GEOMETRY & TRIGONOMETRY

Chapter 8

Trigonometry

8

SYLLABUS 3.5–3.11

Your voice is a wave. So is the light from your screen, the signal carrying this page to your device, the current in the wall socket, the tide in Tokyo Bay, and the beating of your own heart.

They do not look alike. A heartbeat is a sharp spike, a spoken vowel is jagged, a tide is a slow rise and fall over twelve hours. In the early 1800s Joseph Fourier claimed that a repeating signal, however jagged, can be built up from pure sine waves; Peter Dirichlet later proved it for signals that are not too irregular. Choose enough of them, with the right heights and the right timings, and the shape appears. Phone audio compression uses this idea many times a second.

So all of them can be built, at least approximately, from one curve. Where does it come from?

Watch one seat on a turning Ferris wheel. As the wheel turns steadily, the seat rises, slows near the top, falls, slows near the bottom, and rises again. Plot its height against time and the sine curve appears. The seat is a point moving round a circle, and its height is the vertical coordinate of that point. The circle and the wave are the same motion seen two ways: one going round, one going along.

That is why this chapter starts with a circle. In Ch. 7 sine and cosine were defined as ratios in a right-angled triangle. That definition only makes sense for acute angles, even though the sine and cosine rules already used obtuse ones. On the circle they become functions of *any* angle, and they repeat forever.

By the end of this chapter you will be able to read exact values of sine, cosine and tangent from the unit circle. You will use identities to rewrite one expression as another, sketch and transform the graphs of these functions, and use them to model quantities that rise and fall in cycles. You will work with the reciprocal and inverse functions, and solve trigonometric equations, where a single value of the sine or cosine is shared by two different angles, so answers usually have to be looked for in pairs.

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8.1 The Unit Circle

How do you find $\sin 120°$? Or $\cos 360°$, or $\tan(-30°)$?

You already have a definition, and it is SOHCAHTOA. In a right triangle,

$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}.$$

Now try to draw the triangle. For $120°$ there is none: the three angles add to $180°$, one of them is already the right angle, so the other two must each be under $90°$. For $360°$ there is none either, and $-30°$ is worse, because a triangle has no negative angles. Under SOHCAHTOA these three expressions have no meaning at all.

Yet a calculator gives a value for all three, and a turn of $120°$ is an ordinary thing to make. So the definition needs to be extended. Whatever replaces it must agree with SOHCAHTOA wherever SOHCAHTOA works, and keep working where it does not.

The unit circle does exactly that. Take the circle of radius 1 centred at the origin, and a point P on it at angle θ, measured anticlockwise from the positive x-axis.

For an acute θ, drop a perpendicular from P to the x-axis. This makes a right triangle whose hypotenuse is the radius, so the hypotenuse is 1. The old ratios then read

$$\sin\theta = \frac{\text{opposite}}{1} = \text{opposite}, \qquad \cos\theta = \frac{\text{adjacent}}{1} = \text{adjacent},$$

and those two sides are exactly the coordinates of P. So for acute angles the new definition agrees with the old one.

- The new form **never mentions a triangle**. A point can sit anywhere on the circle, at any angle, and it always has coordinates. That is what extends the definition to every real θ.

DEFINITION 8.1 **Sine and cosine on the unit circle**

On the unit circle, for a point P at angle θ:

$$\cos\theta = x\text{-coordinate of } P, \qquad \sin\theta = y\text{-coordinate of } P.$$

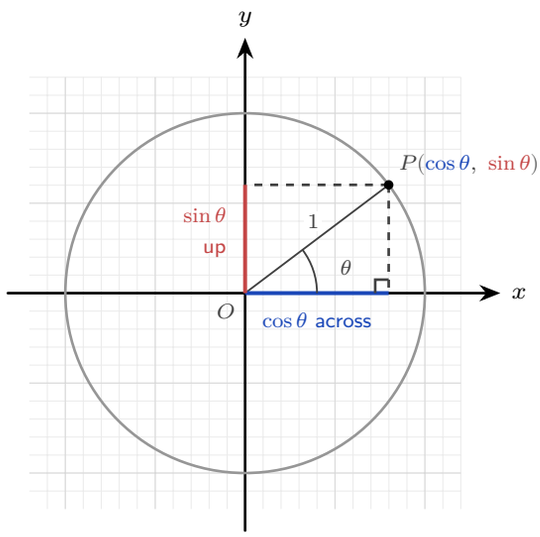


Figure 8.1 P on the unit circle at angle θ.

The tangent is the gradient of the line OP, so $\tan\theta = \dfrac{\text{rise}}{\text{run}} = \dfrac{\sin\theta}{\cos\theta}$. It is undefined wherever $\cos\theta = 0$.

FORMULA 8.1 **Tangent as a ratio** FORMULA BOOKLET · 3.5

$$\tan\theta = \frac{\sin\theta}{\cos\theta}.$$

Undefined where $\cos\theta = 0$.

The line OP passes through the origin with gradient $\tan\theta$, so its equation is $y = x\tan\theta$, a straight line of Ch. 4. The inverse of the tangent, $\arctan$ §8.5, converts a gradient back into an angle. A line of gradient 1 makes an angle of $\frac{\pi}{4}$ with the positive x-axis, because $\tan\frac{\pi}{4} = 1$.

KEY 8.2 **Gradient and angle of a line**

A line through the origin making an angle θ with the positive x-axis has equation

$$y = x\tan\theta.$$

This needs $\cos\theta \neq 0$: when $\theta = \frac{\pi}{2}$ the line is vertical, with equation $x = 0$. Read forwards it gives the gradient from the angle; read backwards, $\theta = \arctan(\text{gradient})$ gives the angle from the gradient. A negative gradient gives a negative angle, measured clockwise from the positive x-axis; add π for the anticlockwise angle.

8.1.1 Exact values

A few angles give exact coordinates, and they come from the 30–60–90 and 45–45–90 triangles.

Cut an equilateral triangle of side 2 in half. The half has angles $30°$, $60°$ and $90°$ (in radians $\frac{\pi}{6}$, $\frac{\pi}{3}$ and $\frac{\pi}{2}$), its hypotenuse is 2 and its shortest side is 1, so Pythagoras gives the third side as $\sqrt{3}$. Hence $\sin\frac{\pi}{6}=\frac{1}{2}$ and $\cos\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, and at $\frac{\pi}{3}$ the two values are exchanged. Now cut a square of side 1 along its diagonal. The half has two angles of $45°$ (that is, $\frac{\pi}{4}$) and hypotenuse $\sqrt{2}$, so $\sin\frac{\pi}{4}=\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$. The values at 0 and $\frac{\pi}{2}$ are the coordinates of the points $(1,0)$ and $(0,1)$ on the circle.

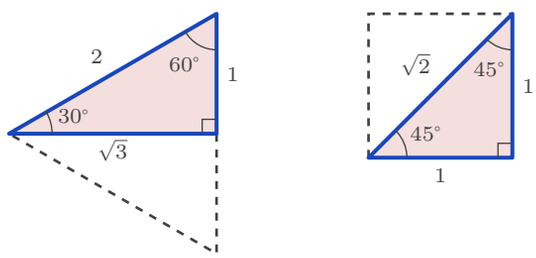


Figure 8.2 Half an equilateral triangle of side 2 gives the values at $30°$ and $60°$. Half a square of side 1 gives the values at $45°$.

θ	$\mathbf{0}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

Only three numbers appear in the shaded row, and they are all that has to be memorised:

$$\frac{1}{2},\qquad \frac{\sqrt{2}}{2},\qquad \frac{\sqrt{3}}{2}.$$

The cosine row is those same three in reverse order. That is not a coincidence. The two acute angles of a right triangle are θ and $\frac{\pi}{2}-\theta$, and the side adjacent to one of them is the side opposite the other. So $\cos\theta=\sin(\frac{\pi}{2}-\theta)$, and reading the sine row backwards reads the cosine row forwards. The tangent row needs nothing new at all: $\tan\theta=\dfrac{\sin\theta}{\cos\theta}$, so divide one row by the other. At $\theta=\frac{\pi}{2}$ the cosine is zero, which is why the tangent has no value there.

The first-quadrant values repeat into the other three quadrants (the four quarters into which the axes cut the plane, numbered anticlockwise from the one where x and y are both positive) with signs changed. Collecting them all onto one circle gives the diagram to learn.

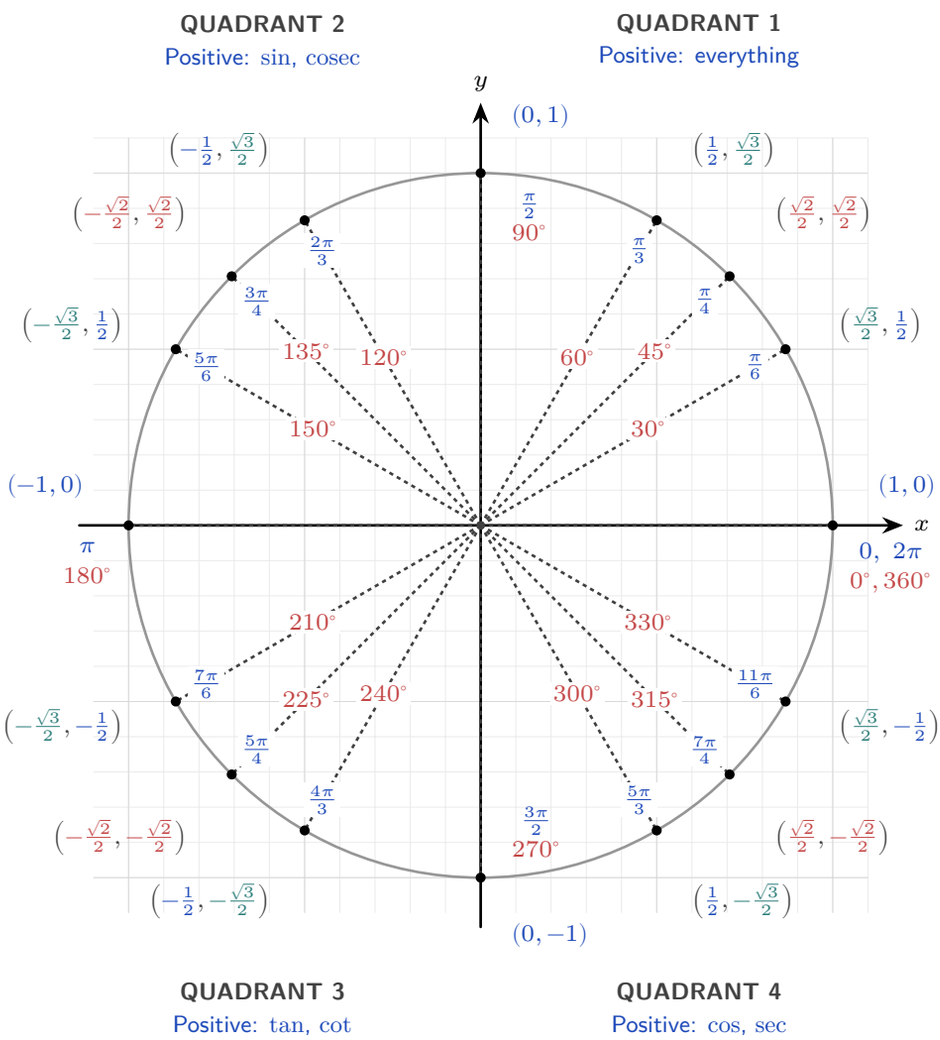


Figure 8.3 Every special angle, in radians (cobalt) and degrees (red), with its point $(\cos\theta,\ \sin\theta)$ outside the circle. Each coordinate is coloured by size: cobalt $\frac{1}{2}$, red $\frac{\sqrt{2}}{2}$, teal $\frac{\sqrt{3}}{2}$. The corners list the functions positive in each quadrant ($\sec$, cosec, $\cot$: §8.4).

Every coordinate labelled in the diagram, away from the axes, is one of those three numbers with a sign attached. Add the four points on the axes and the diagram is complete.

8.1.2 Symmetry and the sign of each function

Since $\cos\theta$ and $\sin\theta$ are the coordinates of P, their signs are just the signs of x and y in each quadrant. Nothing needs to be memorised separately: in the second quadrant x is negative and y is positive, so $\cos\theta < 0$ and $\sin\theta > 0$, and $\tan\theta = \frac{\sin}{\cos}$ is negative.

The unit-circle figure above records the answer at each corner. Reading the four quadrants anticlockwise from the first, the functions that stay positive are all three, then sine alone, then tangent alone, then cosine alone.

The mnemonic **All Students Take Calculus** is the usual way to remember that order. Use it as a check, not as the reason.

Each symmetry of the circle now gives an identity, and each one is a statement about where a reflected point lands.

