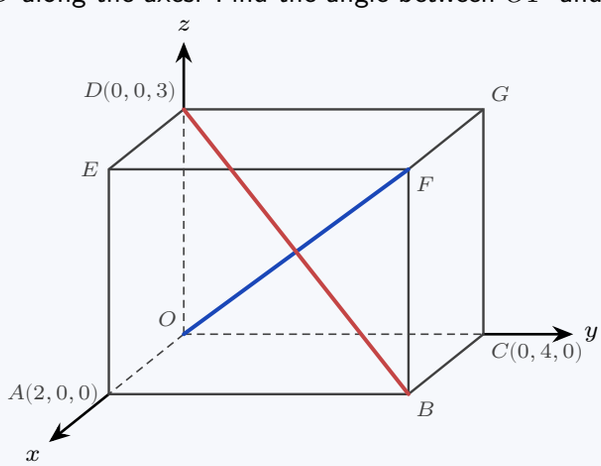
Angles between vectors

14. GDC Find the angle between each pair of vectors.

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ (d) $\begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix}$

(e) The figure shows the cuboid $OABCDEFG$, with O at the origin and the edges OA, OC and OD along the axes. Find the angle between $\overrightarrow{OF}$ and $\overrightarrow{DB}$.



Perpendicular vectors

15. Use the scalar product to find k.

(a) Find k so that $\begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix} = 0$.

(b) Find the possible values of k for which $\begin{pmatrix} k \\ 1 \\ 2 \end{pmatrix}$ is perpendicular to $\begin{pmatrix} k \\ -5 \\ 2 \end{pmatrix}$.

Proofs with the scalar product

16. A rhombus has sides $\mathbf{a}$ and $\mathbf{b}$ drawn from one vertex, so $|\mathbf{a}| = |\mathbf{b}|$.

(a) Write the two diagonals in terms of $\mathbf{a}$ and $\mathbf{b}$.

(b) Expand $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$.

(c) Hence prove that the diagonals of a rhombus are perpendicular.

(d) Verify your result for $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

Concept check

17. For $A(3, 1, 0)$, $B(1, 1, 0)$ and $C(2, 2, 0)$, a student finds angle $A\hat{B}C$ using $\overrightarrow{AB} = \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix}$ and $\overrightarrow{BC} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and obtains $135°$. Identify the error and find the correct angle.

9.4 The Vector Equation of a Line

A line is fixed by two pieces of information: one point it passes through, and a direction it runs along. Start at a known point $\mathbf{a}$ and move any distance along a direction $\mathbf{b}$. As the **parameter** λ takes every real value, this reaches every point on the line.

FORMULA 9.10 **Vector equation of a line** FORMULA BOOKLET · 3.14

$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$, where $\mathbf{a}$ is the position vector of a point on the line and $\mathbf{b}$ is a direction vector.

An equation of a line met earlier, such as $y = 2x + 1$, is a condition: a point either satisfies it or does not. The vector equation is different, because it produces points: each value of λ gives one point of the line. That is why the vector form works in three dimensions, where a single linear equation in x, y and z describes a plane, not a line.

Writing $\mathbf{r} = (x, y, z)$ and reading off each coordinate gives the **parametric form**; eliminating λ between them gives the **Cartesian form**.

FORMULA 9.11 **Parametric and Cartesian forms of a line** FORMULA BOOKLET · 3.14

Parametric: $x = x_0 + \lambda l, \quad y = y_0 + \lambda m, \quad z = z_0 + \lambda n.$

Cartesian: $\dfrac{x - x_0}{l} = \dfrac{y - y_0}{m} = \dfrac{z - z_0}{n}$, where $\mathbf{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}$.

The Cartesian form needs $l, m, n \neq 0$. If a component is 0, that coordinate is constant; for example, $m = 0$ gives $y = y_0$, written beside the other two fractions.

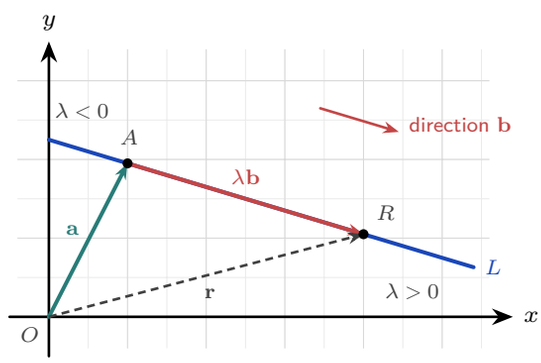


Figure 9.8 A line from a point and a direction: travelling $\lambda\mathbf{b}$ from A reaches a general point R, so $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$. Negative λ runs the other way.

A line has many vector equations. Any point on it can serve as $\mathbf{a}$, and any non-zero multiple of $\mathbf{b}$ can serve as the direction. For example,

$$\mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + t \begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix}$$

are the same line: $(3, 2, 1)$ is the point with $\mu = 1$, and $\begin{pmatrix} -4 \\ 2 \\ -2 \end{pmatrix} = -2 \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. So an answer that looks different from the one in a markscheme can still be right. To check, confirm that the directions are parallel and that a point of one line lies on the other.

How two lines are related depends first on their direction vectors. The **angle between two lines** is the angle between their direction vectors, found by the scalar product (take the acute value). Two lines have the same direction when their direction vectors are scalar multiples.

Two lines in space fall into exactly one of four cases. Answer two questions to decide which: are the directions parallel, and do the lines share a point?

	share a point	**share no point**
directions parallel	**coincident**: the same line written two ways	**parallel**: different lines running the same way
directions not parallel	**intersecting**: they cross at exactly one point	**skew**: they pass without meeting, like an overpass above a road

Only the last case is new. In a plane, two lines are coincident, parallel or crossing. In three dimensions they can also miss each other entirely.

KEY 9.12 **Classifying two lines**

1. Compare the directions. If they are scalar multiples, check whether a point of one lies on the other: yes means **coincident**, no means **parallel**.
2. If the directions differ, set the two equations equal and solve for the parameters. Solve two component equations for the parameters. Use the third equation as the test.
3. A consistent solution means the lines **intersect**, and substituting back gives the point of intersection. An inconsistent system means they are **skew**.

- The two lines **carry different parameters**, λ and μ. Each parameter only measures position along its own line, and a shared point can sit at different parameter values on the two lines. If you used one letter for both, the lines would have to reach the point at the same parameter value. That is a stronger condition. It appears again in §9.4.1, where it describes a collision.

EXAMPLE 9.8

Do the lines $L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ intersect?

SOLUTION

The directions are not multiples of each other, so the lines are not parallel. Set the components equal, one equation for each coordinate:

$$x: \ 2 + \lambda = 1 + 2\mu, \qquad y: \ 1 + \lambda = 3 - \mu, \qquad z: \ \lambda = 2 + \mu.$$

Solve two, check the third. The x and y equations give $\lambda - 2\mu = -1$ and $\lambda + \mu = 2$, so $\mu = 1$ and $\lambda = 1$. Check in the z equation: the left side is 1 and the right side is $2 + 1 = 3$. Since $1 \neq 3$, the system is inconsistent, so the lines are **skew**.

EXAMPLE 9.9

Keep L_1 from the previous example and replace the second line by $L_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Show that L_1 and L_3 intersect, and find the point of intersection, the angle between the lines, and a Cartesian equation of L_3.

SOLUTION

The x and y equations are unchanged, so again $\lambda = 1$ and $\mu = 1$. The z equation is now $\lambda = \mu$, and the check $1 = 1$ holds. The system is consistent, so the lines **intersect**. Substitute $\lambda = 1$ into L_1 to find the point:

$$\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix},$$

so they meet at $\boxed{(3, 2, 1)}$; check with $\mu = 1$ in L_3: $(1+2,\ 3-1,\ 0+1) = (3, 2, 1)$, the same point.

The angle between L_1 and L_3 is the angle between their direction vectors. The scalar product is $(1)(2) + (1)(-1) + (1)(1) = 2$, and the magnitudes are $\sqrt{3}$ and $\sqrt{6}$, so

$$\cos\theta = \frac{2}{\sqrt{3}\sqrt{6}} = \frac{2}{\sqrt{18}} \implies \boxed{\theta \approx 61.9^\circ}$$

This is already acute. Had the scalar product been negative, its modulus would give the acute angle.

To write L_3 in Cartesian form, start from its parametric form, $x = 1 + 2\mu$, $y = 3 - \mu$, $z = \mu$. Make μ the subject of each equation and set the three expressions equal:

$$\boxed{\frac{x-1}{2} = \frac{y-3}{-1} = z}$$

The numerators show the point $(1, 3, 0)$ and the denominators show the direction, with z standing for $\dfrac{z-0}{1}$.

9.4.1 Lines in motion: kinematics

So far the parameter has had no meaning attached to it. Give it one and the line equation describes a moving object. Read λ as *time* t, and $\mathbf{r} = \mathbf{a} + t\mathbf{b}$ says: start at $\mathbf{a}$, and move with constant **velocity** $\mathbf{b}$. The direction of $\mathbf{b}$ is the heading (the direction of travel); its magnitude $|\mathbf{b}|$ is the **speed**. This is the pilot from the opening page: the velocity is the tip-to-tail sum of the aircraft's velocity through the air and the wind velocity.

For example, an aircraft moves through the air with velocity $\binom{0}{200}$ km h^{-1} (due north), in a wind of $\binom{40}{0}$ km h^{-1} (from the west). Its velocity over the ground is $\binom{40}{200}$ km h^{-1}. Its ground speed is $\sqrt{40^2+200^2} \approx 204$ km h^{-1}, and it does not travel due north.

EXAMPLE 9.10

An aircraft passes the point $(20, 10)$ (coordinates in km) at $t = 0$ and flies with constant velocity $\begin{pmatrix}240\\100\end{pmatrix}$ km h^{-1}. Find its speed, its position at $t = 2$ hours, and whether its path passes through the airport at $(500, 210)$.

SOLUTION

The speed is the magnitude of the velocity: $\sqrt{240^2+100^2} = 260$ km h^{-1}. The position at time t is

$$\mathbf{r} = \begin{pmatrix}20\\10\end{pmatrix} + t\begin{pmatrix}240\\100\end{pmatrix}, \qquad \text{so at } t = 2: \quad \mathbf{r} = \begin{pmatrix}500\\210\end{pmatrix}.$$

That *is* the airport, reached at exactly $t = 2$. To test any point, solve the first coordinate for t, then check the others. One value of t must satisfy every equation at once. This is the same test used for intersecting lines.