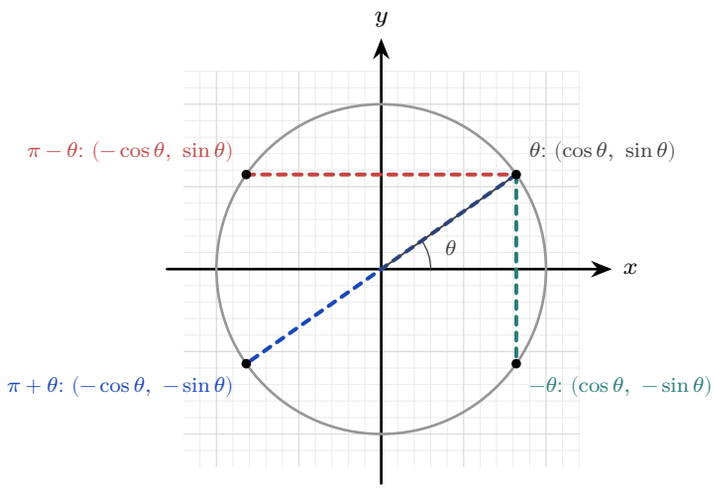


Figure 8.4 The point at angle θ and its three images. Reflecting in the y-axis (red) gives $\pi - \theta$, reflecting in the x-axis (teal) gives $-\theta$, and a half turn about O (cobalt) gives $\pi + \theta$. Each move changes the sign of one coordinate or both.

Reflect in the y-axis. The point at angle $\pi - \theta$ is the mirror image, across the y-axis, of the point at angle θ. Reflecting in the y-axis negates the x-coordinate and leaves y alone, so the two points are $(-\cos\theta,\ \sin\theta)$ and $(\cos\theta,\ \sin\theta)$. Comparing coordinates:

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta.$$

Reflect in the x-axis. Measuring $-\theta$ means turning clockwise instead of anticlockwise, which lands on the mirror image of P across the x-axis. That reflection negates y and leaves x alone, so

$$\cos(-\theta) = \cos\theta, \qquad \sin(-\theta) = -\sin\theta.$$

Turn half a circle. Adding π sends P to the diametrically opposite point, which negates both coordinates:

$$\sin(\theta + \pi) = -\sin\theta, \qquad \cos(\theta + \pi) = -\cos\theta.$$

Turn a full circle. Adding 2π to the angle sends P once round and back to exactly the same point, so both coordinates are unchanged:

$$\sin(\theta + 2\pi) = \sin\theta, \qquad \cos(\theta + 2\pi) = \cos\theta.$$

This is why both functions are periodic with period 2π. Every identity of this kind says the same thing: move the point, see which coordinate changes sign.

You have met the first of these relations before. Two angles, θ and $\pi - \theta$, share the same sine. That shared value is what produced the ambiguous case of the sine rule in §7.3.2. A triangle built from a sine value can sometimes close in two ways, one acute and one obtuse.

An exact value at any special angle now comes out of the circle in two moves: find the acute angle the arm makes with the x-axis, which fixes the size, then look at the quadrant, which fixes the sign.

EXAMPLE 8.1

Find the exact values of $\sin\frac{5\pi}{3}$ and $\cos\frac{5\pi}{3}$.

SOLUTION

Write the angle as $2\pi - \frac{\pi}{3}$. A full turn returns to the positive x-axis, so turning back through $\frac{\pi}{3}$ leaves the point in the fourth quadrant, where the x-coordinate is positive and the y-coordinate negative. The arm makes an acute angle of $\frac{\pi}{3}$ with the axis, so both values have the size they have at $\frac{\pi}{3}$:

$$\sin\frac{5\pi}{3} = -\sin\frac{\pi}{3} = \boxed{-\frac{\sqrt{3}}{2}}, \qquad \cos\frac{5\pi}{3} = \cos\frac{\pi}{3} = \boxed{\frac{1}{2}}$$

Dividing them gives $\tan\frac{5\pi}{3} = -\sqrt{3}$, negative, because only the sine carries a minus sign. That is the fourth quadrant's entry in the mnemonic, recovered from the coordinates rather than remembered.

YOUR TURN 8.1 The Unit Circle

Exact values

1. Write down the exact value of each.

(a) $\sin\frac{\pi}{6}$ (b) $\cos\frac{\pi}{4}$

(c) $\tan\frac{\pi}{3}$ (d) $\tan\frac{\pi}{4}$

2. Use the symmetry of the circle to find each exact value.

(a) $\sin\frac{5\pi}{6}$ (b) $\cos\frac{2\pi}{3}$

(c) $\sin\frac{4\pi}{3}$ (d) $\cos\frac{7\pi}{4}$

Signs in each quadrant

3. State the quadrant or quadrants in which each condition holds.

(a) $\sin\theta > 0$ (b) $\cos\theta < 0$

(c) $\tan\theta > 0$ (d) $\sin\theta < 0$ and $\cos\theta > 0$

The unit circle and gradients

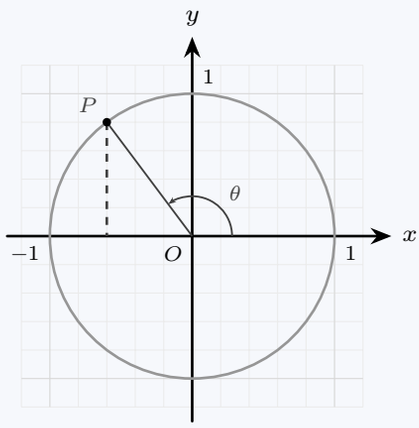
4. The point P on the unit circle has angle θ.

(a) State the coordinates of P in terms of θ.

(b) Explain why $\cos^2\theta + \sin^2\theta = 1$ for every θ.

(c) Explain why $\tan\theta$ is undefined at $\theta = \dfrac{\pi}{2}$.

(d) The diagram shows the point P on the unit circle at angle θ. Use the grid to write down the value of $\cos\theta$, of $\sin\theta$ and of $\tan\theta$.



5. Lines through the origin, using $y = x\tan\theta$. Give each angle in radians.

(a) Write the equation of the line through the origin making an angle of $\dfrac{\pi}{3}$ with the positive x-axis.

(b) Find the angle a line of gradient 1 makes with the positive x-axis.

(c) Find the anticlockwise angle θ from the positive x-axis to the line $y = -x$, with $0 \le \theta < \pi$.

Concept check

6. A student writes $\cos\dfrac{2\pi}{3} = -\cos\dfrac{\pi}{6} = -\dfrac{\sqrt{3}}{2}$, taking the acute angle from the nearest axis. Identify the error and find the correct value.

8.2 Trigonometric Identities

A trigonometric identity is an identity in the sense of §3.1: true for every angle at which both sides are defined. As in the IB formula booklet, these notes write identities with $=$. The first one is Pythagoras' theorem read off the unit circle: the point $(\cos\theta, \sin\theta)$ is distance 1 from the origin, so $\cos^2\theta + \sin^2\theta = 1$.

FORMULA 8.3 **Pythagorean identity** FORMULA BOOKLET · 3.6

$$\cos^2\theta + \sin^2\theta = 1.$$

8.2.1 Double and compound angles

The functions of a sum of two angles can be expanded, and the double-angle formulas follow from the sum formulas. These **compound-angle** identities are usually just quoted, but the proof is short and uses only the unit circle and the cosine rule from Ch. 7.

Put two points on the unit circle, P at angle A and Q at angle B. Their coordinates are $P(\cos A,\ \sin A)$ and $Q(\cos B,\ \sin B)$, and the angle $P\hat{O}Q$ at the centre is $A - B$.

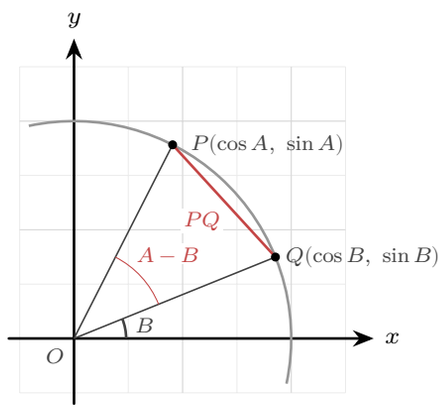


Figure 8.5 P and Q at angles A and B, so $P\hat{O}Q = A - B$. The chord PQ is found two ways: by the distance formula and by the cosine rule.

Now find the distance PQ in two different ways.

By the distance formula:

$$PQ^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2.$$

Expanding gives $\cos^2 A - 2\cos A\cos B + \cos^2 B + \sin^2 A - 2\sin A\sin B + \sin^2 B$. Each Pythagorean pair, $\cos^2 A + \sin^2 A$ and $\cos^2 B + \sin^2 B$, equals 1, so

$$PQ^2 = 2 - 2(\cos A\cos B + \sin A\sin B).$$

By the cosine rule in triangle OPQ, where both sides from the centre have length 1 and the angle between them is $A - B$:

$$PQ^2 = 1^2 + 1^2 - 2(1)(1)\cos(A - B) = 2 - 2\cos(A - B).$$

Both expressions equal PQ^2. Setting them equal and simplifying gives

$$\cos(A - B) = \cos A\cos B + \sin A\sin B.$$

Every other compound-angle formula follows from this one. Replacing B by $-B$, and using $\cos(-B) = \cos B$ with $\sin(-B) = -\sin B$, gives the formula for $\cos(A + B)$.

A sine is a cosine a quarter turn away: $\sin\theta = \cos(\frac{\pi}{2} - \theta)$. The formula just proved shows this for every angle, because $\cos(\frac{\pi}{2} - \theta) = \cos\frac{\pi}{2}\cos\theta + \sin\frac{\pi}{2}\sin\theta = \sin\theta$. Replacing θ by $\frac{\pi}{2} - \theta$ in that result gives $\cos\theta = \sin(\frac{\pi}{2} - \theta)$ as well. These two facts are all the sine version needs. Write the sum in that form, then group the result as a *difference* of two angles:

$$\sin(A+B) = \cos(\tfrac{\pi}{2} - A - B) = \cos((\tfrac{\pi}{2} - A) - B)\,.$$

That is exactly the shape just proved. Expanding it, and using $\cos(\frac{\pi}{2} - A) = \sin A$ together with $\sin(\frac{\pi}{2} - A) = \cos A$, gives

$$\sin(A+B) = \sin A\cos B + \cos A\sin B.$$

The tangent versions come from dividing the sine formula by the cosine formula, then dividing the numerator and the denominator by $\cos A\cos B$:

$$\tan(A+B) = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B} = \frac{\tan A + \tan B}{1 - \tan A\tan B}.$$

FORMULA 8.4 **Compound angle identities** FORMULA BOOKLET · 3.10 HL

$$\sin(A\pm B) = \sin A\cos B \pm \cos A\sin B, \qquad \cos(A\pm B) = \cos A\cos B \mp \sin A\sin B,$$

$$\tan(A\pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}.$$

Setting $B = A$ reduces each of these to a **double-angle** identity. Each one takes a single line to derive.

Put $B = A$ in the sine formula:

$$\sin 2A = \sin(A+A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A.$$

Do the same in the cosine formula:

$$\cos 2A = \cos(A+A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A.$$

That is the first form. The other two come from $\sin^2 A + \cos^2 A = 1$, used to remove one function or the other. Replacing $\sin^2 A$ by $1 - \cos^2 A$ gives

$$\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1,$$

and replacing $\cos^2 A$ by $1 - \sin^2 A$ instead gives

$$\cos 2A = (1 - \sin^2 A) - \sin^2 A = 1 - 2\sin^2 A.$$

The tangent version follows in the same way, by putting $B = A$ in the tangent formula.

FORMULA 8.5 **Double angle identities** FORMULA BOOKLET · 3.6

$$\sin 2\theta = 2\sin\theta\cos\theta, \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta.$$

FORMULA 8.6 **Double angle for tangent** FORMULA BOOKLET · 3.10 HL

$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta}.$$

Choose whichever form of $\cos 2\theta$ leaves you with only sines or only cosines. That choice is what makes the identity so useful when solving equations.

The compound-angle formulas also give exact values beyond the basic table. Write the angle as a sum or difference of two special angles, such as $165^\circ = 120^\circ + 45^\circ$, and expand. The two angles need not be acute; carry the sign of each ratio from its quadrant.

EXAMPLE 8.2

Find the exact value of $\sin 165^\circ$.

SOLUTION

Write $165^\circ = 120^\circ + 45^\circ$ and use the compound-angle formula:

$$\sin 165^\circ = \sin 120^\circ\cos 45^\circ + \cos 120^\circ\sin 45^\circ = \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} + \left(-\frac{1}{2}\right)\cdot\frac{\sqrt{2}}{2} = \boxed{\frac{\sqrt{6}-\sqrt{2}}{4}}$$

Here $\cos 120^\circ = -\frac{1}{2}$ carries a minus sign from the second quadrant, and that minus sign is what turns the sum into a difference.

The tangent formulas work from the tangents alone, with no sine or cosine needed.

EXAMPLE 8.3

The acute angles A and B have $\tan A = 3$ and $\tan B = \frac{1}{2}$. Find (a) the exact value of $A - B$, (b) $\tan 2B$.

SOLUTION

(a) The formula for a difference gives

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} = \frac{3 - \frac{1}{2}}{1 + (3)\left(\frac{1}{2}\right)} = \frac{\frac{5}{2}}{\frac{5}{2}} = 1.$$

Both angles are acute, so $A - B$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the only angle there with tangent 1 is $\frac{\pi}{4}$. Without knowing either angle, you have found $\boxed{A - B = \frac{\pi}{4}}$

(b) The double-angle formula works the same way:

$$\tan 2B = \frac{2\tan B}{1-\tan^2 B} = \frac{2\left(\frac{1}{2}\right)}{1-\frac{1}{4}} = \frac{1}{\frac{3}{4}} = \boxed{\frac{4}{3}}$$

Identities also let you move between ratios *without ever finding the angle*: the Pythagorean identity supplies everything needed. The identity gives the square of the unknown ratio, so it leaves a choice of sign. Use the quadrant of the angle to choose that sign before you use the value.

EXAMPLE 8.4

Given that $\cos x = \frac{3}{4}$ and x is acute, find the exact value of $\sin 2x$ without finding x.

SOLUTION

From $\cos^2 x + \sin^2 x = 1$, $\sin^2 x = 1 - \frac{9}{16} = \frac{7}{16}$, and since x is acute, $\sin x = \frac{\sqrt{7}}{4}$ (positive). Then

$$\sin 2x = 2\sin x \cos x = 2 \cdot \frac{\sqrt{7}}{4} \cdot \frac{3}{4} = \boxed{\frac{3\sqrt{7}}{8}}$$

The word "acute" fixed the sign of $\sin x$. If x had instead been in the fourth quadrant, the same working would give $\sin 2x = -\frac{3\sqrt{7}}{8}$.

Proving an identity uses the same substitutions: start from one side, substitute known identities, and simplify until the other side appears.

EXAMPLE 8.5

Prove that $\cos(A-B) - \cos(A+B) = 2\sin A \sin B$.

SOLUTION

Work on the left side, expanding each cosine by the compound-angle formula:

$$\cos(A-B) = \cos A\cos B + \sin A \sin B, \qquad \cos(A+B) = \cos A \cos B - \sin A \sin B.$$

The two expansions differ only in the sign of the second term, so subtracting removes the $\cos A \cos B$ pair and doubles what is left:

$$\cos(A-B) - \cos(A+B) = 2\sin A \sin B. \quad \blacksquare$$

Expanding the more complicated side first is the usual choice. Once both expansions are written down, the terms that have to cancel are visible before any algebra is done.

When an identity contains $\cos 2\theta$ beside a 1, choose the form of $\cos 2\theta$ that cancels the 1.

EXAMPLE 8.6

Prove that $\dfrac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$.

SOLUTION

Work on the left side. In the numerator use $\cos 2\theta = 1 - 2\sin^2\theta$, and in the denominator use $\cos 2\theta = 2\cos^2\theta - 1$:

$$\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)} = \frac{2\sin^2\theta}{2\cos^2\theta} = \tan^2\theta. \qquad \blacksquare$$

Each form was chosen so that the 1 disappears and a single squared term is left. The identity holds wherever both sides are defined, that is, where $\cos\theta \neq 0$.

8.2.2 One wave from two

This subsection goes beyond the syllabus. It is included because the result has many real-world uses: two waves of the same frequency, whether sound, light or alternating current, add to give a single wave.

A sum such as $8\sin x + 15\cos x$ mixes a sine and a cosine of the same angle. Both have period 2π, so the sum repeats every 2π as well. More is true: the sum is a single sine wave, stretched and translated, and reading the compound-angle formula backwards shows why.

Expanding $R\sin(x+\alpha)$ gives

$$R\sin(x+\alpha) = R\cos\alpha\,\sin x + R\sin\alpha\,\cos x,$$

which already has the shape $a\sin x + b\cos x$, with $a = R\cos\alpha$ and $b = R\sin\alpha$. Squaring those two and adding removes α, since $\cos^2\alpha + \sin^2\alpha = 1$, and leaves $a^2 + b^2 = R^2$. Dividing them removes R instead and leaves $\tan\alpha = \frac{b}{a}$. So any such sum can be written as one wave, and the two constants can be recovered from a and b alone. The cosine form on the card below is found in the same way, with the same R and α:
$R\cos(x-\alpha) = R\cos\alpha\,\cos x + R\sin\alpha\,\sin x = a\cos x + b\sin x$.

KEY 8.7 **The wave form**

For constants a and b not both zero,

$$a\sin x + b\cos x = R\sin(x+\alpha), \qquad R = \sqrt{a^2+b^2}, \qquad \cos\alpha = \frac{a}{R}, \quad \sin\alpha = \frac{b}{R},$$

with $R > 0$. When $a \neq 0$ this gives $\tan\alpha = \frac{b}{a}$, with α taken in the quadrant where $\cos\alpha$ has the sign of a and $\sin\alpha$ the sign of b. The same R and α give the cosine form

$$a\cos x + b\sin x = R\cos(x-\alpha).$$

Since $\sin$ and $\cos$ run between -1 and 1, the expression has maximum R and minimum $-R$.

This form is not printed in the formula booklet, so rebuild it from the expansion of $R\sin(x+\alpha)$ whenever it is needed. Its value is that the variable now appears once instead of twice. The maximum and minimum can then be read off at once, and an equation becomes the ordinary work of §8.6.

EXAMPLE 8.7

Write $8\sin x + 15\cos x$ in the form $R\sin(x+\alpha)$ with $R > 0$ and $0 < \alpha < \frac{\pi}{2}$, and state its maximum and minimum values.

SOLUTION

Take $a = 8$ and $b = 15$. Both are positive, so α is in the first quadrant:

$$R = \sqrt{8^2+15^2} = \sqrt{289} = 17, \qquad \tan\alpha = \frac{15}{8} \implies \alpha = \arctan\frac{15}{8} \approx 1.0808,$$

giving $\boxed{8\sin x + 15\cos x \approx 17\sin(x+1.0808)}$

The sine factor runs between -1 and 1, so the maximum and the minimum are $\boxed{17 \text{ and } -17}$

YOUR TURN 8.2 Trigonometric Identities

Recognising the identities

7. Simplify each expression.

(a) $\sin^2\theta + \cos^2\theta$

(b) $2\sin\theta\cos\theta$

(c) $\cos^2\theta - \sin^2\theta$

(d) $1 - 2\sin^2\theta$

Values without the angle

8. Given $\sin\theta = \frac{3}{5}$ with θ acute, find each exact value.

(a) $\cos\theta$ (b) $\tan\theta$

(c) $\sin 2\theta$ (d) $\cos 2\theta$

9. Given $\cos\theta = -\frac{3}{5}$ with θ in the second quadrant, find each exact value.

(a) $\sin\theta$ (b) $\tan\theta$

(c) $\sin 2\theta$ (d) $\cos 2\theta$

Compound-angle values

10. Use a compound-angle formula to find each exact value.

(a) $\cos 15°$, from $45° - 30°$

(b) $\cos 105°$, from $60° + 45°$

(c) $\cos 165°$, from $120° + 45°$

Tangent formulas

11. Use the tangent compound-angle and double-angle formulas.

(a) Find the exact value of $\tan 75°$, from $45° + 30°$.

(b) Given $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, with A and B acute, find $\tan(A+B)$, and hence $A + B$.

(c) Given $\tan\theta = \frac{3}{4}$, find the exact value of $\tan 2\theta$.

(d) Given $\tan\theta = -2$, find the exact value of $\tan 2\theta$.

Proving identities

12. Prove each identity.

(a) $\dfrac{\sin 2\theta}{1+\cos 2\theta} = \tan\theta$

(b) $\dfrac{1-\cos 2\theta}{\sin 2\theta} = \tan\theta$

(c) $(\sin\theta + \cos\theta)^2 = 1 + \sin 2\theta$

(d) $\dfrac{\cos 2\theta}{\cos\theta + \sin\theta} = \cos\theta - \sin\theta$

(e) $\tan\theta \sin 2\theta = 2\sin^2\theta$

(f) $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2$

One wave from two

13. Consider $\sin x + \sqrt{3}\cos x$.

(a) Write it in the form $R\sin(x+\alpha)$, with $R>0$ and $0<\alpha<\dfrac{\pi}{2}$, giving R and α exactly.

(b) State its maximum and minimum values.

14. GDC Consider $7\sin x + 24\cos x$.

(a) Write it in the form $R\sin(x+\alpha)$, with $R>0$ and $0<\alpha<\dfrac{\pi}{2}$, giving α to three significant figures.

(b) State its maximum and minimum values, and find the smallest positive value of x at which the maximum occurs.

Concept check

15. A student writes $\sin(A+B)=\sin A+\sin B$.

(a) Give a counter-example.

(b) State the correct expansion of $\sin(A+B)$.

8.3 Graphs of Trigonometric Functions

Send the point P round the circle and plot its y-coordinate against the angle. On a graph the angle is usually called x, so that plot *is* $y=\sin x$. Plot the x-coordinate of P instead and you get $y=\cos x$.

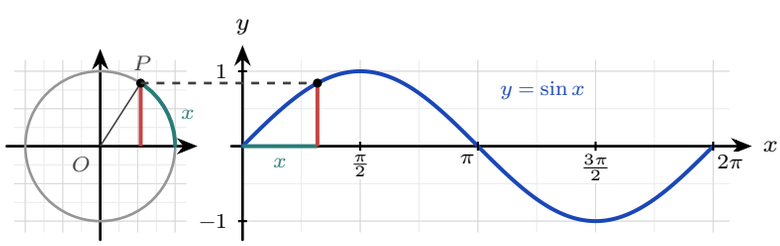


Figure 8.6 As P turns through an angle x (the teal arc), its height $\sin x$ (red) is carried across to the graph. Laid along the axis, the arc of length x gives the input, and the height gives the output.

Every feature of the sine graph can be read off the circle before you plot a single point.

- The y-coordinate of P never leaves -1 to 1, so the curve stays between those two heights. Its **amplitude**, the distance from the middle to the top, is 1.
- P returns to its starting place after one full turn, so the curve repeats every 2π. Its period, the width of one complete cycle, is 2π.
- $y=0$ when P sits on the x-axis, at $x=0$, π and 2π in the first turn, so the curve crosses the axis there.
- y is largest at the top of the circle, so the maximum is 1 and it occurs at $x=\frac{\pi}{2}$.

- $y > 0$ everywhere in the upper half of the circle, so the curve is above the axis for $0 < x < \pi$.

Read the x-coordinate instead and the cosine graph appears the same way. It starts at 1, because P starts on the positive x-axis, and falls to 0 at $\frac{\pi}{2}$, where P is at the top and its x-coordinate is zero.

So both curves have amplitude 1 and period 2π. They differ only in where a cycle begins. The x-coordinate at angle θ is the y-coordinate a quarter turn later, so $\cos\theta = \sin(\theta + \frac{\pi}{2})$, and the cosine curve is the sine curve translated left by $\frac{\pi}{2}$. The compound-angle formula confirms it: $\sin(\theta + \frac{\pi}{2}) = \sin\theta\cos\frac{\pi}{2} + \cos\theta\sin\frac{\pi}{2} = \cos\theta$.

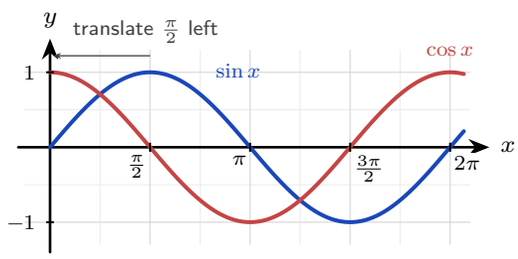


Figure 8.7 Sine (cobalt) and cosine (red) over one turn. The cosine curve is the sine curve translated $\frac{\pi}{2}$ left.

Unlike sine and cosine, the tangent has vertical asymptotes and a period of π. Since $\tan\theta = \frac{y}{x}$, it is undefined wherever $x = 0$, which is at the top and the bottom of the circle. That gives vertical asymptotes at $x = \frac{\pi}{2} + n\pi$, where n stands for any integer.

Its period is π, not 2π. Half a turn sends P to the point diametrically opposite, whose coordinates are $(-x, -y)$. The ratio $\frac{-y}{-x}$ equals $\frac{y}{x}$, so the tangent repeats after only half a revolution.

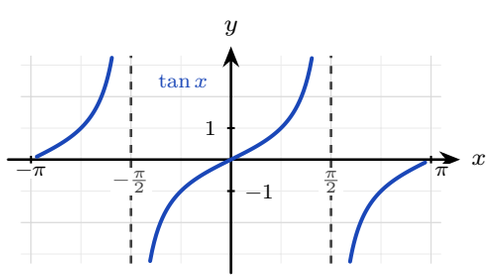


Figure 8.8 The graph of $y = \tan x$.

That is the basic behaviour, collected in one place.

	$\sin x$	$\cos x$	$\tan x$
Domain	all x	all x	$x \neq \frac{\pi}{2} + n\pi$
Range	$-1 \leq y \leq 1$	$-1 \leq y \leq 1$	all y
Period	2π	2π	π
Amplitude	1	1	none
Zeros	$n\pi$	$\frac{\pi}{2} + n\pi$	$n\pi$

The last row gives the values of x at which the curve crosses the axis. The tangent has no amplitude, because it rises without limit.

8.3.1 Transforming the wave

The general sinusoid $(\text{any stretched and translated sine wave})$ stretches and slides the basic sine, and each parameter has a plain meaning.