Two moving objects raise three different questions: whether their paths intersect, whether the objects collide, and how close they come. Their *paths intersect* when the two lines share a point; find it with separate parameters, as in §9.4. The objects *collide* only if they are at that point at the *same time*: one value of t must satisfy every coordinate equation at once. Paths can cross without a collision, when the objects pass the crossing point at different times. Their *closest approach* is the least distance between them over the times allowed.

EXT For two objects moving at different constant velocities, the distance between them usually falls, reaches a least value, and then rises again. The moment of **closest approach** is found from the vector between them. Subtracting the position vectors gives $\mathbf{r}_B - \mathbf{r}_A$, whose components are linear in t, so the squared distance

$$d^2 = |\mathbf{r}_B - \mathbf{r}_A|^2$$

is a quadratic in t. Completing the square gives its least value and the time at which that value occurs. State the times allowed, usually $t \geq 0$. If the vertex lies outside that interval, the least distance occurs at an endpoint, so check the endpoints.

Work with d^2 rather than with d. Squaring clears the square root, and the two are least at the same instant, since squaring preserves order among non-negative numbers.

EXAMPLE 9.11

Two birds fly in straight lines at constant velocity. Taking t in seconds and distances in metres, their position vectors are

$$\mathbf{r}_A = \begin{pmatrix} 8 \\ 5 \\ 1 \end{pmatrix} + t \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + t \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix}.$$

For $t \geq 0$, find the time at which they are closest, and the least distance between them.

SOLUTION

Subtract, component by component:

$$\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix} 2-8 \\ 3-5 \\ 3-1 \end{pmatrix} + t \begin{pmatrix} 3-1 \\ 1-(-1) \\ 2-3 \end{pmatrix} = \begin{pmatrix} 2t-6 \\ 2t-2 \\ 2-t \end{pmatrix}.$$

Its squared length is

$$\begin{aligned} d^2 &= (2t-6)^2 + (2t-2)^2 + (2-t)^2 \\ &= (4t^2 - 24t + 36) + (4t^2 - 8t + 4) + (t^2 - 4t + 4) \\ &= 9t^2 - 36t + 44. \end{aligned}$$

Complete the square:

$$d^2 = 9\left(t^2 - 4t\right) + 44 = 9(t-2)^2 - 36 + 44 = 9(t-2)^2 + 8.$$

The squared bracket is never negative, and it is zero only at $t = 2$. That time lies in the allowed interval $t \geq 0$. So the least value of d^2 is 8, reached at $\boxed{t = 2 \text{ s}}$, the moment of closest approach. The least distance is

$$\boxed{d = \sqrt{8} = 2\sqrt{2} \approx 2.83 \text{ m}}$$

Check either side: at $t = 1$ and at $t = 3$ the formula gives $d^2 = 17$ both times. Both values are larger than 8 and equal to each other, as they must be for a parabola symmetric about $t = 2$.

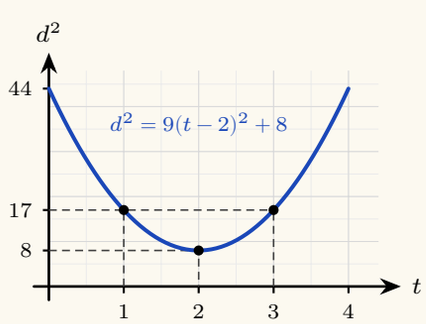


Figure 9.9 The squared distance against time. The least value is 8, at $t = 2$, and the values at $t = 1$ and $t = 3$ are equal.

YOUR TURN 9.4 The Vector Equation of a Line

Writing the equation of a line

18. Writing the equation of a line.

(a) Write a vector equation of the line through $A(1, 2, 3)$ with direction $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(b) Find a direction vector of the line through $A(1, 0, 2)$ and $B(4, 3, 5)$.

(c) Write a vector equation of that line.

19. Forms of the equation.

(a) Write $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$ in parametric form.

(b) Write $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$ in Cartesian form.

(c) Does $(3, 3, 2)$ lie on $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$? Justify your answer.

Classifying two lines

20. Decide whether each pair of lines is coincident, parallel, intersecting or skew. Where the lines intersect, find the point of intersection.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix}$

(c) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(d) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -2 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

Angle between two lines

21. GDC Find the acute angle between each pair of lines.

(a) $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 0 \\ 5 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$

(b) $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the x-axis, $\mathbf{r} = \mu \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

Lines in motion

22. A boat leaves the point $(2, -1)$, in km, at $t = 0$ and moves with constant velocity $\begin{pmatrix} 6 \\ 8 \end{pmatrix}$ km h^{-1}.

(a) Find the speed of the boat.

(b) Write down its position vector at time t.

(c) Find its position after 30 minutes.

(d) A lighthouse stands at $(20, 15)$. Show that the boat does not pass through it.

(e) Sketch the path of the boat on a coordinate grid for $0 \leq t \leq 2$. Mark its positions at $t = 0$, 1 and 2, and the lighthouse.

23. Two drones fly in straight lines at constant velocity. With t in seconds and distances in metres, their position vectors are

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 7 \\ 0 \\ 5 \end{pmatrix} + t \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}.$$

(a) Show that $\mathbf{r}_B - \mathbf{r}_A = \begin{pmatrix} 6-3t \\ t-2 \\ 5 \end{pmatrix}$.
(b) Show that the square of the distance between the drones is $d^2 = 10t^2 - 40t + 65$.
(c) Find the time at which the drones are closest, and their least distance apart.
(d) A third drone has position vector $\mathbf{r}_C = \begin{pmatrix} 4 \\ 2 \\ 4 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}$. For $t \geq 0$, find the least distance between drones A and C, and the time at which it occurs.

Concept check

24. A student says: "*the direction vectors of the two lines are not parallel, so the lines intersect.*" State when this is false, and describe the check that decides.

9.5 The Vector Product

The second way to multiply vectors returns not a number but a *vector*. The **vector product** (also called the cross product) $\mathbf{v} \times \mathbf{w}$ points perpendicular to *both* of the vectors it is built from, and its length is the area of the parallelogram they form. The scalar product measures how closely two vectors point the same way. The vector product describes the plane that contains them.

FORMULA 9.13 **Vector product** FORMULA BOOKLET · 3.16

For $\mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$,

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}, \qquad |\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin\theta,$$

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

The magnitude and the direction can be combined into one statement,

$$\mathbf{v} \times \mathbf{w} = |\mathbf{v}|\,|\mathbf{w}| \sin\theta\, \hat{\mathbf{n}},$$

where $\hat{\mathbf{n}}$ is a unit vector perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, pointing in the direction given by the rule below. Taking magnitudes gives the second formula in the box, since $|\hat{\mathbf{n}}| = 1$.

Two non-parallel vectors have exactly two directions perpendicular to both of them, and they are opposite to each other. Which one you get depends on the order of the two vectors. The direction of $\mathbf{v} \times \mathbf{w}$ is fixed by the **right-hand screw rule**. Curl the fingers of your right hand from $\mathbf{v}$ towards $\mathbf{w}$, and your thumb points along $\mathbf{v} \times \mathbf{w}$. An ordinary screw gives the same direction: turn it from $\mathbf{v}$ towards $\mathbf{w}$, and it moves along $\mathbf{v} \times \mathbf{w}$. Reversing the order reverses the direction.

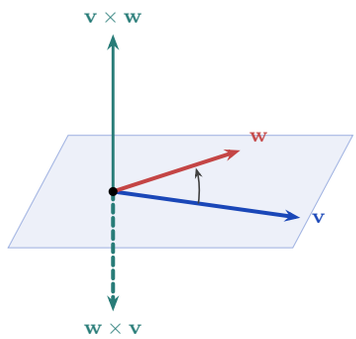


Figure 9.10 Curl the fingers of the right hand from $\mathbf{v}$ towards $\mathbf{w}$: the thumb gives $\mathbf{v} \times \mathbf{w}$. The opposite direction is $\mathbf{w} \times \mathbf{v}$.

Four properties are used often.

- **Perpendicular to both.** The result is at right angles to the plane of $\mathbf{v}$ and $\mathbf{w}$. This is what makes it the tool for building normal vectors in the next section.
- **Not commutative.** It is **anti-commutative**: $\mathbf{w} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{w})$. The order matters, and reversing it reverses the answer. Turning the screw from $\mathbf{w}$ towards $\mathbf{v}$ is the opposite turn, so the screw advances the opposite way, and the magnitude $|\mathbf{v}|\,|\mathbf{w}| \sin\theta$ is the same.
- **Distributive, and scalars come out.** $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = \mathbf{u} \times \mathbf{v} + \mathbf{u} \times \mathbf{w}$ and $(k\mathbf{v}) \times \mathbf{w} = k(\mathbf{v} \times \mathbf{w})$. Both rules are accepted here without proof.
- **Zero for parallel vectors.** If $\mathbf{v}$ and $\mathbf{w}$ are parallel then $\sin\theta = 0$, so $\mathbf{v} \times \mathbf{w} = \mathbf{0}$. In particular $\mathbf{v} \times \mathbf{v} = \mathbf{0}$, the counterpart of $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$. The converse also holds. For non-zero vectors, $\mathbf{v} \times \mathbf{w} = \mathbf{0}$ forces $|\mathbf{v}|\,|\mathbf{w}| \sin\theta = 0$, so $\sin\theta = 0$ and the vectors are parallel.

In practice, compute each component by covering its own row in both vectors and multiplying the remaining four numbers crosswise. The subtraction then runs one way for the outer two components and the other way for the middle one. Covering row 1 leaves rows 2 and 3, and covering row 3 leaves rows 1 and 2; both pairs are in the cyclic order $1 \to 2 \to 3 \to 1$. Covering row 2 leaves rows 1 and 3, which run against that order, so the two products swap places before subtracting. Taking all three the same way is a common error. The vector it produces is usually not perpendicular to both $\mathbf{v}$ and $\mathbf{w}$, so a scalar-product check will usually find the mistake. When a question asks for a vector perpendicular to both, the correct vector product is already the answer; a *unit* vector perpendicular to both needs one further step.

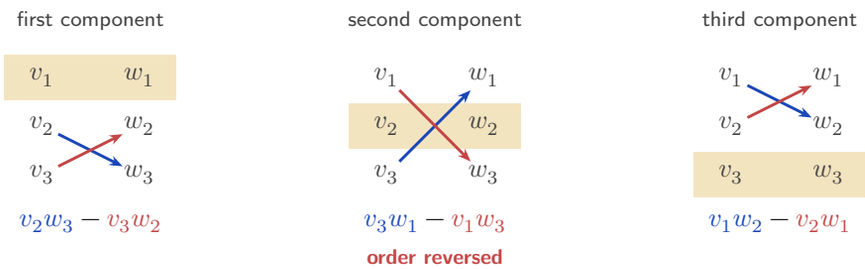


Figure 9.11 Cover each row in turn and cross-multiply the other four numbers. The middle component subtracts the other way round.

EXAMPLE 9.12

For $\mathbf{v} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}$, find $\mathbf{v} \times \mathbf{w}$ and a unit vector perpendicular to both.

SOLUTION

Each component of $\mathbf{v} \times \mathbf{w}$ leaves out its own row:

$$\mathbf{v} \times \mathbf{w} = \begin{pmatrix} (1)(1) - (0)(2) \\ (0)(2) - (2)(1) \\ (2)(2) - (1)(2) \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

The middle entry is $v_3w_1 - v_1w_3$, not $v_1w_3 - v_3w_1$; reading it in the same direction as the other two would give $+2$ there instead of -2. Check it against both vectors: $(1)(2) + (-2)(1) + (2)(0) = 0$ and $(1)(2) + (-2)(2) + (2)(1) = 0$. The rejected version fails this test, since $(1)(2) + (2)(1) + (2)(0) = 4$. For a unit vector perpendicular to both, note that the magnitude is $\sqrt{1 + 4 + 4} = 3$, so divide by it:

$$\hat{\mathbf{n}} = \frac{1}{3} \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

The vector $-\hat{\mathbf{n}} = \frac{1}{3} \begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}$ is equally correct. It is $\mathbf{w} \times \mathbf{v}$ divided by its magnitude, and it runs the opposite way along the same perpendicular line, so unless the question fixes a direction either answer is complete.

9.5.1 Why the formula works

This part is optional. It explains where the component formula comes from; the calculations above do not depend on it.

Start with the base vectors, as for the scalar product. Each is perpendicular to the other two. So the vector product of two different base vectors is a unit vector along the third axis, and the right-hand screw rule decides which way it points. Taking the vector product of one with itself gives $\mathbf{0}$, since the angle is 0 and $\sin 0 = 0$.

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}, \qquad \mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}.$$

Read the first three in the cyclic order $\mathbf{i} \to \mathbf{j} \to \mathbf{k} \to \mathbf{i}$: going forwards gives a plus, and going backwards gives a minus, so $\mathbf{j} \times \mathbf{i} = -\mathbf{k}$.

Both products are built from the same nine numbers: the products $v_i w_j$ of each component of $\mathbf{v}$ with each component of $\mathbf{w}$. Lay them out in a square.

	$\boldsymbol{w_1}$	$\boldsymbol{w_2}$	$\boldsymbol{w_3}$
$\boldsymbol{v_1}$	v_1w_1	v_1w_2 (3)	v_1w_3 (2)
$\boldsymbol{v_2}$	v_2w_1 (3)	v_2w_2	v_2w_3 (1)
$\boldsymbol{v_3}$	v_3w_1 (2)	v_3w_2 (1)	v_3w_3

The **diagonal** is the scalar product. Those are the three terms where a base vector meets itself. Because $\mathbf{i} \cdot \mathbf{i} = 1$, these terms stay. Because $\mathbf{i} \cdot \mathbf{j} = 0$, every term off the diagonal is zero. Adding the three blue entries gives $v_1w_1 + v_2w_2 + v_3w_3$.

The **off-diagonal** entries are the vector product, taken in opposite pairs. Here the diagonal terms are zero, since $\mathbf{i} \times \mathbf{i} = \mathbf{0}$, and each pair of mirror-image entries (shaded red and marked with the same small number) subtracts to give one component:

$$\underbrace{v_2w_3 - v_3w_2}_{\text{the pair missing } 1}, \qquad \underbrace{v_3w_1 - v_1w_3}_{\text{the pair missing } 2}, \qquad \underbrace{v_1w_2 - v_2w_1}_{\text{the pair missing } 3}.$$

So the component of $\mathbf{v} \times \mathbf{w}$ in a given direction is built from the two rows and columns that leave that direction out. That the result is perpendicular to both vectors is checked by the scalar product: $\mathbf{v} \cdot (\mathbf{v} \times \mathbf{w})$ expands to six terms that cancel in pairs,

$$v_1(v_2w_3 - v_3w_2) + v_2(v_3w_1 - v_1w_3) + v_3(v_1w_2 - v_2w_1) = 0,$$

and $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{w}) = 0$ in the same way. The two products are the same array read two ways: down the diagonal, or across it.

9.5.2 Area of a parallelogram and a triangle

The magnitude has a simple geometric meaning: it is the area of the parallelogram with $\mathbf{v}$ and $\mathbf{w}$ as sides. To see why, read the two factors of $|\mathbf{v} \times \mathbf{w}| = |\mathbf{v}|\,|\mathbf{w}| \sin\theta$ separately. Take $|\mathbf{v}|$ as the base. Then $|\mathbf{w}| \sin\theta$ is the perpendicular height, because $\mathbf{w}$ is the slanted side and

$\sin\theta$ keeps only the part of it at right angles to the base. So the magnitude is base $\times$ height, which is the area.

FORMULA 9.14 **Area of a parallelogram** FORMULA BOOKLET · 3.16

Area of a parallelogram $= |\mathbf{v} \times \mathbf{w}|$, where $\mathbf{v}$ and $\mathbf{w}$ are two adjacent sides.

A triangle is half of such a parallelogram, so its area is $\frac{1}{2}|\mathbf{v} \times \mathbf{w}|$; remember the half, because the booklet prints only the parallelogram.

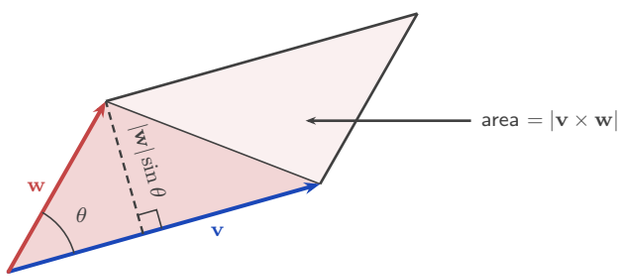


Figure 9.12 The parallelogram on $\mathbf{v}$ and $\mathbf{w}$; the darker triangle is half of it.

EXAMPLE 9.13

Find the area of the triangle with vertices $A(0,0,0)$, $B(3,1,0)$, $C(1,0,2)$.

SOLUTION

Take two sides from the same vertex, $\overrightarrow{AB} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$. Their vector product is

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (1)(2) - (0)(0) \\ (0)(1) - (3)(2) \\ (3)(0) - (1)(1) \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ -1 \end{pmatrix}.$$

Its magnitude is $\sqrt{4 + 36 + 1} = \sqrt{41}$, so the triangle's area is $\frac{1}{2}\sqrt{41}$; with the vector product no angle is needed, only one product and one square root.

YOUR TURN 9.5 The Vector Product

Calculating the vector product

25. Evaluate each vector product.

(a) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ (b) $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$

(c) $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ (d) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \times \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$

26. Use $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$ and $\mathbf{k} \times \mathbf{i} = \mathbf{j}$ to find each product without writing column vectors.

(a) $\mathbf{j} \times \mathbf{k}$ (b) $\mathbf{k} \times \mathbf{j}$

(c) $\mathbf{i} \times (\mathbf{i} + \mathbf{j})$ (d) $(2\mathbf{i}) \times (3\mathbf{k})$

Perpendicular to both

27. Let $\mathbf{a} = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$.

(a) Find $\mathbf{a} \times \mathbf{b}$.

(b) Verify that $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$.

(c) Find a unit vector perpendicular to both.

(d) Write down $\mathbf{b} \times \mathbf{a}$, without further calculation.

Parallel vectors

28. Use the vector product to decide whether each pair of vectors is parallel.

(a) $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} -4 \\ 2 \\ -6 \end{pmatrix}$

(b) $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ 5 \end{pmatrix}$

Areas

29. Areas and magnitudes.

(a) Given $|\mathbf{a}| = 3$, $|\mathbf{b}| = 5$ and an angle of $30°$, find $|\mathbf{a} \times \mathbf{b}|$.

(b) Find the area of the parallelogram with sides $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$.

(c) Find the area of the triangle with vertices $A(1, 0, 1)$, $B(3, 2, 1)$, $C(2, -1, 3)$.

(d) Explain why $\mathbf{v} \times \mathbf{v} = \mathbf{0}$ for every vector $\mathbf{v}$.

Concept check

30. A student calculates $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 4 \\ -1 \\ -6 \end{pmatrix}$.

(a) Use a scalar product to show that the answer is wrong.

(b) Find the correct vector product, and state the likely error.

9.6 Planes

A plane is a flat, endless sheet, and like a line it can be built from a point and some directions. A line needed only one direction, because a line is one-dimensional: one number, the parameter, is enough to say where you are on it. A plane is two-dimensional, so it needs two directions and two numbers. Anchor at $\mathbf{a}$, then travel λ times $\mathbf{b}$ and μ times $\mathbf{c}$. The pair (λ, μ) acts as a coordinate system inside the sheet, and every point of the plane has exactly one such pair. The two directions must not be parallel: if they were, every

combination of them would still point along a single line, and the points would all lie on that line.

FORMULA 9.15 **Vector equation of a plane** FORMULA BOOKLET · 3.17

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}.$$

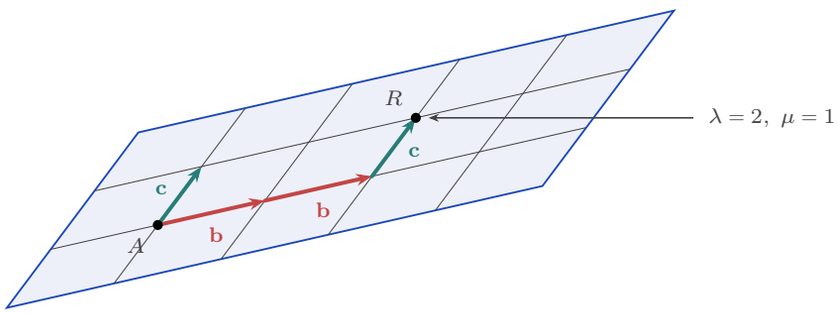


Figure 9.13 Each point of the plane is reached by one pair (λ, μ); here $\lambda = 2$, $\mu = 1$.

With two parameters, this form is awkward to use. It is hard to tell whether a point lies on the plane, or whether two such equations describe the same plane. A better description uses a **normal vector** $\mathbf{n}$: a direction perpendicular to every direction within the plane. Take a known point A on the plane and any point R on it. The displacement $\mathbf{r} - \mathbf{a}$ from A to R lies in the plane, so it is perpendicular to $\mathbf{n}$, and the scalar product says exactly that:

$$(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0.$$

Expanding the bracket gives $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, and writing $\mathbf{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ turns it into one equation in x, y and z.