KEY 8.8 **Transformations of a sinusoid**

For $y = a\sin\big(b(x + c)\big) + d$, with $a \neq 0$ and $b \neq 0$:

$$\text{amplitude} = |a|, \qquad \text{period} = \frac{2\pi}{|b|},$$

$$\text{horizontal translation} = -c, \qquad \text{vertical translation (midline)} = d.$$

These are the transformations of §5.2 applied to $y = \sin x$. The factor a stretches the wave vertically, and d translates it vertically. Inside the bracket, $b(x + c)$ changes by 2π when x changes by $\frac{2\pi}{|b|}$, so that is the length of one full cycle. The $+c$ acts on the input, so it moves the graph c to the left, which is a translation of $-c$.

A question can also run the other way, giving you the graph's largest and smallest values and asking for a and d.

- **Amplitude.** From the equation it is $|a|$. From a graph it is $\dfrac{\max - \min}{2}$, half the total rise and fall.
- **Midline.** From the equation it is the line $y = d$. From a graph it is $y = \dfrac{\max + \min}{2}$, the level halfway between the top and the bottom.
- **Maximum and minimum.** They are $d + |a|$ and $d - |a|$: the midline, plus and minus the amplitude.
- **Period.** From the equation it is $\dfrac{2\pi}{|b|}$. From a graph it is the horizontal distance from one repeat to the next.

For example, $y = 5\sin(2x) - 3$ has maximum 2 and minimum -8. The amplitude is $\frac{2-(-8)}{2} = 5$ and the midline is $\frac{2+(-8)}{2} = -3$, which is what the equation says.

EXAMPLE 8.8

State the amplitude, the period and the midline of $y = 2\sin(3x) + 1$, and find its range.

SOLUTION

The amplitude is 2, so the wave rises and falls 2 either side of its midline. The period is $\frac{2\pi}{3}$, so it completes a full cycle three times as fast as $\sin x$. The midline is $y = 1$. The graph therefore runs from $1 - 2 = -1$ up to $1 + 2 = 3$, giving the range $\boxed{-1 \leq y \leq 3}$

A real quantity that rises and falls in a regular cycle can be modelled by a sinusoid. Four measurements fix the four constants: the largest value, the smallest value, the length of one cycle, and the time of one maximum.

EXAMPLE 8.9

In a harbour, high water is 9.2 m deep at $t = 2$, where t is the time in hours after midnight. The next low water is 2.4 m deep at $t = 8.2$. (a) Model the depth by $h(t) = a\sin(b(t + c)) + d$ with $a > 0$ and $b > 0$. (b) Use the model to estimate the depth at 05:00.

SOLUTION

(a) Find the constants one at a time. The amplitude and the midline come from the largest and smallest depths:

$$a = \frac{9.2 - 2.4}{2} = 3.4, \qquad d = \frac{9.2 + 2.4}{2} = 5.8.$$

From a maximum to the next minimum is half a cycle, so the period is $2(8.2 - 2) = 12.4$ hours, and

$$b = \frac{2\pi}{12.4} \approx 0.507.$$

A sine wave reaches its maximum a quarter of a period after its cycle begins. A quarter of 12.4 is 3.1, so this cycle begins at $t = 2 - 3.1 = -1.1$. The bracket $t + c$ must be zero there, so $c = 1.1$. The model is

$$\boxed{h(t) = 3.4\sin(0.507(t + 1.1)) + 5.8}$$

Check it at high water: $0.507(2 + 1.1) \approx 1.57 \approx \frac{\pi}{2}$, and $\sin\frac{\pi}{2} = 1$, so $h(2) = 3.4 + 5.8 = 9.2$, as measured.

(b) The model now predicts other depths. At 05:00, with the calculator in radians,

$$h(5) = 3.4\sin(0.507 \times 6.1) + 5.8 \approx 3.4(0.0489) + 5.8 \approx \boxed{5.97 \text{ m}}$$

The value of c is not unique: adding a whole period, 12.4, to it gives the same wave. Regression technology fitted to a table of readings may also give the model in another form, such as $a\sin(bt + c) + d$, which is the same wave with the bracket multiplied out.

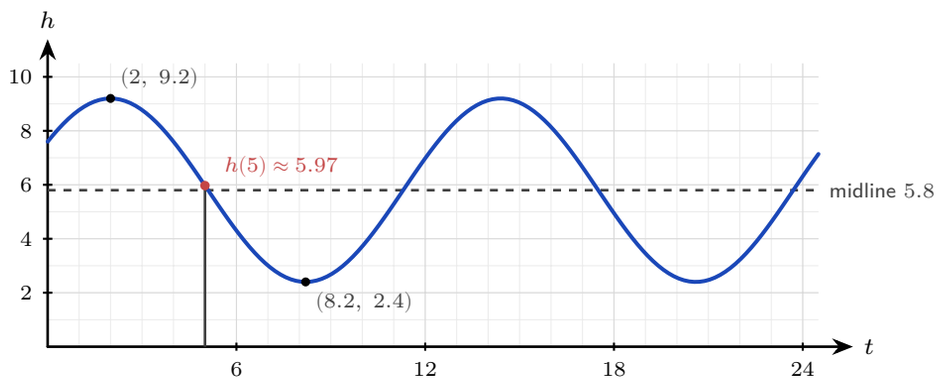


Figure 8.9 The harbour model, with depth h in metres and t in hours after midnight. High water and low water fix the amplitude and the midline, and their spacing fixes the period. The 05:00 estimate is marked in red.

8.3.2 Symmetry of the graphs HL

Each reflection symmetry of the circle from §8.1 becomes a visible symmetry of the graphs. The cosine curve is mirror-symmetric in the y-axis, so $\cos(-x) = \cos x$: cosine is an even function. The sine and tangent curves map onto themselves under a half-turn about the origin, so $\sin(-x) = -\sin x$ and $\tan(-x) = -\tan x$: both are odd, in the sense of Ch. 5. The same circle symmetries give the supplementary-angle relations.

KEY 8.9 **Supplementary angle identities** HL

$$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta, \qquad \tan(\pi - \theta) = -\tan\theta.$$

You can see all three on the graphs directly: the sine curve is symmetric about the vertical line $x = \frac{\pi}{2}$, while cosine and tangent change sign across it. These relations give values such as $\cos\frac{3\pi}{4}$ without a calculator: $\cos\frac{3\pi}{4} = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$.

YOUR TURN 8.3 Graphs of Trigonometric Functions

Amplitude, period and midline

16. State the amplitude, the period and the midline of each graph.

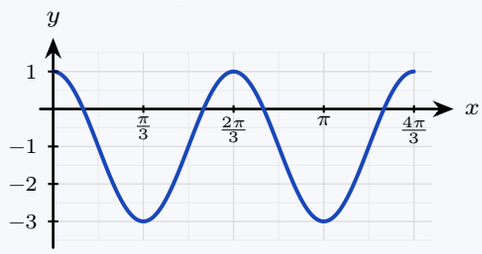
(a) $y = 3\sin x$

(b) $y = \sin 2x$

(c) $y = 4\sin(3x) - 2$

(d) $y = 5\sin(4x) + 3$

(e) The graph of a function of the form $y = a\cos(bx) + d$ is shown for $0 \le x \le \frac{4\pi}{3}$. Use the graph to write down the amplitude, the period and the midline.



17. State the range of each function.

(a) $y = 2\sin x + 1$

(b) $y = 3\sin(2x) - 1$

(c) $y = 2\cos\left(\frac{x}{2}\right) - 1$

Horizontal translation

18. State the amplitude, the period, the horizontal translation and the midline of each graph. *(Hint: write the argument in the form $b(x + c)$ first)*

(a) $y = 3\sin\left(x - \frac{\pi}{4}\right) + 1$

(b) $y = 2\sin\left(2x + \frac{\pi}{3}\right)$

(c) $y = 5\sin(3x - \pi) - 2$

(d) Sketch the graph of $y = 2\sin\left(2x - \frac{\pi}{3}\right) + 1$ for $0 \le x \le \pi$. Show the maximum and minimum points, the midline, and the points where the graph meets the axes.

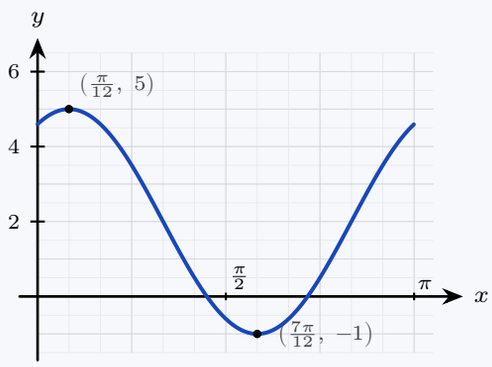
From the maximum and minimum

19. Find the constants from the information given.

(a) The graph of $y = a\sin(bx) + d$, with $a > 0$ and $b > 0$, has maximum value 7, minimum value -1 and period π. Find a, b and d.

(b) The depth of water in a harbour is modelled by $h(t) = a\sin(bt) + d$ metres, t hours after midnight, with $a > 0$ and $b > 0$. The greatest depth is 12.5 m, the least depth is 2.5 m, and the pattern repeats every 12 hours. Find a, b and d.

(c) The graph of $y = a\sin(b(x+c)) + d$, with $a > 0$, $b > 0$ and $0 < c < \pi$, is shown for $0 \le x \le \pi$. It has a maximum point at $\left(\frac{\pi}{12},\ 5\right)$ and a minimum point at $\left(\frac{7\pi}{12},\ -1\right)$. Find a, b, c and d.



The tangent graph

20. Answer each question about $y = \tan x$.

(a) State its period.

(b) State the equations of two of its asymptotes.

(c) Explain why the asymptotes occur where they do.

Symmetry of the graphs

21. An angle θ has $\sin\theta = 0.6$ and $\cos\theta = 0.8$. Use the symmetries of the graphs to write down each value.

(a) $\sin(\pi - \theta)$

(b) $\cos(\pi - \theta)$

(c) $\tan(\pi - \theta)$

(d) $\sin(-\theta)$

Modelling

22. A wheel of diameter 40 m has its lowest point 2 m above the ground and turns once every 20 seconds. A seat starts at the bottom. Its height in metres after t seconds is modelled by $h(t) = a\sin(b(t+c)) + d$, with $a > 0$ and $b > 0$.

(a) State the amplitude a and the midline value d.

(b) Find the value of b.

(c) The seat is at its lowest point when $t = 0$. Show that $c = -5$ is a suitable value, and write down $h(t)$.

(d) Find the height after 8 seconds.

Concept check

23. A student says: "*the graph of* $y = \sin\left(2x - \frac{\pi}{3}\right)$ *is the graph of* $y = \sin 2x$ *translated* $\frac{\pi}{3}$ *to the right.*" Identify the error and state the correct translation.

24. A sinusoidal graph has maximum value 7 and minimum value -1. A student says: "*the amplitude is* 8." Identify the error, and find the correct amplitude and the equation of the midline.

8.4 Reciprocal Trigonometric Functions HL

Divide 1 by a sine, a cosine or a tangent and three more functions appear. Each has a name of its own, a graph with vertical asymptotes, and an identity used to solve certain equations.

FORMULA 8.10 **Reciprocal trigonometric functions** FORMULA BOOKLET · 3.9 HL

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \operatorname{cosec}\theta = \frac{1}{\sin\theta}.$$

The third is the **cotangent**, $\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$. Where $\tan\theta$ is undefined, use the second form: $\cot\frac{\pi}{2} = \frac{0}{1} = 0$. Its definition is not printed in the formula booklet, so learn it.

The pairing looks wrong at first: *sec*ant goes with *cos*ine, and *cosec*ant with *sin*e. The *co* means complement $(\text{the angle that, added to } \theta, \text{ makes a right angle})$: $\operatorname{cosec}\theta = \sec(\frac{\pi}{2} - \theta)$, just as $\cos\theta = \sin(\frac{\pi}{2} - \theta)$. As a quick check, use the third letter. In **sec** the third letter is c, for cosine. In **cosec** it is s, for sine. In **cot** it is t, for tangent.

Three facts follow from the reciprocal alone: an asymptote wherever the parent is zero, no value of $\sec$ or cosec strictly between -1 and 1, and a meeting point wherever the parent is ± 1. With them each graph can be sketched from its parent.

First, a reciprocal has an asymptote wherever the original function is zero, because you cannot divide by zero. So $\sec$ has asymptotes where $\cos\theta = 0$, at $\theta = \frac{\pi}{2} + n\pi$, while cosec and $\cot$ have them where $\sin\theta = 0$, at $\theta = n\pi$.

Second, $\sin$ and $\cos$ never exceed 1 in size, so their reciprocals are never smaller than 1 in size. Neither $\sec\theta$ nor $\operatorname{cosec}\theta$ ever takes a value strictly between -1 and 1. State that gap whenever you give the range.

Third, where the parent equals 1 or -1, so does the reciprocal, so the two graphs meet there. For $\sec$ and cosec, a maximum of 1 on the parent is a minimum of 1 on the reciprocal, and a minimum of -1 becomes a maximum of -1. As the parent approaches 0, the reciprocal grows without limit towards an asymptote.