FORMULA 9.16 **Normal and Cartesian forms of a plane** FORMULA BOOKLET · 3.17

Normal (scalar-product) form: $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$,

Cartesian form: $ax + by + cz = d$, where $\mathbf{n} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $d = \mathbf{a} \cdot \mathbf{n}$.

The Cartesian form uses the booklet's letters. There the plain letters a, b and c are three numbers, the components of the normal. They are not the bold $\mathbf{a}$, which is the position vector of a point on the plane.

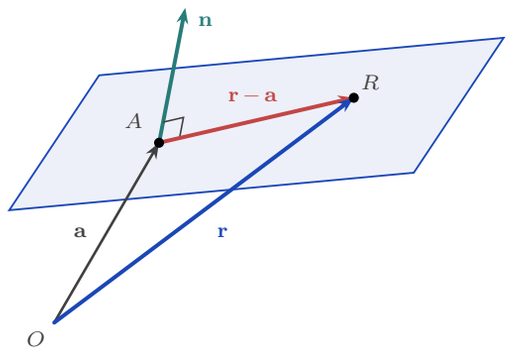


Figure 9.14 A plane fixed by a point A and a normal $\mathbf{n}$.

A Cartesian plane equation can now be read directly: the coefficients of x, y, z *are* the normal vector. To find a plane through three points, use the vector product of two edge vectors to build $\mathbf{n}$, then fix d from any one point.

EXAMPLE 9.14

Find the Cartesian equation of the plane with normal $\mathbf{n} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix}$ through $A(4, 8, -1)$.

SOLUTION

The Cartesian equation is $3x - 2y + 2z = d$. Substitute A to find d:

$$d = 3(4) - 2(8) + 2(-1) = 12 - 16 - 2 = -6, \qquad \text{so} \qquad \boxed{3x - 2y + 2z = -6}$$

The normal supplies the coefficients and a single point supplies the constant.

EXAMPLE 9.15

Find the Cartesian equation of the plane through $A(3, 0, 0)$, $B(0, 4, 0)$, $C(0, 0, 2)$.

SOLUTION

Two edge vectors span the plane: $\overrightarrow{AB} = \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix}$ and $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}$. Their vector product is normal to both, hence to the plane:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} (4)(2) - (0)(0) \\ (0)(-3) - (-3)(2) \\ (-3)(0) - (4)(-3) \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 12 \end{pmatrix} = 2 \begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}.$$

Any scalar multiple of a normal is a normal, so use the cleaner $\begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$. Through A: $d = 4(3) = 12$, giving

$$\boxed{4x + 3y + 6z = 12}$$

Check with B: $3(4) = 12$; and C: $6(2) = 12$. Both lie on it.

That example explains the tripod from the opening page. Take three points that are not in a line. The two edge vectors from one of them are not parallel, so their vector product is not $\mathbf{0}$, and it is a normal. A normal and a point fix one plane. So three such points lie in exactly one plane, and three feet can always rest on a flat floor together. A fourth point has no such guarantee, so a fourth foot need not touch the floor when the other three do.

The plane of that example also has a vector equation. Take A as the point and the two edge vectors as the directions:

$$\mathbf{r} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix}.$$

Here $\lambda = 1$, $\mu = 0$ gives B, and $\lambda = 0$, $\mu = 1$ gives C.

To go the other way, from a vector equation to the Cartesian form, take the vector product of the two direction vectors as the normal, then find d from the point $\mathbf{a}$. Here $\begin{pmatrix} -3 \\ 4 \\ 0 \end{pmatrix} \times \begin{pmatrix} -3 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \\ 12 \end{pmatrix}$, a multiple of $\begin{pmatrix} 4 \\ 3 \\ 6 \end{pmatrix}$, so the same Cartesian equation returns.

EXAMPLE 9.16

Decide whether $D(-3, 4, 2)$ lies on this plane.

SOLUTION

Look for one pair (λ, μ) that fits all three coordinates. The y-coordinate needs $4\lambda = 4$, so $\lambda = 1$. The z-coordinate needs $2\mu = 2$, so $\mu = 1$. Test the pair in the x-coordinate: $3 - 3(1) - 3(1) = -3$, which is the x-coordinate of D. So D lies on the plane. The Cartesian equation agrees: $4(-3) + 3(4) + 6(2) = 12$. Two coordinates fix the parameters and the third is the test, as for a point on a line.

9.6.1 Angles with a plane

Every angle in this section comes from the same calculation: the scalar product of two direction vectors, with a modulus to make the angle acute,

$$\cos\theta = \frac{|\mathbf{u} \cdot \mathbf{v}|}{|\mathbf{u}|\,|\mathbf{v}|}.$$

What changes is which two vectors you use. A plane has no single direction of its own, but it has a normal, and one fact from plane geometry lets you use the normal in place of the plane:

In a plane, the angle between two lines is the same as the angle between two lines perpendicular to them.

The reason is that turning each arm of an angle through $90°$, in the same sense, turns the whole angle without changing its size. To use the fact for two planes, look along the line where they meet, so that each plane is seen edge-on as a line. The normals are perpendicular to those two lines, so the angle between the normals is the angle between the planes.

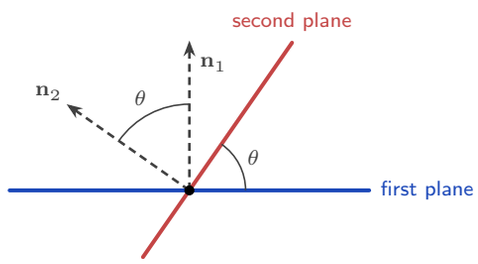


Figure 9.15 Two planes seen edge-on, looking straight along the line where they meet. Each plane shows as a line. The angle between the planes and the angle between their normals are the same angle.

The modulus makes the answer the acute angle, which is the one meant by the angle between two planes.

KEY 9.17 **Angle between two planes**

Angle between two planes (normals $\mathbf{n}_1, \mathbf{n}_2$): $\cos\theta = \dfrac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{|\mathbf{n}_1|\,|\mathbf{n}_2|}$.

EXAMPLE 9.17

Find the acute angle between the planes $2x + 2y - z = 4$ and $x - 2y + 2z = 1$.

SOLUTION

Read the normals straight off the coefficients: $\mathbf{n}_1 = \begin{pmatrix} 2 \\ 2 \\ -1 \end{pmatrix}$ and $\mathbf{n}_2 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$. Then $\mathbf{n}_1 \cdot \mathbf{n}_2 = 2 - 4 - 2 = -4$, and both magnitudes are 3, so

$$\cos\theta = \frac{|-4|}{3 \times 3} = \frac{4}{9} \implies \boxed{\theta \approx 63.6°}$$

The scalar product came out negative, which means the angle between the normals is obtuse. Taking the modulus gives the acute angle instead, which is what the question asks for.

For a line and a plane there is one extra step: find the angle with the normal, then subtract it from $90°$. Call the angle you want θ. It is the angle between the line and its projection *in* the plane: the line formed by the feet of perpendiculars dropped from the line to the plane §7.1.1. The scalar product of the line's direction $\mathbf{b}$ with the normal $\mathbf{n}$ gives a different angle, φ, **the angle between the line and the *normal***. The normal is perpendicular to the projection, so the two angles add to $90°$, as the figure below shows. In symbols:

$$\cos\varphi = \frac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}, \qquad \theta = 90° - \varphi.$$

Because $\cos\varphi = \sin\theta$, the same calculation is often written in one step as $\sin\theta = \dfrac{|\mathbf{b}\cdot\mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}$.

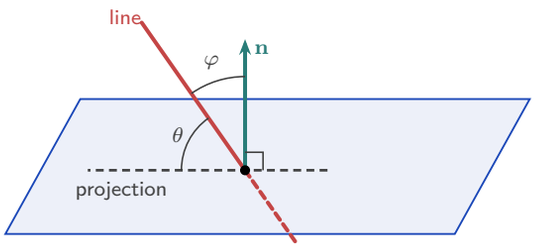


Figure 9.16 The normal is at 90° to the projection, so $\theta + \varphi = 90^\circ$.

The four cases side by side; each needs non-zero vectors.

Angle between	Calculation	Range
two vectors	$\cos\theta$, no modulus	0° to 180°
two lines (directions)	$\cos\theta$, with modulus	0° to 90°
two planes (normals)	$\cos\theta$, with modulus	0° to 90°
a line and a plane	$\cos\varphi$ with modulus, then $\theta = 90^\circ - \varphi$	0° to 90°

With the modulus, the formula gives the smaller of the two angles the objects make, which is acute unless they are perpendicular or parallel.

EXAMPLE 9.18

Find the angle between the line $\mathbf{r} = \begin{pmatrix}1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\2\end{pmatrix}$ and the plane $2x + 2y + z = 4$.

SOLUTION

The direction is $\mathbf{b} = \begin{pmatrix}2\\1\\2\end{pmatrix}$, and the normal is read straight off the coefficients, $\mathbf{n} = \begin{pmatrix}2\\2\\1\end{pmatrix}$. Then $\mathbf{b}\cdot\mathbf{n} = 4 + 2 + 2 = 8$, with $|\mathbf{b}| = 3$ and $|\mathbf{n}| = 3$. The scalar product gives the angle φ between the line and the normal:

$$\cos\varphi = \frac{|8|}{3\times 3} = \frac{8}{9} \implies \varphi \approx 27.3^\circ.$$

The angle with the plane is the complement:

$$\theta = 90^\circ - 27.3^\circ \approx 62.7^\circ$$

A common mistake is to stop at φ: 27.3° is the angle to the *normal*, not to the plane.

9.6.2 Intersections: line and plane, plane and plane

To find where a line meets a plane, substitute the line's parametric coordinates into the plane's Cartesian equation. That gives one equation in λ. Solve it, then substitute back to get the point. If the λ terms cancel, the line is parallel to the plane. The line then misses the plane when the constants disagree, and lies entirely in it when they agree.

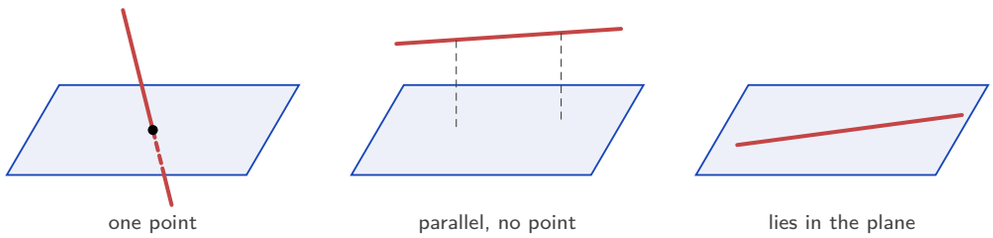


Figure 9.17 A line crosses a plane once, is parallel and misses it, or lies in it. In the last two cases the λ terms cancel.

EXAMPLE 9.19

Find where the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ meets the plane $3x + y - z = 9$.

SOLUTION

Parametrically, $x = 1 + 2\lambda$, $y = 2 - \lambda$, $z = 3\lambda$. Substituting into the plane,

$$3(1 + 2\lambda) + (2 - \lambda) - 3\lambda = 9 \implies 2\lambda + 5 = 9 \implies \lambda = 2,$$

so the point is $\boxed{(5, 0, 6)}$, and as a check, $3(5) + 0 - 6 = 9$.

Two non-parallel planes do not meet at a point; they meet along a whole line, the **line of intersection**. That line runs within both planes. So its direction is perpendicular to both normals, and the vector product of the two normals gives it directly. For a point on the line, set one coordinate to zero and solve the two plane equations for the other two. If that system has no solution, the line never crosses that coordinate plane; set a different coordinate to zero instead.

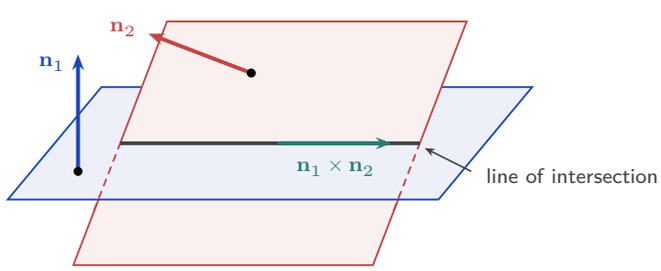


Figure 9.18 Two planes meet along a line. Its direction $\mathbf{n}_1 \times \mathbf{n}_2$ is perpendicular to both normals.

EXAMPLE 9.20

Find a vector equation of the line of intersection of the planes $x + y + 2z = 9$ and $3x - y + z = 7$.

SOLUTION

Note that the direction is perpendicular to both normals:

$$\mathbf{b} = \mathbf{n}_1 \times \mathbf{n}_2 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}.$$

For a point on the line, pick a coordinate and set it to zero. With $z = 0$: $x + y = 9$ and $3x - y = 7$, so $x = 4$, $y = 5$. Hence

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$$

Check the direction: $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = 0$ and $\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix} \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} = 0$.

9.6.3 Three planes

Three planes form a system of three linear equations, the same systems solved by row reduction in §4.3. Nothing new has to be calculated here. What is new is that every algebraic outcome now has a shape, and the shape is often easier to remember than the algebra.

There are five arrangements of three distinct planes. The first two have solutions; the last three do not, and they are told apart by how many of the normals are parallel.

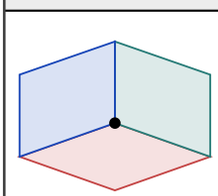
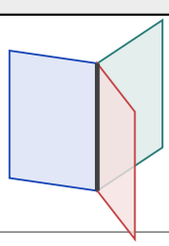
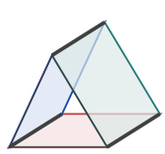
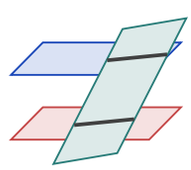
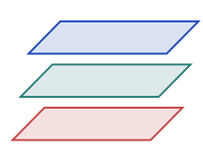
Unique solution	Infinitely many solutions	No solution (inconsistent system)		
		no two normals parallel	two normals parallel	all three normals parallel
				
meet at one point	meet along one line	form a triangular prism	one plane cuts two parallel planes	three parallel planes

Figure 9.19 The five ways three distinct planes can be arranged. The dark lines are lines of intersection.

Three further cases arise when two or more of the equations are multiples of one another, so that they describe the same plane.

- All three describe the same plane: every point of that plane is a solution.
- Two describe the same plane and the third crosses it: the solutions form the line where they meet.
- Two describe the same plane and the third is parallel to it: there is no solution.

Row reduction tells you which case you are in without any picture. First look for a row $(0\ 0\ 0 \mid c)$ with $c \neq 0$. It means the system is inconsistent, so the three planes have no common point. If there is no such row, a row $(0\ 0\ 0 \mid 0)$ means infinitely many solutions: the planes share a line, or a whole plane if two rows become all zeros.

To tell a common line from a prism, take the line where two of the planes meet and substitute it into the third plane's equation. If the parameter cancels, the third plane is parallel to that line. Equal constants mean the third plane contains the line, so all three planes share it. Such planes are sometimes called a *sheaf*. Unequal constants mean a prism, provided no two of the planes are parallel.

EXAMPLE 9.21

The planes $x + y + 2z = 9$ and $3x - y + z = 7$, whose line of intersection was found above, are joined by a third, $4x + 3z = c$. (a) Describe the configuration of the three planes for $c = 16$ and for $c \neq 16$. (b) For $c = 16$, solve the system by row reduction, and write the general solution as a vector equation of a line.

SOLUTION

(a) The third equation has no y-term, so look for a combination of the first two that cancels y. Adding them gives $4x + 3z = 16$, so every point on their line of intersection satisfies $4x + 3z = 16$. Indeed, check the line $\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$ in the third plane: $4(4 + 3\lambda) + 3(-4\lambda) = 16 + 12\lambda - 12\lambda = 16$, for every λ.
So when $c = 16$ the third plane contains the entire line, and all three planes share it. When $c \neq 16$ the third plane is parallel to that line but does not contain it. No two of the normals $\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$, $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$ are parallel, so no two of the planes are parallel. The three planes therefore form a triangular prism: each pair meets in a line, the three lines are parallel, and no point lies on all three. In the language of §4.3.2, the system is consistent with infinitely many solutions when $c = 16$, and inconsistent otherwise.
(b) Row-reduce the augmented matrix, as in §4.3.1. Use $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - 4R_1$, then $R_3 \to R_3 - R_2$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 3 & -1 & 1 & 7 \\ 4 & 0 & 3 & 16 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -4 & -5 & -20 \\ 0 & -4 & -5 & -20 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & -4 & -5 & -20 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The last row says $0 = 0$, so one unknown is free. No pivot lies in the z-column, so let the free variable be $z = t$, with $t \in \mathbb{R}$. Row 2 gives $-4y - 5t = -20$, so $y = 5 - \frac{5}{4}t$. Row 1 then gives $x = 9 - y - 2t = 4 - \frac{3}{4}t$.

Write the solution as a column, and separate the terms without t from the terms with t:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{3}{4} \\ -\frac{5}{4} \\ 1 \end{pmatrix}.$$

This is a vector equation of a line. The constant terms give a point on it, and the coefficients of t give its direction. The point $(4, 5, 0)$ is the one found above, and the direction is $-\frac{1}{4}\begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$, parallel to the vector product of the normals. So row reduction gives the same line as the vector product.

Choosing $z = -4s$ instead clears the fractions and gives

$$\mathbf{r} = \begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix} + s \begin{pmatrix} 3 \\ 5 \\ -4 \end{pmatrix}$$

The same line can be written with different parameters.

YOUR TURN 9.6 Planes

Equations of a plane

31. Equations of planes.

(a) Write the Cartesian equation of the plane with normal $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ through the origin.

(b) State a normal vector to the plane $2x - y + 3z = 5$.

(c) Find the Cartesian equation of the plane with normal $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ through $(2, 0, 1)$.

(d) Does $(1, 1, 1)$ lie on the plane $x + y + z = 3$?

32. Forms of the equation of a plane.

(a) Write the equation of the plane through $(1, -2, 3)$ with normal $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ in the form $\mathbf{r} \cdot \mathbf{n} = \mathbf{a} \cdot \mathbf{n}$, and in Cartesian form.

(b) A plane has vector equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix}$. Find a normal vector and the Cartesian equation of the plane.

(c) Does the point $(2, 1, -2)$ lie on the plane in part (b)? Justify your answer.

33. Planes from points.

(a) Find the Cartesian equation of the plane through $A(1,0,2)$, $B(3,1,0)$ and $C(0,2,1)$.

(b) Find the Cartesian equation of the plane that contains the line $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and the point $(2,1,3)$.

Angles with a plane

34. GDC Find each acute angle.

(a) The angle between the planes $x + 2y - z = 3$ and $2x - y + z = 1$.

(b) The angle between the planes $x + y = 3$ and $y + z = 1$.

(c) The angle between the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ and the plane $3x - 2y + 2z = 11$.

(d) The angle between the line $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and the plane $z = 0$.

Intersections

35. Intersections.

(a) Find where the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$ meets the plane $3x - 2y + 2z = 11$.

(b) Find a direction vector of the line of intersection of $x + y + z = 6$ and $2x - y + z = 3$.

(c) Find a point on both planes in part (b), and hence a vector equation of their line of intersection.

(d) Find a vector equation of the line of intersection of $x + y + z = 4$ and $x + y - z = 2$.

Three planes

36. For each system of three planes, decide whether they meet at a single point, in a line, or not at all, and describe the arrangement (for example, a sheaf or a triangular prism). Give the point or the line where there is one.

(a) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + y - z = 1$

(b) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + 2z = 8$

(c) $x + y + z = 6, \quad 2x + 2y + 2z = 10, \quad x - y + z = 2$

(d) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + 2z = 3$

(e) Solve $x + 2y - z = 3, \quad 2x + 5y - z = 7, \quad x + 3y = 4$ by row reduction, and write the solution as a vector equation of a line.

Concept check

37. GDC To find the angle between the line $\mathbf{r} = \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and the plane $z = 0$, a student uses the direction vector and the normal $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ in $\cos\theta = \dfrac{|\mathbf{b} \cdot \mathbf{n}|}{|\mathbf{b}|\,|\mathbf{n}|}$, and gets $\theta \approx 48.2°$. Identify the error and find the correct angle.

9.7 Distances EXT

The distance between a point and a line, or between two lines or planes that do not meet, always means the *shortest* distance between them. Every case in this section uses one idea: the shortest link is perpendicular, so a scalar product of zero gives the equation to solve. No new formula is needed.

Shortest distance is not itself listed in the syllabus, which asks for intersections and angles. Every step below is built from tools that are listed, and exam questions can reach the same answers by asking for the foot of a perpendicular (the point where the perpendicular meets the line or plane). Treat this section as the natural extension of what comes before, not as separate material.

KEY 9.18 **The shortest link is perpendicular**

Of all the segments joining a point to a line or a plane, or joining two lines or planes that do not meet, the shortest is perpendicular to each line or plane it meets. Perpendicular means a scalar product of zero, so the condition becomes an equation.

Nothing here is printed in the formula booklet. These methods find distances to whole lines and planes, which have no ends. Sometimes a question is about a finite segment, or a bounded piece of a plane. Then the foot found this way may lie outside it. In that case the nearest point is at an endpoint or an edge, so check those as well.