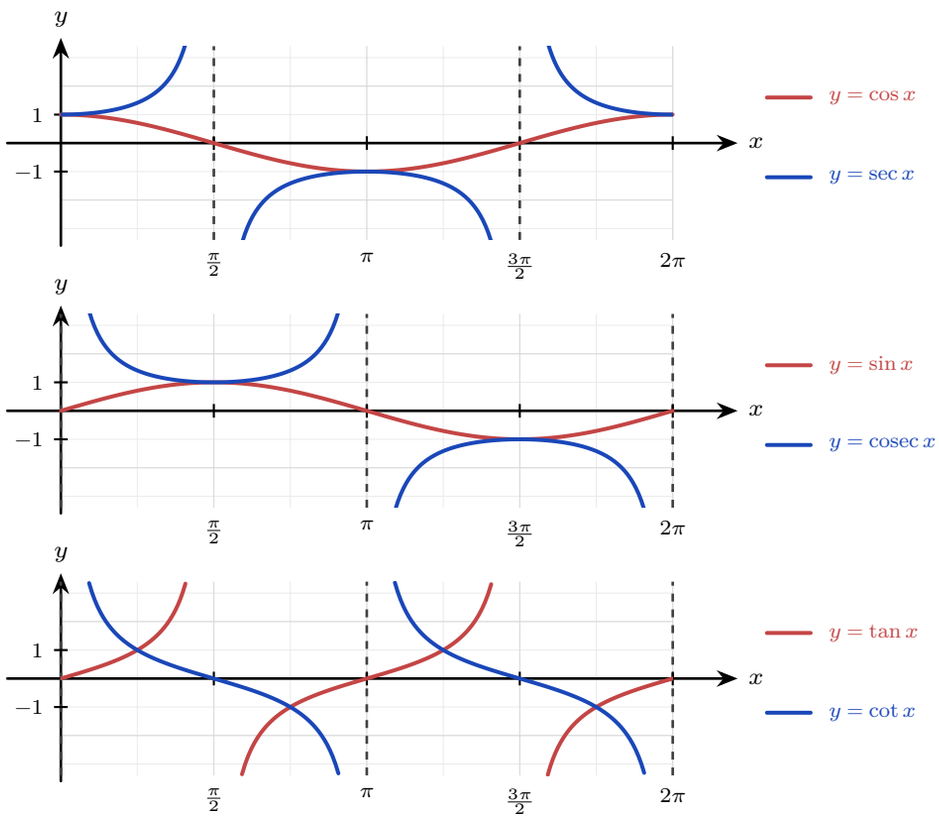


Figure 8.10 Each parent (red) with its reciprocal (cobalt), which meets the parent wherever it equals ± 1.

Taking n to be any integer, the three read as follows.

Function	Domain	Range	Period
$\sec\theta = \dfrac{1}{\cos\theta}$	all θ except $\frac{\pi}{2} + n\pi$	$y \le -1$ or $y \ge 1$	2π
$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$	all θ except $n\pi$	$y \le -1$ or $y \ge 1$	2π
$\cot\theta = \dfrac{\cos\theta}{\sin\theta}$	all θ except $n\pi$	all real y	π

8.4.1 The two Pythagorean variants

The identity $\cos^2\theta + \sin^2\theta = 1$ has two companions, one for each reciprocal pair.

Dividing $\cos^2\theta + \sin^2\theta = 1$ by $\cos^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta,$$

since $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{1}{\cos\theta} = \sec\theta$. Dividing the same identity by $\sin^2\theta$ instead gives $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. Both are in the formula booklet.

FORMULA 8.11 **Reciprocal Pythagorean identities** FORMULA BOOKLET · 3.9 HL

$$1+\tan^2\theta=\sec^2\theta, \qquad 1+\cot^2\theta=\operatorname{cosec}^2\theta.$$

Use these identities to rewrite an equation that mixes two functions in terms of one function only, as the double-angle identities do in §8.6.

EXAMPLE 8.10

Solve $\tan^2 x+\sec x=1$ for $0\le x\le 2\pi$.

SOLUTION

The equation mixes $\tan$ and $\sec$; the identity $\tan^2 x=\sec^2 x-1$ rewrites it in $\sec x$ alone:

$$\sec^2 x-1+\sec x=1 \implies \sec^2 x+\sec x-2=0 \implies (\sec x+2)(\sec x-1)=0.$$

So $\sec x=-2$ or $\sec x=1$, which is $\cos x=-\frac{1}{2}$ or $\cos x=1$. Read the angles from the unit-circle diagram of §8.1:

$$x=0,\ \frac{2\pi}{3},\ \frac{4\pi}{3} \text{ or } 2\pi$$

Converting back to $\cos$ at the end is a sensible final step, because the unit circle is read in sine and cosine, not in secant.

YOUR TURN 8.4 Reciprocal Trigonometric Functions HL

Exact values

25. Evaluate each exactly.

(a) $\sec\dfrac{\pi}{3}$ (b) $\operatorname{cosec}\dfrac{\pi}{2}$

(c) $\cot\dfrac{\pi}{4}$ (d) $\sec\dfrac{2\pi}{3}$

(e) $\cot\dfrac{2\pi}{3}$ (f) $\operatorname{cosec}\dfrac{7\pi}{6}$

Graphs of the reciprocal functions

26. Use the graphs of the reciprocal functions.

(a) State the range of $y=\sec x$.

(b) State the equations of the asymptotes of $y=\operatorname{cosec} x$ for $0\le x\le 2\pi$.

(c) State the domain and the range of $y=\cot x$.

(d) Find the equations of the asymptotes of $y=\sec 2x$ for $0\le x\le \pi$.

(e) Sketch $y = \operatorname{cosec} x$ for $0 < x < 2\pi$ from the graph of $y = \sin x$, marking the points where the two graphs meet.

Simplifying

27. Simplify each expression.

(a) $\sec^2\theta - \tan^2\theta$

(b) $\cot\theta \sin\theta$

(c) $\dfrac{1}{\cot\theta}$

(d) $(\sec\theta - 1)(\sec\theta + 1)$

(e) $\dfrac{\sec\theta}{\operatorname{cosec}\theta}$

(f) $\sin\theta \operatorname{cosec}\theta + \cos\theta \sec\theta$

Proving identities

28. Prove each identity.

(a) $\cot\theta \sec\theta = \operatorname{cosec}\theta$

(b) $\dfrac{\operatorname{cosec}\theta}{\cot\theta + \tan\theta} = \cos\theta$

(c) $\sec^2\theta + \operatorname{cosec}^2\theta = \sec^2\theta \operatorname{cosec}^2\theta$

(d) $\dfrac{1 + \cot^2\theta}{1 + \tan^2\theta} = \cot^2\theta$

Solving equations

29. Solve for $0 \le x \le 2\pi$. *(Hint: find one solution, then use the symmetry of the unit circle from §8.1 to find the others; §8.6 covers this in full)*

(a) $\sec x = 2$

(b) $2\tan^2 x + \sec^2 x = 4$

(c) $\cot^2 x + \operatorname{cosec} x = 1$

8.5 Inverse Trigonometric Functions HL

Everything so far has gone one way: give an angle, get a ratio. The inverse functions run the other way, recovering the angle from the ratio.

You have already used them without naming them. When you solve $\sin x = \frac{1}{2}$ on a calculator and it returns $\frac{\pi}{6}$, it is using the inverse function.

Two notations, one function. The inverse of sine is written either $\arcsin x$ or $\sin^{-1} x$. They mean exactly the same thing. These notes use $\arcsin$, because the other form is easy to misread:

$$\sin^{-1} x \text{ means the } \textit{inverse}, \qquad \text{but} \qquad (\sin x)^{-1} = \frac{1}{\sin x} = \operatorname{cosec} x.$$

The -1 is not an exponent here. Writing $\arcsin$, $\arccos$ and $\arctan$ removes the ambiguity.

8.5.1 Why a restriction is needed

A function has an inverse only if it is one-to-one (Ch. 5), and $\sin$, $\cos$ and $\tan$ are not. Each takes every value in its range infinitely often, so the horizontal line $y = \frac{1}{2}$ meets $y = \sin x$ infinitely many times.

If you ask "which angle has sine $\frac{1}{2}$?", there is no single answer: $\frac{\pi}{6}$, $\frac{5\pi}{6}$, $\frac{13\pi}{6}$ and infinitely many others all qualify. A function must return one output, so it has to choose.

The fix is the one from Ch. 5: cut the domain down until the function is one-to-one, then invert what is left. Which piece to keep is a choice. For sine, mathematicians have agreed to keep $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$.

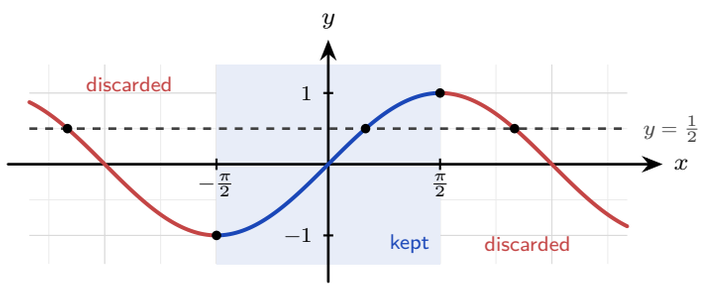


Figure 8.11 The line $y = \frac{1}{2}$ meets $y = \sin x$ again and again. The kept piece, from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$, is one-to-one and meets it once.

The same is done for the other two: cosine keeps $0 \le x \le \pi$, where it falls steadily, and tangent keeps $-\frac{\pi}{2} < x < \frac{\pi}{2}$, one branch between two asymptotes. Those kept pieces become the *ranges* of the inverses, because domain and range swap when a function is inverted.

Function	Domain	Range
$\arcsin x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$\arccos x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$\arctan x$	all real x	$-\frac{\pi}{2} < y < \frac{\pi}{2}$

Each inverse is a reflection of its restricted parent in the line $y = x$ (Ch. 5), so the graphs are the familiar curves reflected in that line. Arcsine increases from $(-1, -\frac{\pi}{2})$ to $(1, \frac{\pi}{2})$. Arccosine decreases from $(-1, \pi)$ to $(1, 0)$. Arctangent approaches the horizontal asymptotes $y = \pm\frac{\pi}{2}$, which are the vertical asymptotes of tangent after the reflection.

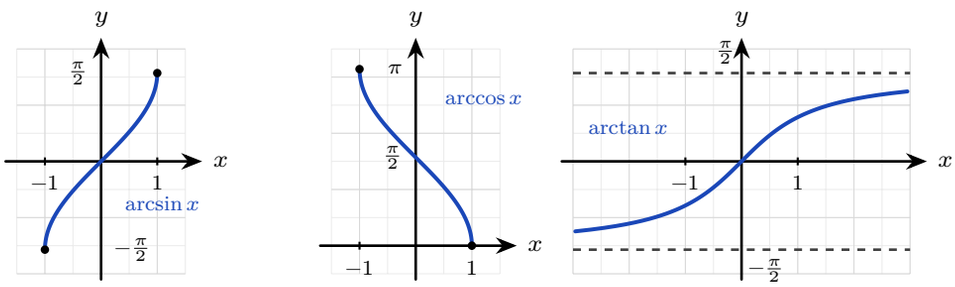


Figure 8.12 The graphs of $\arcsin x$, $\arccos x$ and $\arctan x$. The endpoints and dashed asymptotes mark the range of each.

EXAMPLE 8.11

Find (a) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$, (b) $\arccos\left(-\frac{\sqrt{3}}{2}\right)$, (c) $\arcsin\left(\sin\frac{5\pi}{4}\right)$.

SOLUTION

(a) Look for the angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $-\frac{\sqrt{3}}{2}$: it is $\boxed{-\frac{\pi}{3}}$

Not $\frac{4\pi}{3}$ and not $\frac{5\pi}{3}$: both have sine $-\frac{\sqrt{3}}{2}$, but they lie outside arcsine's range.

(b) In the same way, $\arccos\left(-\frac{\sqrt{3}}{2}\right)$ is the angle in $[0, \pi]$ whose cosine is $-\frac{\sqrt{3}}{2}$, which is $\pi - \frac{\pi}{6} = \boxed{\frac{5\pi}{6}}$

For a negative input, arccosine gives an angle in the second quadrant.

(c) First, $\sin\frac{5\pi}{4} = -\frac{\sqrt{2}}{2}$, and $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$, so $\arcsin\left(\sin\frac{5\pi}{4}\right) = \boxed{-\frac{\pi}{4}}$, not $\frac{5\pi}{4}$: arcsine can only answer with angles from its own range.

CAUTION $\arcsin(\sin x)$ does not always give back x. Every answer from arcsin must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and 2 does not. So $\arcsin(\sin 2) = \pi - 2$, an angle with the same sine as 2 but inside the range.

8.5.2 Compositions the other way round

A composition can also run with the inverse on the inside, as in $\cos(\arcsin k)$. The angle is never found here, and it never needs to be. Name it: let $\theta = \arcsin k$, so $\sin\theta = k$. A right triangle with opposite k and hypotenuse 1 has exactly that sine, and Pythagoras gives its adjacent side as $\sqrt{1-k^2}$, so

$$\cos(\arcsin k) = \sqrt{1-k^2}.$$

The positive root is the right one because $\arcsin$ answers in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and cosine is never negative there.

A side of a triangle cannot be negative, so for a negative k draw the triangle with $|k|$ and take each sign from the range of the inverse function. For example, $\theta = \arcsin\left(-\frac{3}{5}\right)$ lies between $-\frac{\pi}{2}$ and 0; the 3–4–5 triangle gives the sizes, so $\cos\theta = \frac{4}{5}$ and $\tan\theta = -\frac{3}{4}$.