KEY 9.19 **Finding a shortest distance**

Two constructions cover every case.

1. **To a line** (a point to a line, or two skew lines): make the joining vector perpendicular to the line. Choose a general point on each object, form the displacement between them, set its scalar product with each line's direction to zero, solve for the parameters, and take the magnitude.
2. **To a plane** (a point to a plane, or two parallel planes): travel along the normal. Walk from the point along $\mathbf{p} + \lambda\mathbf{n}$, substitute into the plane's equation to find λ, and the distance is $|\lambda|\,|\mathbf{n}|$.

9.7.1 From a point to a line

Let P be the given point. A general point on the line is R, with position vector $\mathbf{a} + \lambda\mathbf{b}$. As λ varies, R moves along the line and the length PR changes. It is shortest exactly when $\overrightarrow{PR}$ is perpendicular to the line, so $\overrightarrow{PR} \cdot \mathbf{b} = 0$. That is one equation in one unknown. The point F where this happens is the **foot of the perpendicular** from P.

The reason is Pythagoras. Let G be any point of the line other than F. The triangle PFG has its right angle at F, so $PG^2 = PF^2 + FG^2$, which is greater than PF^2. The same argument applies to a point and a plane, with G any other point of the plane.

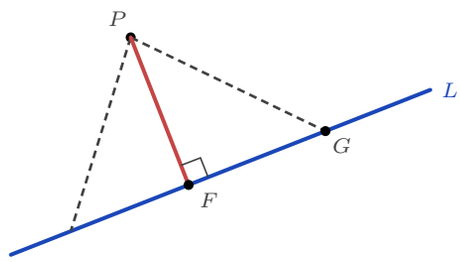


Figure 9.20 Each segment joins P to the line L. The shortest, PF, meets L at right angles.

EXAMPLE 9.22

Find the distance from $P(0,1,3)$ to the line $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$.

SOLUTION

Choose the point. A general point on the line is $R = (1+\lambda,\ 2\lambda,\ -1+2\lambda)$.

Form the displacement.

$$\overrightarrow{PR} = \begin{pmatrix} 1+\lambda \\ 2\lambda - 1 \\ 2\lambda - 4 \end{pmatrix}.$$

Impose perpendicularity to the direction:

$$\overrightarrow{PR} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = (1+\lambda) + 2(2\lambda - 1) + 2(2\lambda - 4) = 9\lambda - 9 = 0,$$

Solve: $\lambda = 1$, so the foot is $F(2,2,1)$.

Take the magnitude. $\overrightarrow{PF} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, so the distance is $\sqrt{4+1+4} = \boxed{3}$. The method also gives the point of the line closest to P, the foot $F(2,2,1)$, which questions often ask for as well.

9.7.2 From a point to a plane

For a plane the perpendicular direction is already known: it is the normal $\mathbf{n}$. So no scalar product needs to be set to zero. Instead, walk from P along $\mathbf{n}$ until you reach the plane. The general point on that walk is $\mathbf{p} + \lambda\mathbf{n}$, where $\mathbf{p}$ is the position vector of P, and substituting it into $\mathbf{r} \cdot \mathbf{n} = d$ gives one equation in λ. The distance travelled is then $|\lambda|\,|\mathbf{n}|$.

EXAMPLE 9.23

Find the distance from $P(1,1,1)$ to the plane $2x - y + 2z = 12$.

SOLUTION

Travel along the normal. The normal is $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, so a point on the walk from P is

$$\mathbf{p} + \lambda\mathbf{n} = (1 + 2\lambda,\ 1 - \lambda,\ 1 + 2\lambda).$$

Solve. This lies on the plane when

$$2(1 + 2\lambda) - (1 - \lambda) + 2(1 + 2\lambda) = 12 \implies 9\lambda + 3 = 12 \implies \lambda = 1.$$

Distance. The foot is $(3, 0, 3)$, and the distance is $|\lambda|\,|\mathbf{n}| = 1 \times 3 = \boxed{3}$; check the foot on the plane: $6 - 0 + 6 = 12$. ✓

9.7.3 Between two skew lines

Skew lines never meet, yet there is still a shortest gap between them, and the same idea finds it. Take a general point on each, P_1 on the first and P_2 on the second. The joining vector $\overrightarrow{P_1P_2}$ is shortest when it is perpendicular to *both* directions at once. If it were not perpendicular to one of the lines, the point-to-line argument above would give a closer point on that line, and so a shorter segment. That is two scalar products set to zero, so two equations in the two unknowns λ and μ.

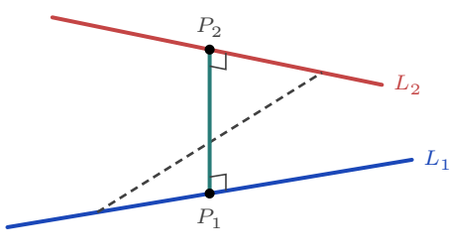


Figure 9.21 Skew lines L_1 and L_2. The shortest segment P_1P_2 meets both lines at right angles.

EXAMPLE 9.24

Find the shortest distance between $L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ and

$L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

SOLUTION

Choose the points. A general point on each line: $P_1 = (\lambda,\ \lambda,\ 1 + \lambda)$ and $P_2 = (1 + \mu,\ 1,\ \mu)$.

Form the displacement.

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 1+\mu-\lambda \\ 1-\lambda \\ \mu-1-\lambda \end{pmatrix}.$$

Impose perpendicularity to both directions, which gives two equations:

$$\overrightarrow{P_1P_2} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1-3\lambda+2\mu = 0, \qquad \overrightarrow{P_1P_2} \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2\mu-2\lambda = 0.$$

Solve. The second gives $\mu = \lambda$; the first then gives $1-\lambda = 0$, so $\lambda = \mu = 1$. The feet are $P_1(1,1,2)$ and $P_2(2,1,1)$.

Take the magnitude.

$$\overrightarrow{P_1P_2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \qquad \boxed{\text{distance} = \sqrt{2}}$$

The method needs only two simultaneous equations, and it also gives both endpoints of the shortest segment.

9.7.4 The remaining cases

Nothing else needs a new method. Every other pairing either meets, and so has distance zero, or is parallel, and then any convenient point reduces it to one of the three cases above.

The two objects	What to do
Two parallel lines	Take any point on one line, then use point to line
Two intersecting lines	They meet, so the distance is 0
Two skew lines	Two scalar products set to zero, as above
A line parallel to a plane	Take any point on the line, then use point to plane
A line meeting a plane	They meet, so the distance is 0
Two parallel planes	Take any point on one plane, then use point to plane
Two non-parallel planes	They meet along a line, so the distance is 0

EXAMPLE 9.25

Find the distance between the parallel planes $2x - y + 2z = 12$ and $2x - y + 2z = 3$.

SOLUTION

Both have normal $\mathbf{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$, so they are parallel and never meet. Take any point on the second plane; setting $y = z = 0$ gives $x = \frac{3}{2}$, so $A(\frac{3}{2}, 0, 0)$ lies on it. Now walk from A along the normal to the first plane:

$$2\left(\tfrac{3}{2} + 2\lambda\right) - (-\lambda) + 2(2\lambda) = 12 \implies 9\lambda + 3 = 12 \implies \lambda = 1,$$

so the distance is $|\lambda|\,|\mathbf{n}| = \boxed{3}$; any other starting point on the second plane gives the same answer, because the two planes are parallel.

YOUR TURN 9.7 Distances EXT

From a point to a line

38. Distance from a point to a line. Throughout, $L_1 : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$.

(a) Find the foot of the perpendicular from $P(6, 2, 1)$ to L_1, and hence the distance from P to L_1.

(b) Find the distance from $Q(3, 3, 0)$ to L_1.

(c) The line L_2 has equation $\mathbf{r} = \begin{pmatrix} 6 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -4 \\ 2 \\ -4 \end{pmatrix}$. Explain why L_2 is parallel to L_1, and write down the distance between them.

(d) Check your answer to part (c) by repeating the method with the point on L_2 where $\mu = 1$.

From a point to a plane

39. Distance from a point to a plane. Throughout, $\Pi : x + 2y + 2z = 14$.

(a) Find the foot of the perpendicular from $P(0, -1, -1)$ to Π, and hence the distance from P to Π.

(b) Find the distance from the origin to Π.

(c) Find the distance between Π and the parallel plane $x + 2y + 2z = 5$.

(d) Show that the line $\mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}$ is parallel to Π, and write down the distance between the line and the plane.

Between two lines

40. Distance between two lines. Throughout, $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 4 \\ -2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$.

(a) Show that L_1 and L_2 are skew.

(b) Write down general points P_1 on L_1 and P_2 on L_2, and find the values of λ and μ for which $\overrightarrow{P_1P_2}$ is perpendicular to both direction vectors.

(c) Hence state the two feet of the common perpendicular and the shortest distance between L_1 and L_2.

(d) The lines $L_3 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $L_4 : \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ are not parallel. Apply the same method to them and explain what your answer shows.

Concept check

41. A student says: "*the distance from the origin to the plane* $x + 2y + 2z = 14$ *is* 14." Identify the error and find the correct distance.

Chapter 9 · Vectors

Reflections

TOK The same plane can be a drawing, a vector equation, or the single equation $2x - y + 3z = 7$. The symbols are easier to calculate with. The picture is easier to believe. If the two disagree in your head, which one do you trust? And what happens in four dimensions, where the picture is no longer available to you?

IM The products of this chapter were chosen, not found. In October 1843 the Irish mathematician William Rowan Hamilton was walking beside a canal in Dublin. Suddenly he saw how to multiply numbers of the form $a + bi + cj + dk$. He was so excited that he carved the rule $i^2 = j^2 = k^2 = ijk = -1$ into the stone of Brougham Bridge. His *quaternions* multiply in a way that depends on the order, as the vector product does. About forty years later the American Josiah Willard Gibbs and the English engineer Oliver Heaviside kept only the two operations that physicists actually used, the scalar product and the vector product. Their vector notation replaced Hamilton's. What should decide which of two definitions survives: which is more elegant, or which is easier to use?

RW Three-dimensional games and animated films use the mathematics of this chapter. A program finds how brightly each surface is lit from the scalar product of two unit vectors: one pointing towards the light, and one normal to the surface. A value near 1 means the surface faces the light, so it is bright. A value near 0 means the light arrives edge-on, so the surface is dark. A negative value means the surface faces away from the light. The vector product builds the normals in the first place. At the same time the physics engine adds force and velocity vectors tip to tail, many times a second, so a thrown object moves the way your eye expects.

EXT The vector product, as defined here, works only in three dimensions. In two dimensions there is no room for a vector perpendicular to two others. In four dimensions or more there is too much room: no single perpendicular direction is picked out, because a whole family of them works. Three dimensions is the only case where two vectors determine exactly one perpendicular line. That is why $\mathbf{v} \times \mathbf{w}$ can be a vector at all. (A product with most of the same properties also exists in seven dimensions, and in no others.)

TOK A vector keeps the direction that a scalar throws away. Does that make a vector more valuable, or does each simply answer a different question?

End-of-Chapter Problems

1. Given $\mathbf{a} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$:

(a) find $\mathbf{a} + 2\mathbf{b}$; [2]

(b) find $|\mathbf{a}|$; [1]

(c) find the unit vector in the direction of $\mathbf{a}$. [2]

2. For $A(1, -1, 2)$ and $B(3, 2, 4)$:

(a) find $\overrightarrow{AB}$; [1]

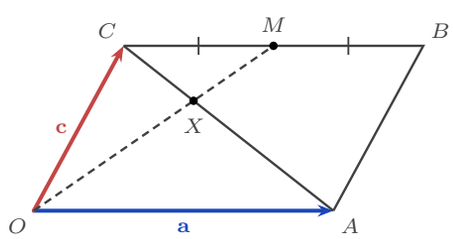
(b) find $|\overrightarrow{AB}|$; [2]

(c) find the midpoint of AB. [2]

3. The point $M(3, -1, 4)$ is the midpoint of AB, where A is $(1, 2, 5)$. Find the coordinates of B. [3]

4. Show that the points $A(1, 2, 3)$, $B(2, 4, 5)$ and $C(4, 8, 9)$ are collinear. [3]

5. The figure shows the parallelogram $OABC$, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{c}$. The point M on $[BC]$ has $CM = MB$, and the line OM meets the diagonal AC at X.



(a) Write down $\overrightarrow{OB}$ and $\overrightarrow{OM}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. [2]

(b) Write down $\overrightarrow{AC}$ in terms of $\mathbf{a}$ and $\mathbf{c}$. [1]

(c) The point X satisfies $\overrightarrow{OX} = \lambda\overrightarrow{OM}$ and $\overrightarrow{AX} = \mu\overrightarrow{AC}$. Write down two expressions for $\overrightarrow{OX}$ in terms of $\mathbf{a}$ and $\mathbf{c}$, and hence find λ and μ. [4]

(d) Hence show that X divides both AC and OM in the ratio $2 : 1$. [2]

6. The vectors $\mathbf{a} = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ k \end{pmatrix}$ are perpendicular. Find k. [3]

7. Find the angle between $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$. [4]

8. GDC Given $|\mathbf{a}| = 5$, $|\mathbf{b}| = 3$ and $\mathbf{a} \cdot \mathbf{b} = 9$, find the angle between $\mathbf{a}$ and $\mathbf{b}$. [3]

9. GDC Find, in degrees, the acute angle between two lines with direction vectors $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$. [3]

10. Find a vector equation of the line through $A(2, -1, 3)$ and $B(4, 1, 7)$. [3]

11. Consider the line $L : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$.

(a) Find the point on L when $\lambda = 2$. [2]

(b) Determine whether $P(7, 3, 4)$ lies on L. [2]

12. The lines $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ intersect. Find the point of intersection. [5]

13. Show that the lines $L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$ intersect, and find the point of intersection. [6]

14. At time $t = 0$ seconds a drone is at the point $(2, -1, 10)$ (coordinates in metres) and moves with constant velocity $\begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ m s^{-1}.

(a) Write down a vector equation for the drone's position at time t. [2]

(b) Find the drone's speed. [2]

(c) Determine whether the drone passes through the point $(17, 19, 10)$, and if so, at what time. [3]

15. GDC Two aircraft fly at the same altitude. At time t hours their position vectors, in km, are

$$\mathbf{r}_A = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 300 \\ 400 \\ 0 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 600 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 0 \\ 500 \\ 0 \end{pmatrix}.$$

(a) Find the speed of each aircraft. [3]

(b) Find $\overrightarrow{AB}$ at time t. [3]

(c) Show that the distance d between them satisfies $d^2 = 100\,000t^2 - 360\,000t + 360\,000$. [4]

(d) Find the time at which the aircraft are closest, and the least distance between them. [4]

16. GDC Two drones move with constant velocity. At time t seconds their positions, in metres, are

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + t \begin{pmatrix} 4 \\ -3 \\ 12 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 3 \\ -13 \\ 6 \end{pmatrix} + t \begin{pmatrix} 2 \\ 3 \\ 6 \end{pmatrix}.$$

(a) Find the speed of each drone. [3]

(b) Show that the two flight paths cross, and find the point where they cross. [5]

(c) Explain why the drones nevertheless do not collide. [3]

(d) Find the distance between the drones at $t = 2$. [3]

(e) Find the acute angle between the two flight paths. [3]

17. Given $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$, find $\mathbf{a} \times \mathbf{b}$ and $|\mathbf{a} \times \mathbf{b}|$. [4]

18. Find a unit vector perpendicular to both $\mathbf{a} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. [5]

19. The line $L : \mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ meets the plane $z = 4$. Find the point of intersection. [3]

20. A plane Π has normal $\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$ and passes through $A(1, 2, -1)$.

(a) Find the Cartesian equation of Π. [3]
(b) Determine whether $B(2, 1, 0)$ lies on Π. [1]

21. The plane Π has equation $2x + y - 2z = 6$.

(a) State a normal vector to Π. [1]
(b) Find where the line $\mathbf{r} = \lambda \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$ meets Π. [4]

22. GDC Find the acute angle between the planes $x + y + z = 1$ and $x - y + z = 3$. [4]

23. The points $A(1, 0, 0)$, $B(0, 2, 0)$ and $C(0, 0, 3)$ are given.

(a) Find the Cartesian equation of the plane ABC. [3]
(b) Find the area of triangle ABC. [3]

24. Find a vector equation of the line of intersection of the planes $x + y - z = 2$ and $2x - y + z = 1$. [6]

25. GDC Find the angle that the line $L : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ makes with the plane $x + 2y + 2z = 5$. [5]

26. Consider the planes $\Pi_1 : x + y + z = 3$, $\Pi_2 : x - y + z = 1$ and $\Pi_3 : x + z = c$, where $c \in \mathbb{R}$.

(a) Find a vector equation of the line of intersection of Π_1 and Π_2. [4]
(b) Show that when $c = 2$ the three planes intersect in this line. [2]
(c) Describe geometrically the configuration of the three planes when $c \neq 2$. [2]

27. The points $A(1, 2, 1)$, $B(3, 1, 2)$ and $C(2, 4, 3)$ are given.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]
(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]
(c) Hence find the Cartesian equation of the plane Π through A, B and C. [3]

(d) The line L_1 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. Find the coordinates of the point where L_1 meets Π. [3]

(e) The line L_2 has equation $\mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ k \\ 1 \end{pmatrix}$. Find the value of k for which L_2 is parallel to Π. [3]

28. The point P is $(1, 2, 3)$ and the plane Π has equation $2x - y + 2z = 0$. Find the coordinates of the foot of the perpendicular from P to Π, and hence the shortest distance from P to Π. [6]

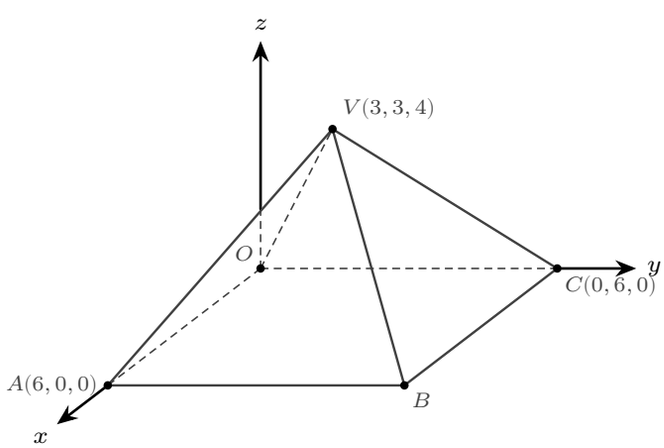
29. The line L has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$, and P is the point $(4, 3, 2)$. The point R lies on L.

(a) Write down $\overrightarrow{PR}$ in terms of λ. [2]

(b) Find the value of λ for which $\overrightarrow{PR}$ is perpendicular to L, and hence the coordinates of the foot F of the perpendicular from P to L. [3]

(c) Hence find the shortest distance from P to L. [2]

30. GDC The figure shows the pyramid $OABCV$. Its base $OABC$ is a square in the plane $z = 0$, and the vertex V lies above the centre of the base.



(a) Write down the coordinates of B. [1]

(b) Show that the plane VAB has Cartesian equation $4x + 3z = 24$. [4]

(c) Find the angle between the edge OV and the plane VAB. [4]

(d) Find the shortest distance from O to the plane VAB. [3]

31. GDC The points $A(1, 2, -1)$, $B(4, 0, 3)$ and $C(-2, 5, 1)$ are the vertices of a triangle.

(a) Find $|\overrightarrow{AB}|$ and $|\overrightarrow{AC}|$. [2]

(b) Find the angle $B\hat{A}C$. [3]

(c) Find the area of triangle ABC, and hence the shortest distance from C to the line AB. [3]

32. Consider the lines

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 4 \end{pmatrix} + \mu \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. [2]
(b) Determine whether the lines intersect, and if so find the point of intersection. [5]
(c) Find the angle between the lines. [3]
(d) Find the Cartesian equation of the plane containing both lines. [4]
(e) Explain why such a plane exists only because of your answer to part (b). [2]

33. The points $A(2, 0, 1)$, $B(1, 3, 2)$ and $C(4, 1, 0)$ are given, and O is the origin.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]
(b) Find $\overrightarrow{AB} \times \overrightarrow{AC}$. [3]
(c) Find the exact area of triangle ABC. [2]
(d) Find the Cartesian equation of the plane through A, B and C. [4]
(e) Find the exact shortest distance from O to that plane. [3]
(f) Hence find the volume of the tetrahedron $OABC$. [2]

34. The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 3 \\ 5 \\ -5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. [2]
(b) Show that L_1 and L_2 do not intersect. [4]
(c) Find $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \times \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. [2]
(d) The point P on L_1 has parameter λ, and the point Q on L_2 has parameter μ. Find $\overrightarrow{PQ}$ in terms of λ and μ. [2]
(e) The line PQ is perpendicular to both L_1 and L_2. Show that $\lambda = 1$ and $\mu = 1$, and find the coordinates of P and Q. [4]
(f) Hence find the shortest distance between L_1 and L_2, and verify that $\overrightarrow{PQ}$ is parallel to your answer to part (c). [2]

35. Consider the planes

$$\Pi_1 : x + y + z = 1, \qquad \Pi_2 : x + 2y + 3z = 4, \qquad \Pi_3 : x + y + kz = m,$$

where k and m are real constants.