The same triangle serves $\arccos$ and $\arctan$: place the known ratio on the two sides it names, find the third by Pythagoras, then read off whatever the question asks for.

EXAMPLE 8.12

Find the exact value of (a) $\cos\left(\arcsin\dfrac{20}{29}\right)$, (b) $\sin\left(\arccos\dfrac{9}{41}+\arctan\dfrac{20}{21}\right)$.

SOLUTION

(a) Let $\theta=\arcsin\frac{20}{29}$, so $\sin\theta=\frac{20}{29}$: opposite 20, hypotenuse 29. The adjacent side is $\sqrt{29^2-20^2}=\sqrt{441}=21$, so

$$\cos\left(\arcsin\frac{20}{29}\right)=\boxed{\frac{21}{29}}$$

This is positive because θ lies in $\left[-\frac{\pi}{2},\ \frac{\pi}{2}\right]$.

(b) Name both angles: $A=\arccos\frac{9}{41}$ and $B=\arctan\frac{20}{21}$. A 9–40–41 triangle gives $\cos A=\frac{9}{41}$ and $\sin A=\frac{40}{41}$, taken positive because $\arccos$ answers in $[0,\ \pi]$, where sine is never negative. A 20–21–29 triangle, the same one as in part (a), gives $\sin B=\frac{20}{29}$ and $\cos B=\frac{21}{29}$, both positive because $\tan B>0$ puts B in the first quadrant. The sum is now a compound angle:

$$\begin{aligned}\sin(A+B)&=\sin A\cos B+\cos A\sin B\\&=\frac{40}{41}\cdot\frac{21}{29}+\frac{9}{41}\cdot\frac{20}{29}\\&=\frac{840+180}{1189}\\&=\boxed{\frac{1020}{1189}}\end{aligned}$$

Neither A nor B was ever evaluated. The range of each inverse function is what fixes the sign of every side, and that is the only place in the working where a decision is made.

YOUR TURN 8.5 Inverse Trigonometric Functions HL

Inverse or reciprocal

30. A student writes: "$\cos^{-1}0=\dfrac{1}{\cos 0}=1$." Explain the difference between $\cos^{-1}x$ and $\sec x$, and evaluate both at $x=0$.

Exact values

31. Evaluate each exactly.

(a) $\arcsin\dfrac{1}{2}$ (b) $\arccos 0$

(c) $\arctan(-1)$ (d) $\arccos\left(-\dfrac{1}{2}\right)$

Domains, ranges and graphs

32. Use the graphs of the inverse functions.

(a) Sketch the graph of $y = \arccos x$, and label the coordinates of its endpoints.
(b) Write down the equations of the horizontal asymptotes of $y = \arctan x$.
(c) Find the domain of $y = \arcsin(x - 2)$.
(d) Find the range of $y = 2\arctan x + \pi$.
(e) Sketch the graph of $y = 2\arctan x + \pi$. Show its asymptotes and the point where it meets the y-axis.

The inverse on the outside

33. An inverse undoes its function only on the piece that was kept.

(a) State the range of $\arcsin x$, of $\arccos x$ and of $\arctan x$.
(b) Evaluate $\arcsin\left(\sin\frac{3\pi}{4}\right)$, and explain why the answer is not $\frac{3\pi}{4}$.
(c) Evaluate $\arccos\left(\cos\frac{4\pi}{3}\right)$.
(d) Evaluate $\arctan\left(\tan\frac{5\pi}{6}\right)$.
(e) State every value of x for which $\arcsin(\sin x) = x$.

34. Find the exact value, or write **undefined**.

(a) $\arcsin\left(\sin\frac{2\pi}{3}\right)$
(b) $\arccos(\cos 4)$
(c) $\arctan\left(\tan\left(-\frac{3\pi}{4}\right)\right)$
(d) $\arcsin(\sin 3)$
(e) $\arcsin\left(\tan\frac{\pi}{3}\right)$
(f) $\arctan\left(2\sin\frac{\pi}{3}\right)$

The inverse on the inside

35. Find the exact value, or write **undefined**.

(a) $\tan(\arctan 12)$
(b) $\cos\left(\arccos\frac{2\pi}{3}\right)$
(c) $\sin(\arcsin\pi)$
(d) $\sin\left(\arctan\frac{3}{4}\right)$
(e) $\cos\left(\arcsin\frac{7}{25}\right)$
(f) $\cos\left(\arctan\frac{1}{2}\right)$
(g) $\cos(\arcsin 0.6)$

Sums of inverse angles

36. Use a compound-angle formula to find each exact value.

(a) $\sin\left(\arccos\frac{3}{5} + \arctan\frac{5}{12}\right)$
(b) $\cos\left(\arctan 3 + \arcsin\frac{1}{3}\right)$

Concept check

37. A student says: "$\arcsin\left(\sin\dfrac{5\pi}{6}\right) = \dfrac{5\pi}{6}$, *because* $\arcsin$ *undoes* $\sin$." Explain why this is false, and find the correct value.

8.6 Solving Trigonometric Equations

An equation like $\sin x = \frac{1}{2}$ has not one solution but infinitely many, because the sine repeats every 2π. Your calculator returns only one of them, the principal value. Everything else you must find yourself, and the circle is what tells you how.

Start by counting the solutions in a single turn. Solving $\sin x = k$ means finding the points on the circle whose y-coordinate is k, so draw the horizontal line at height k and see where it meets the circle. Three things can happen.

- If $-1 < k < 1$ the line cuts the circle in **two** places, so two angles in each turn have sine k.
- If $k = 1$ or $k = -1$ the line touches the circle at **one** point, the top or the bottom. The two angles have merged into one.
- If $|k| > 1$ the line misses the circle, and there is **no** solution at all.

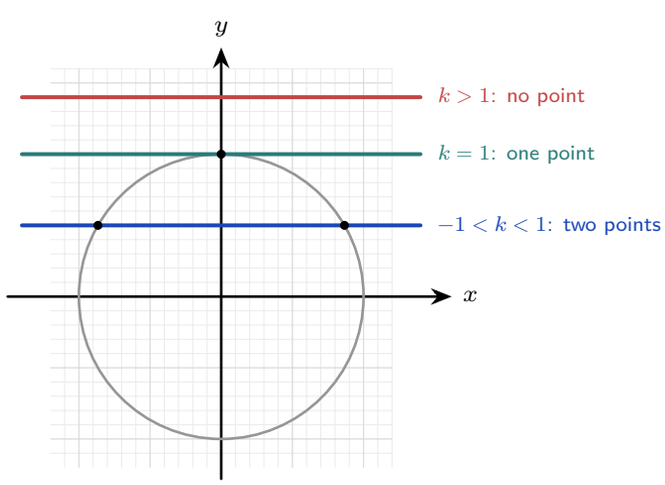


Figure 8.13 The line $y = k$ and the unit circle: the three cases.

The same three cases hold for $\cos x = k$, read with a vertical line at $x = k$ instead. Tangent behaves differently: it takes every real value, so $\tan x = k$ has solutions whatever k is.

This section uses two terms throughout: the reference angle and the principal value.

DEFINITION 8.2 **Reference angle**

The **reference angle** $\hat{\theta}$ is the angle between the arm OP and the x-axis, taken between 0 and $\frac{\pi}{2}$ inclusive. It is never negative, and it does not depend on the sign of k: you get it from the size of k alone, as $\arcsin|k|$, $\arccos|k|$ or $\arctan|k|$. It is acute whenever the arm lies off the axes.

The **principal value** is the single angle your calculator returns. It is the one solution lying inside the range of the matching inverse function:

$$\arcsin k \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right], \qquad \arccos k \in [0, \pi], \qquad \arctan k \in \left(-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right).$$

The two are easy to confuse. A principal value can be negative, and for a cosine it can be obtuse; a reference angle is never either. Take $\cos x = -\frac{1}{2}$. The calculator returns the principal value $\frac{2\pi}{3}$, while the reference angle is $\frac{\pi}{3}$, the acute angle the arm makes with the negative x-axis. Work from the reference angle, and it no longer matters which quadrant the calculator chose.

Equation	Principal value	Reference angle
$\sin x = -\frac{1}{2}$	$-\frac{\pi}{6}$	$\frac{\pi}{6}$
$\cos x = -\frac{1}{2}$	$\frac{2\pi}{3}$	$\frac{\pi}{3}$
$\tan x = -1$	$-\frac{\pi}{4}$	$\frac{\pi}{4}$

The method. Every solution is built from the reference angle and the quadrant it sits in. The sign of k tells you which two quadrants to use, by the rule of §8.1.2, and each quadrant has one construction.

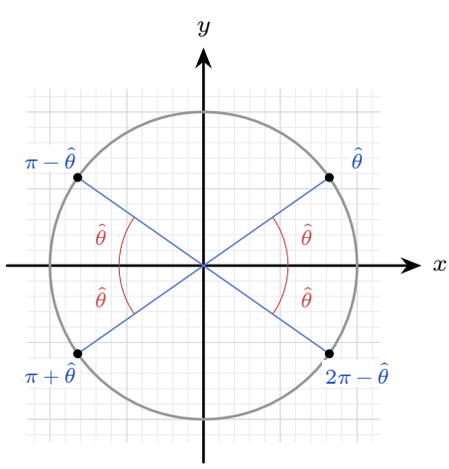


Figure 8.14 One reference angle $\hat{\theta}$, measured from the x-axis, gives an angle in each quadrant. The red arcs are all the same size; only the direction they are measured from changes.

KEY 8.12 **Solving a trigonometric equation**

1. Rearrange until a single trigonometric function is alone on one side.
2. If the argument is not plain x, substitute u for it and **transform the interval as well**.
3. Find the reference angle from the *size* of k: $\hat{\theta} = \arcsin|k|$, $\arccos|k|$ or $\arctan|k|$.
4. Use the sign of k to choose the two quadrants, then build the angle in each: $\hat{\theta}$, $\pi - \hat{\theta}$, $\pi + \hat{\theta}$ or $2\pi - \hat{\theta}$.
5. Add or subtract whole turns of 2π until you have every solution the interval asks for.
6. If you substituted, convert back to x at the very end.

There is a shorter route once you have one solution. If a solution exists (for sine and cosine, when $|k| \leq 1$), the calculator gives you one. Call it θ. Its partner in the same turn is $\pi - \theta$ for sine, $2\pi - \theta$ for cosine and $\pi + \theta$ for tangent, whatever quadrant θ is in.

KEY 8.13 **The partner solution**

If θ is one solution, then the other solution is

$$\sin x = k: \quad \pi - \theta, \qquad \cos x = k: \quad 2\pi - \theta, \qquad \tan x = k: \quad \pi + \theta.$$

Then add or subtract whole turns of 2π (for the tangent, half turns of π) to reach every solution the interval asks for.

Both routes give the same angles. The reference angle is the one to fall back on when the signs get confusing, or when the equation has been rearranged so far that you no longer trust which quadrant you are in.

A few checks belong in every solution.

- Your two angles should lie in the quadrants where that function has the sign of k. If only one does, you have built the wrong pair.
- For sine and cosine, when $k = \pm 1$ the two constructions land on the same point of the circle, so report that angle once in each turn.
- When $k = 0$ the solutions lie on the axes and not in a quadrant, so read them from the circle directly.
- Keep both ends when the interval includes them: $\cos x = 1$ on $0 \leq x \leq 2\pi$ gives $x = 0$ and $x = 2\pi$.

EXAMPLE 8.13

Solve $\cos x = -\frac{1}{2}$ for $0 \leq x \leq 2\pi$.

SOLUTION

Here $k = -\frac{1}{2}$. Take the reference angle from the size of k alone:

$$\hat{\theta} = \arccos\left|-\tfrac{1}{2}\right| = \arccos\tfrac{1}{2} = \frac{\pi}{3}.$$

Cosine is negative in the second and third quadrants, so build one angle in each:

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}, \qquad \pi + \frac{\pi}{3} = \frac{4\pi}{3}.$$

Both lie in the interval, so $x = \frac{2\pi}{3}$ or $x = \frac{4\pi}{3}$

Notice that the calculator alone would have given only $\frac{2\pi}{3}$. The reference angle gives both, and the sign of k says which quadrants they are in. The same steps work when k is not an exact value: only the reference angle now comes from the calculator.

TIP On a calculator paper, always check your solution count graphically: plot both sides and count the intersections in the interval. If your algebra found three solutions and the screen shows four, look for the one you missed.

- When the argument is not plain x but $2x$ or $3(x-4)$, substitute for the whole argument and, crucially, **transform the interval along with it**. A doubled argument sweeps the circle twice as fast, so the equation has twice as many solutions.

EXAMPLE 8.14

Solve $\cos 2x = \dfrac{\sqrt{3}}{2}$ for $0 \le x \le 2\pi$.

SOLUTION

Let $u = 2x$. As x runs over $[0, 2\pi]$, u runs over $[0, 4\pi]$: two full turns.