(a) Find a vector equation of the line of intersection of Π_1 and Π_2. [4]
(b) Show that when $k = 2$ the three planes meet at a single point, and find that point in terms of m. [4]
(c) Find the values of k and m for which the three planes meet in a line. [3]

(d) Find the values of k and m for which the three planes have no common point, and describe that arrangement geometrically. [3]

36. GDC The lines L_1 and L_2 have equations

$$L_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}, \qquad L_2 : \mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}.$$

(a) Show that L_1 and L_2 are not parallel. [2]
(b) Determine whether L_1 and L_2 intersect. [4]
(c) Find the acute angle between L_1 and L_2. [3]

37. Two boats move with constant velocity. At time t hours, $t \geq 0$, their positions in km are

$$\mathbf{r}_A = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix} 4 \\ 6 \end{pmatrix} + t \begin{pmatrix} 3 \\ 3 \end{pmatrix}.$$

(a) Show that the square of the distance between the boats is $d^2 = 5t^2 + 22t + 25$. [3]
(b) Find the least distance between the boats for $t \geq 0$, and the time at which it occurs. [3]

38. The planes $\Pi_1 : x - y + z = 2$ and $\Pi_2 : x - y + 2z = 5$ meet in the line L.

(a) Find a direction vector of L. [2]
(b) Explain why L has no point with $z = 0$, and find a point on L. [3]
(c) Write down a vector equation of L. [1]
(d) Write L in Cartesian form. [2]

Challenge Questions

C1. The triple product and volume. Combining both products gives $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$, a single number called the *scalar triple product*. It measures volume, and it decides whether three planes meet in a single point.

In parts (a) to (e), take $\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ and $\mathbf{w} = \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$.

(a) Find $\mathbf{v} \times \mathbf{w}$, then $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. [3]
(b) The parallelepiped with edges $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ has base area $|\mathbf{v} \times \mathbf{w}|$. Explain why its volume is $|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$. *(Hint: volume = base area × perpendicular height)* [3]
(c) Evaluate $\mathbf{v} \cdot (\mathbf{w} \times \mathbf{u})$, $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$ and $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$. Make a conjecture about what happens to the triple product when the three vectors are moved round in order, and when two of them are swapped. Prove both parts of your conjecture for any three vectors. [5]

(d) Three vectors are *coplanar* when they all lie in one plane. Prove that three vectors are coplanar if and only if their triple product is zero, proving both directions. Test $\begin{pmatrix}1\\2\\0\end{pmatrix}$, $\begin{pmatrix}0\\1\\2\end{pmatrix}$, $\begin{pmatrix}2\\5\\2\end{pmatrix}$. [5]

(e) A tetrahedron on the same three edges has one sixth of the parallelepiped's volume. Find the volume of the tetrahedron with vertices at the origin and at the three points $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ above. [2]

(f) Three planes have normals $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_3$. Explain why they meet in exactly one point when $\mathbf{n}_1 \cdot (\mathbf{n}_2 \times \mathbf{n}_3) \neq 0$, and why they cannot meet in exactly one point when it is zero. Use this to show that the planes $x + 2y = 3$, $y + 2z = 1$ and $2x + 5y + 2z = k$ never meet in a single point, and find the value of k for which they share a line. [5]

(g) The sign of the triple product has been set aside so far. Explain what the sign of $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ tells you about the three vectors, and suggest how this could be investigated further. [2]

C2. Varignon's theorem in three dimensions. Varignon's theorem states that joining the midpoints of the sides of a quadrilateral in a plane, in order, gives a parallelogram. This investigation asks whether the four corners need to lie in a plane at all.

Let A, B, C, D have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$, and let P, Q, R, S be the midpoints of AB, BC, CD and DA.

(a) Take $A(0,0,0)$, $B(4,0,0)$, $C(6,4,2)$ and $D(0,2,6)$. Show that these four points do not lie in one plane. *(Hint: use the triple product of Challenge Question C1)* [2]

(b) For the points in part (a), find P, Q, R and S, and show that $PQRS$ is a parallelogram with four equal sides. [3]

(c) Prove that for any four points $\overrightarrow{PQ} = \overrightarrow{SR} = \frac{1}{2}(\mathbf{c} - \mathbf{a})$ and $\overrightarrow{QR} = \overrightarrow{PS} = \frac{1}{2}(\mathbf{d} - \mathbf{b})$. Deduce that $PQRS$ is always a parallelogram, lying in one plane even when A, B, C, D do not. [3]

(d) Show that the diagonals PR and QS bisect each other at the point G with position vector $\frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c} + \mathbf{d})$, and that G is also the midpoint of the segment joining the midpoints of AC and BD. Find G for the points in part (a). [3]

(e) Compare the lengths of AC and BD in part (a). Conjecture a condition on AC and BD for $PQRS$ to be a rhombus, and another for it to be a rectangle, and prove both. Hence state what shape $PQRS$ is when $ABCD$ are the vertices of a regular tetrahedron, justifying your answer. [5]

(f) Show that the area of $PQRS$ is $\frac{1}{4}|(\mathbf{c} - \mathbf{a}) \times (\mathbf{d} - \mathbf{b})|$, and find it for the points in part (a). Compare it with the area of $ABCD$ when $ABCD$ is a convex quadrilateral in a plane, and suggest how this could be investigated further. [3]

C3. Closest approach of two moving objects. Two drones fly in straight lines at constant velocity. Taking $t \geq 0$ in seconds and distances in metres, their positions are

$$\mathbf{r}_A = \begin{pmatrix}2\\3\\3\end{pmatrix} + t\begin{pmatrix}2\\2\\-1\end{pmatrix}, \qquad \mathbf{r}_B = \begin{pmatrix}2\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\2\\1\end{pmatrix}.$$

Section 9.4 finds a closest approach by completing the square for one pair of objects. This investigation turns the method into a formula, links it to the vector product, and compares it with the distance between the two flight paths.

In general write $\mathbf{r}_A = \mathbf{a}_A + t\mathbf{v}_A$ and $\mathbf{r}_B = \mathbf{a}_B + t\mathbf{v}_B$, with $\Delta\mathbf{a} = \mathbf{a}_B - \mathbf{a}_A$ and $\Delta\mathbf{v} = \mathbf{v}_B - \mathbf{v}_A \neq \mathbf{0}$, and let d be the distance between the two objects at time t.

(a) For the drones, show that $d^2 = 4(t-2)^2 + 4$. State the time of closest approach and the least distance. [4]

(b) In general, show that $d^2 = |\Delta\mathbf{v}|^2 t^2 + 2(\Delta\mathbf{a}\cdot\Delta\mathbf{v})\,t + |\Delta\mathbf{a}|^2$. By completing the square, show that d^2 is least at

$$t^* = -\frac{\Delta\mathbf{a}\cdot\Delta\mathbf{v}}{|\Delta\mathbf{v}|^2}, \qquad \text{with} \qquad d_{\min}^2 = |\Delta\mathbf{a}|^2 - \frac{(\Delta\mathbf{a}\cdot\Delta\mathbf{v})^2}{|\Delta\mathbf{v}|^2},$$

and state what it means for the two objects when $t^* < 0$, and where the least distance for $t \geq 0$ then occurs. [5]

(c) Use $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}||\mathbf{v}|\sin\theta$ and $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}||\mathbf{v}|\cos\theta$ to show that $|\mathbf{u}\times\mathbf{v}|^2 = |\mathbf{u}|^2|\mathbf{v}|^2 - (\mathbf{u}\cdot\mathbf{v})^2$. Hence show that $d_{\min} = \dfrac{|\Delta\mathbf{a}\times\Delta\mathbf{v}|}{|\Delta\mathbf{v}|}$, and check this against part (a). [4]

(d) The two flight paths are lines. Use the method of Section 9.7 to find the two points where the common perpendicular of the paths meets them, and show that the shortest distance between the paths is $\sqrt{2}$ metres. [4]

(e) Explain why the least distance between two moving objects can never be less than the distance between their paths. Find the times at which A and B pass the two points found in part (d), and use them to explain why the drones never come as close as $\sqrt{2}$ metres. State when the two distances are equal. [3]

(f) Suppose drone B starts s seconds late, so that $\mathbf{r}_B = \mathbf{a}_B + (t-s)\mathbf{v}_B$, where a negative s means an early start; take both formulas to hold for every t. Investigate how the least distance between the drones depends on s, and suggest how this model of a near miss could be taken further. [3]

C4. How far apart are two skew lines? Two skew lines never meet, yet there is still a shortest distance between them. This investigation finds it, and then deals with the case the formula cannot handle.

Let $L_1 : \mathbf{r} = \mathbf{a}_1 + \lambda\mathbf{b}_1$ and $L_2 : \mathbf{r} = \mathbf{a}_2 + \mu\mathbf{b}_2$ be skew.

(a) Show that $L_1 : \mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $L_2 : \mathbf{r} = \begin{pmatrix}2\\3\\1\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\2\end{pmatrix}$ are skew. [4]

(b) The shortest join between the two lines is perpendicular to both. Explain why its direction must be parallel to $\mathbf{n} = \mathbf{b}_1 \times \mathbf{b}_2$. [3]

(c) Hence explain why the shortest distance is the length of the projection of $\mathbf{a}_2 - \mathbf{a}_1$ onto $\mathbf{n}$, that is

$$\text{distance} = \frac{|(\mathbf{a}_2 - \mathbf{a}_1)\cdot\mathbf{n}|}{|\mathbf{n}|}.$$

(Hint: let the shortest join run from P_1 on L_1 to P_2 on L_2, and write $\mathbf{a}_2 - \mathbf{a}_1$ in terms of $\overrightarrow{P_1P_2}$, $\mathbf{b}_1$ and $\mathbf{b}_2$) [4]

(d) Use the formula on the two lines in part (a), giving an exact answer. [5]

(e) What does the formula give when the two lines intersect? Explain why your answer is what it should be. [3]

(f) Now let $L_3 : \mathbf{r} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$, which is parallel to L_1. Explain why the formula of part (c) cannot be used for L_1 and L_3. Show that the distance between two parallel lines is $\dfrac{|(\mathbf{a}_3 - \mathbf{a}_1) \times \mathbf{b}_1|}{|\mathbf{b}_1|}$, and find the exact distance between L_1 and L_3. [4]

(g) The lines here are infinite. Suggest how the method would have to change to find the shortest distance between two line segments, such as two rods or two edges of a solid, and how this could be investigated further. [2]