The reference angle is $\hat{\theta} = \arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$. Cosine is positive, so the first turn gives

$$u = \frac{\pi}{6} \quad \text{and} \quad u = 2\pi - \frac{\pi}{6} = \frac{11\pi}{6}.$$

The interval covers a second turn, so add 2π to each:

$$\frac{\pi}{6} + 2\pi = \frac{13\pi}{6}, \qquad \frac{11\pi}{6} + 2\pi = \frac{23\pi}{6}.$$

A third turn would pass 4π, so the list is complete:

$$u = \frac{\pi}{6}, \ \frac{11\pi}{6}, \ \frac{13\pi}{6}, \ \frac{23\pi}{6} \implies x = \frac{\pi}{12}, \ \frac{11\pi}{12}, \ \frac{13\pi}{12}, \ \frac{23\pi}{12}$$

Four solutions, not two. Note where the doubling acts: the step in u is still 2π, because $\cos u$ has period 2π whatever u came from. What doubling changes is the *interval*, and that is why two extra solutions appear. In x the period has become π.

8.6.1 When the interval is not 0 to 2π

The interval is part of the question, and it is not always $0 \le x \le 2\pi$. Another interval you will meet is $-\pi \le x \le \pi$. That is still one whole turn, but measured differently: half a turn clockwise from the positive x-axis, and half a turn anticlockwise.

The method does not change. Build the two angles from the reference angle exactly as before, then change either one by a whole turn if it falls outside the interval: subtract 2π from an angle above π, add 2π to an angle below $-\pi$. A whole turn returns the point to exactly where it was, so the value of the function is unchanged. Because cosine is even, its two solutions on $[-\pi, \pi]$ form a $\pm$ pair: for $\cos x = -\frac{1}{2}$ they are $x = \pm\frac{2\pi}{3}$. For sine the pair is $\hat{\theta}$ and $\pi - \hat{\theta}$ when $k > 0$, and $-\hat{\theta}$ and $-\pi + \hat{\theta}$ when $k < 0$.

An interval given in degrees asks for answers in degrees. The method is the same, with 180° in place of π and 360° in place of 2π. Set the calculator to degrees first: a radian answer such as 2.21 is the correct angle in the wrong unit.

8.6.2 Using identities

An equation mixing $\sin 2x$ with $\sin x$, or $\cos 2x$ with $\cos x$, cannot be solved as it stands. An identity rewrites everything in terms of a single function, turning the equation into an algebraic one, usually a quadratic.

- Choose the form of $\cos 2x$ that leaves one function only. When both sides share a factor, **factorise instead of dividing**: dividing by $\sin x$ loses every solution where $\sin x = 0$.

EXAMPLE 8.15

Solve for $0 \le x \le 2\pi$: (a) $\cos 2x = \sin x$, (b) $\sin 2x = \sin x$.

SOLUTION

(a) The right side is a sine, so use $\cos 2x = 1 - 2\sin^2 x$. The equation becomes a quadratic in $\sin x$:

$$1 - 2\sin^2 x = \sin x \implies 2\sin^2 x + \sin x - 1 = 0 \implies (2\sin x - 1)(\sin x + 1) = 0.$$

So $\sin x = \frac{1}{2}$ or $\sin x = -1$. The first gives $x = \frac{\pi}{6}$ or $\pi - \frac{\pi}{6} = \frac{5\pi}{6}$. The second gives one angle, $x = \frac{3\pi}{2}$, at the bottom of the circle:

$$x = \frac{\pi}{6},\ \frac{5\pi}{6} \text{ or } \frac{3\pi}{2}$$

(b) Use $\sin 2x = 2\sin x\cos x$, take everything to one side, and factorise:

$$2\sin x\cos x - \sin x = 0 \implies \sin x\,(2\cos x - 1) = 0.$$

So $\sin x = 0$ or $\cos x = \frac{1}{2}$. The first gives $x = 0$, π or 2π. The second gives $x = \frac{\pi}{3}$ or $2\pi - \frac{\pi}{3} = \frac{5\pi}{3}$:

$$\boxed{x = 0,\ \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3} \text{ or } 2\pi}$$

Dividing both sides by $\sin x$ would have left only $\cos x = \frac{1}{2}$ and lost the three solutions where $\sin x = 0$.

The wave form of §8.2.2 also reduces an equation that contains both $\sin x$ and $\cos x$ to an equation in a single sine.

EXAMPLE 8.16

Solve $8\sin x + 15\cos x = 6$ for $0 \le x \le 2\pi$.

SOLUTION

Use $8\sin x + 15\cos x = 17\sin(x + \alpha)$ with $\alpha \approx 1.0808$, found in the example of §8.2.2. Then $17\sin(x+\alpha) = 6$, so $\sin(x+\alpha) = \frac{6}{17}$. Put $u = x + \alpha$. As x runs over $[0,\ 2\pi]$, u runs over $[1.0808,\ 7.3640]$. The reference angle is $\arcsin\frac{6}{17} \approx 0.3607$, and $\frac{6}{17}$ is positive, so sine takes it in the first and second quadrants:

$$u \approx 0.3607 \qquad \text{and} \qquad u \approx \pi - 0.3607 \approx 2.7809.$$

The first sits below 1.0808, so add a turn: $0.3607 + 2\pi \approx 6.6439$, which is inside. The second is already inside, and a further turn from it would pass 7.3640. So $u \approx 2.7809$ or $u \approx 6.6439$, and subtracting α,

$$\boxed{x \approx 1.70 \qquad \text{or} \qquad x \approx 5.56} \quad (3\text{ s.f.})$$

Transforming the interval along with the substitution is what saved the second answer. Had u been left to run over $[0,\ 2\pi]$, the pair 0.3607 and 2.7809 would have given one negative x and lost 5.56 altogether.

YOUR TURN 8.6 Solving Trigonometric Equations

Exact solutions

38. Solve for $0 \leq x \leq 2\pi$.

(a) $\sin x = \frac{1}{2}$ (b) $\cos x = 0$

(c) $\tan x = 1$ (d) $\cos x = 1$

39. Solve for $0 \leq x \leq 2\pi$.

(a) $\sin x = -\frac{1}{2}$ (b) $2\cos x = \sqrt{3}$

(c) $\sqrt{2}\sin x = -1$ (d) $\tan x = -\sqrt{3}$

Other intervals

40. Solve for $-\pi \leq x \leq \pi$ unless a part gives another interval. Give every answer inside the interval.

(a) $\cos x = \frac{\sqrt{3}}{2}$ (b) $\sin x = -\frac{1}{2}$

(c) $\tan x = -1$ (d) $2\sin x = \sqrt{2}$

(e) $\cos x = -\frac{1}{2}$, but now for $0^\circ \leq x \leq 360^\circ$, with answers in degrees

Answers that are not exact

41. GDC Solve for $0 \leq x \leq 2\pi$, giving your answers to three significant figures. If there is no solution, say why.

(a) $\cos x = 0.3$

(b) $3\sin x = -2$

(c) $4\cos x + 5 = 0$

42. GDC Solve $\sin x = -0.8$ for $0 \leq x \leq 2\pi$, giving your answers to three significant figures.

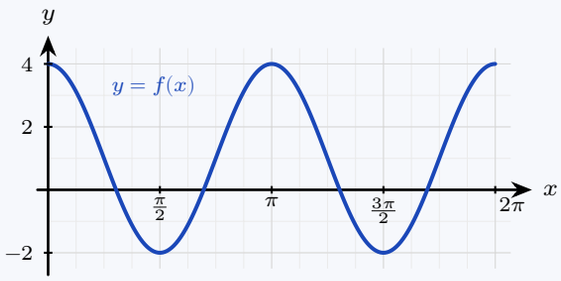
Multiple angles

43. Solve for $0 \leq x \leq 2\pi$. Transform the interval before solving.

(a) $\sin 2x = 0$ (b) $\cos 2x = \frac{1}{2}$

(c) $\tan 2x = 1$

(d) The graph of $f(x) = 3\cos 2x + 1$ is shown for $0 \leq x \leq 2\pi$. Use the graph to write down the number of solutions of $f(x) = 2$, and the value of k for which $f(x) = k$ has exactly three solutions.



Translated arguments

44. Solve for $0 \leq x \leq 2\pi$. Transform the interval before solving.

(a) $\sin\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

(b) $\cos\left(x - \frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2}$

(c) $2\sin\left(2x - \frac{\pi}{3}\right) = \sqrt{3}$

Factorising

45. Factorise, then solve for $0 \leq x \leq 2\pi$.

(a) $2\sin^2 x - \sin x = 0$

(b) $2\cos^2 x + \cos x - 1 = 0$

(c) $2\sin^2 x + 3\sin x + 1 = 0$

(d) $\tan^2 x - \tan x = 0$

Using an identity first

46. Use an identity to rewrite each equation, then solve for $0 \leq x \leq 2\pi$.

(a) $\cos 2x + \sin x = 0$

(b) $\sin 2x = \cos x$

(c) Explain why dividing both sides of the equation in (b) by $\cos x$ would lose solutions.

(d) $2\sin^2 x = 3\cos x$

(e) On the same axes, sketch the graphs of $y = \sin x$ and $y = \cos 2x$ for $0 \leq x \leq 2\pi$. Hence write down the number of solutions of $\sin x = \cos 2x$ for $0 \leq x \leq 2\pi$.

Using the wave form

47. Write the left-hand side in the form $R\sin(x + \alpha)$, then solve for $0 \leq x \leq 2\pi$. Transform the interval before solving.

(a) $\sin x + \sqrt{3}\cos x = 1$

(b) $\sqrt{3}\sin x + \cos x = \sqrt{2}$

Concept check

48. To solve $\sin 2x = \frac{1}{2}$ for $0 \le x \le 2\pi$, a student writes: "$2x = \frac{\pi}{6}$ *or* $\frac{5\pi}{6}$, *so* $x = \frac{\pi}{12}$ *or* $\frac{5\pi}{12}$." Identify the error and give the full solution set.

Chapter 8 · Trigonometry

Reflections

TOK A sine wave repeats forever, identical in every cycle, and nothing in the physical world does that exactly. Yet sums of sine waves describe voices, tides and heartbeats closely enough to compress, predict and diagnose them. Is the sinusoid a human tool, or an 'alphabet' of nature, a basic set of shapes from which the repeating patterns of the physical world are built?

IM Why is a full turn $360°$? The number goes back to Babylonian astronomers, and it divides exactly by 2, 3, 4, 5, 6, 8, 9, 10 and 12, which makes fractions of a turn easy. During the French Revolution, the reformers who gave the world the metre also proposed a decimal angle, the *grade*: 400 in a full turn and 100 in a right angle. Most of the world kept degrees, but surveyors in parts of continental Europe still use the grade, now called the *gon*, and many calculators still have a grade mode. Which unit is "right" depends on who is measuring, and why.

RW Because a sine wave is what steady rotation looks like from the side, it models oscillations everywhere. Alternating current rises and falls as a sine while the generator spins. Sound is air pressure oscillating in the same way, and its pitch is the frequency of the wave. Light and radio are sinusoids too, and the signals your phone sends and receives are built from them. Fourier analysis, which splits a signal into its sine components, is how music and photos are compressed and how earthquake and heart signals are read.

EXT In radians, $\sin x$ and x almost agree for small x: $\sin 0.1 = 0.09983\ldots$ On the unit circle x is the arc and $\sin x$ is the height, and over a tiny arc there is almost no difference between them. Physicists use this as the **small-angle approximation**: for $|x|$ small and in radians,

$$\sin x \approx x, \qquad \tan x \approx x, \qquad \cos x \approx 1 - \tfrac{1}{2}x^2.$$

At $x = 0.1$ these give 0.1, 0.1 and 0.995, against true values 0.09983, 0.10033 and 0.99500. The pendulum equation can be solved in elementary functions only when $\sin\theta$ is replaced by θ, which is why the simple period formula holds for small swings and not for large ones. All three fail in degrees, and that is one reason calculus uses radians. Chapter 13 turns the first into $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$, the limit that gives $\dfrac{d}{dx}\sin x = \cos x$.

End-of-Chapter Problems
Chapter 8: Trigonometry

1. Use the unit circle to write down the exact value of each.

(a) $\sin\frac{2\pi}{3}$ [1] (b) $\cos\frac{2\pi}{3}$ [1]

(c) $\tan\frac{2\pi}{3}$ [1] (d) $\cos\frac{5\pi}{4}$ [1]

(e) $\sin\frac{11\pi}{6}$ [1] (f) $\tan\frac{3\pi}{4}$ [1]

2. The point P lies on the unit circle at angle θ.

(a) State the coordinates of P when $\theta = \frac{5\pi}{6}$. [2]

(b) P has coordinates $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$. Find θ for $0 \le \theta \le 2\pi$. [2]

(c) P lies in the third quadrant. State the sign of $\sin\theta$, of $\cos\theta$ and of $\tan\theta$. [2]

(d) Given $\sin\theta = \frac{\sqrt{3}}{2}$, write down the two possible values of θ in $0 \le \theta \le 2\pi$, and explain why there are exactly two. [3]

3. Use the symmetry of the unit circle to answer each part.

(a) Given $\sin 0.6 \approx 0.565$, write down $\sin(\pi - 0.6)$. [1]

(b) Given $\cos 1.2 \approx 0.362$, write down $\cos(-1.2)$ and $\cos(2\pi - 1.2)$. [2]

(c) Explain why $\sin(\theta + 2\pi) = \sin\theta$ for every θ. [2]

(d) Explain why $\tan(\theta + \pi) = \tan\theta$, but $\sin(\theta + \pi) = -\sin\theta$. [3]

4. Given $\sin\theta = \frac{5}{13}$ with θ in the second quadrant, find the exact value of each.

(a) $\cos\theta$ [2] (b) $\tan\theta$ [1]

(c) $\sin 2\theta$ [2] (d) $\cos 2\theta$ [2]

5. Given $\tan\theta = 2$ with θ acute, find the exact value of each.

(a) $\sin\theta$ and $\cos\theta$ [2] (b) $\sin 2\theta$ [2]

(c) $\cos 2\theta$ [2]

6. Use a compound-angle formula to find each exact value.

(a) $\tan 15°$ [3] (b) $\sin 15°$ [2]

(c) $\sin 105°$ [2] (d) $\tan 105°$ [3]

7. GDC

(a) Show that $3\sin x + 4\cos x$ can be written as $5\sin(x+\alpha)$, where $\tan\alpha = \frac{4}{3}$ and $0 < \alpha < \frac{\pi}{2}$. Find α. [3]

(b) Hence write down the maximum value of $3\sin x + 4\cos x$, and find the smallest positive value of x at which it occurs. [2]

8. GDC Let $g(x) = 5\sin x + 12\cos x$ for $0 \le x \le 2\pi$.

(a) Write $g(x)$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. [3]

(b) Hence find the maximum value of g, and the value of x at which it occurs. [2]

(c) Solve $g(x) = 6.5$. [4]

9. Consider $f(x) = 3\sin(2x) + 1$.

(a) State the amplitude and the period. [2]

(b) State the maximum and minimum values. [2]

10. The graph of $y = a\sin(bx) + d$ has maximum 7, minimum 1, and period π, where $a > 0$ and $b > 0$. Find a, b and d. [4]

11. Sketch $y = \cos x - 1$ for $0 \le x \le 2\pi$, stating the maximum, the minimum, and the intercepts. [4]

12. The depth of water in a harbour is $d(t) = 5 + 3\sin\left(\frac{\pi t}{6}\right)$ metres, t hours after midnight.

(a) State the maximum and minimum depths. [2]

(b) Find the period, and say what it represents. [2]

13. A seat on a Ferris wheel has height $h(t) = 10 - 8\cos\left(\frac{\pi t}{15}\right)$ metres, t seconds after the start.

(a) State the minimum and maximum heights. [2]

(b) Find the time for one full revolution. [2]

14. GDC Let $f(x) = 2\sin(2x) + 1$ and $g(x) = \frac{x}{2}$, for $0 \le x \le 2\pi$.

(a) Write down the period and the range of f. [2]

(b) Find the x-coordinates of the points where the graphs of f and g meet. [4]

(c) Hence solve $f(x) > g(x)$. [3]

(d) Find the greatest value of $f(x) - g(x)$, and the value of x at which it occurs. [2]

15. GDC The mean monthly temperature in a coastal city is modelled by

$$T(t) = a\cos\left(b(t-8)\right) + d,$$

where T is in °C, t is the time in months from the start of the year, and $a, b > 0$. The model has a maximum of $27\,°\text{C}$ at $t = 8$, and its next minimum is $5\,°\text{C}$ at $t = 14$.

(a) Find a and d. [2]
(b) Find b. [2]
(c) Find $T(4)$. [2]
(d) Solve $T(t) > 20$ for $0 \le t \le 12$, and hence find the length of time in each year for which the model gives a mean temperature above $20\,°\text{C}$. [5]

16. GDC **Daylight in Sapporo.** The number of hours of daylight is modelled by

$$L(t) = 12.2 + 3.4\sin\left(\frac{2\pi(t-81)}{365}\right),$$

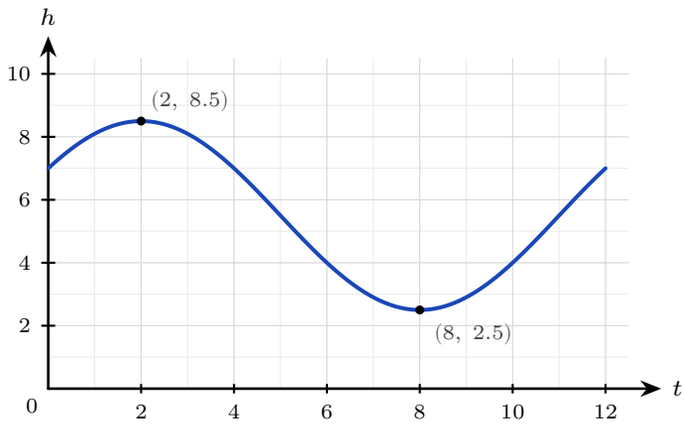
where t is the day of the year, with $t = 1$ on 1 January.

(a) State the maximum and minimum daylight, and find the value of t at which each occurs. [4]
(b) State the period of L, and say what it represents. [2]
(c) Find the daylight on 1 January, to three significant figures. [2]
(d) Find the two days of the year on which the daylight is exactly 12.2 hours, and explain what is special about them. [4]
(e) Find the values of t for which $L(t) \ge 15$, and state how many days of the year have at least 15 hours of daylight. [6]

17. The depth of water at the entrance to a harbour is modelled by

$$h(t) = a\cos(b(t-c)) + d,$$

where h is in metres, t is the time in hours after midnight, $0 \le t \le 12$, $a > 0$, $b > 0$ and $0 \le c < 12$. The graph of h is shown. It has a maximum point at $(2,\ 8.5)$ and a minimum point at $(8,\ 2.5)$.



(a) Find the values of a and d. [2]

(b) Find the value of b. [2]

(c) Write down the value of c. [1]

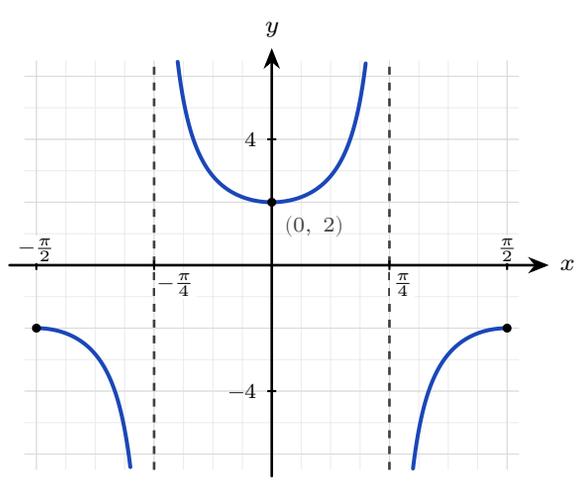
(d) Find the exact depth of water at 03:00. [2]

(e) A boat can pass through the entrance only when the depth is at least 4 m. Find the times between midnight and noon when the boat can pass, and the total length of that time. [5]

(f) Assume that the model also holds from noon to midnight. Sketch the graph of h for $0 \leq t \leq 24$. Label the maximum and minimum points and the points at $t = 0$ and $t = 24$. [3]

18. Solve $\sec^2 x = 4$ for $0 \leq x < \frac{\pi}{2}$. [2]

19. The graph of $y = f(x)$, where $f(x) = a\sec(bx)$ with $a > 0$ and $b > 0$, is shown for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The graph has vertical asymptotes $x = -\frac{\pi}{4}$ and $x = \frac{\pi}{4}$, a local minimum point at $(0,\ 2)$, and end points at $\left(-\frac{\pi}{2},\ -2\right)$ and $\left(\frac{\pi}{2},\ -2\right)$.



(a) Write down the range of f. [2]

(b) Find the value of a and the value of b. [3]

(c) Solve $f(x) = 4$. [3]

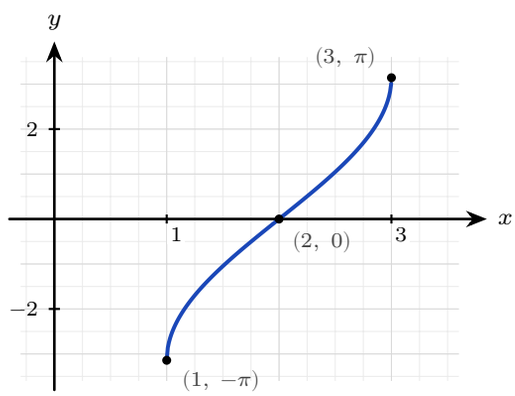
(d) Write down the values of k for which the equation $f(x) = k$ has no solution. [1]

20. Evaluate, giving exact values.

(a) $\arctan 1$ [1] (b) $\arcsin\left(-\frac{1}{2}\right)$ [1]

(c) $\arccos(-1)$ [1] (d) $\arcsin\left(\sin\frac{4\pi}{3}\right)$ [2]

21. The graph of $y = f(x)$, where $f(x) = p\arcsin(x - q)$, is shown. Its end points are $(1,\ -\pi)$ and $(3,\ \pi)$, and it passes through $(2,\ 0)$.



(a) Write down the domain and the range of f. [2]

(b) Find the value of p and the value of q. [3]

(c) Solve $f(x) = \frac{\pi}{3}$. [2]

(d) Find $f^{-1}(x)$, and state its domain. [3]

22. Solve for $0 \le x \le 2\pi$.

(a) $2\cos x + 1 = 0$ [2] (b) $\sqrt{3}\tan x = 1$ [2]

(c) $2\cos^2 x = 1$ [3] (d) $\tan^2 x = 3$ [3]

23. Solve each equation for $-\pi \le x \le \pi$. Build every answer from the reference angle, then add or subtract a whole turn where the interval requires it.

(a) $\sqrt{3}\tan x = -1$ [3] (b) $\sin 2x = \frac{\sqrt{3}}{2}$ [4]

(c) $\tan\left(x + \frac{\pi}{4}\right) = \sqrt{3}$ [4]

24. Solve each equation on the interval given. Take care: the same equation has different answers on different intervals.

(a) $\cos x = -\frac{\sqrt{2}}{2}$ for $0 \le x \le 2\pi$, and then for $-\pi \le x \le \pi$. [4]

(b) $\sin x = -\frac{\sqrt{3}}{2}$ for $-\pi \le x \le \pi$. [3]

(c) Explain why an equation of the form $\cos x = k$, with $|k| < 1$, always has two solutions on $-\pi \leq x \leq \pi$, and why they are negatives of each other. [3]

25. Solve for $0 \leq x \leq 2\pi$. Each becomes a quadratic, or factorises, after using an identity.

(a) $2\sin^2 x + \sin x - 1 = 0$ [4] (b) $4\sin^2 x - 3 = 0$ [3]

(c) $\cos 2x = \cos x$ [4] (d) $\sin 2x + \sin x = 0$ [4]

26. Solve for $0 \leq x \leq 2\pi$.

(a) $2\sin x \cos x = \dfrac{1}{2}$ [3]

(b) $\cos 2x + 3\sin x = 2$ [5]

(c) $\sin x + \cos x = 1$ [5]

27. Find the exact value of each.

(a) $\arccos\left(\cos\dfrac{7\pi}{4}\right)$ [2]

(b) $\tan\left(\arcsin\dfrac{8}{17}\right)$ [2]

(c) $\cos\left(\arcsin\dfrac{3}{5} + \arctan\dfrac{4}{3}\right)$, and hence $\arcsin\dfrac{3}{5} + \arctan\dfrac{4}{3}$ [4]

(d) $\tan\left(2\arctan\dfrac{1}{3}\right)$ [3]

28.

(a) Prove that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. [5]

(b) Hence solve $8\cos^3\theta - 6\cos\theta = 1$ for $0 \leq \theta \leq \pi$. [4]

29. In this question, θ takes only values for which every expression is defined.

(a) Prove that $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$. [4]

(b) Show that $\dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \tan\left(\theta + \dfrac{\pi}{4}\right)$. [3]

(c) Hence solve $\dfrac{1+\sin 2\theta}{\cos 2\theta} = \sqrt{3}$ for $0 \leq \theta \leq 2\pi$. [4]

30. In this question, θ takes only values for which every expression is defined.

(a) Prove that $\cot\theta - \tan\theta = 2\cot 2\theta$. [4]

(b) Prove that $\operatorname{cosec}\theta - \cot\theta = \tan\dfrac{\theta}{2}$. [5]

(c) Use part (a) with $\theta = \dfrac{\pi}{8}$ to show that $\tan\dfrac{\pi}{8} = \sqrt{2} - 1$, and check the result with part (b). [4]

31. **One function, two forms.** Let $f(x) = 2\sin x + \cos 2x$ for $0 \le x \le 2\pi$.

(a) Show that $f(x) = -2\sin^2 x + 2\sin x + 1$. [3]

(b) Let $u = \sin x$. Write f as a quadratic in u, and state the interval of values u can take. [3]

(c) Hence show that the maximum value of f is $\frac{3}{2}$. [4]

(d) Find the values of x in $[0, 2\pi]$ at which that maximum occurs. [3]

(e) Solve $f(x) = 1$ for $0 \le x \le 2\pi$. [5]

(f) State the range of f. [3]

32. Let $f(x) = -\sin x + \sqrt{3}\cos x$.

(a) Write $f(x)$ in the form $R\sin(x + \alpha)$, where $R > 0$ and $-\pi < \alpha \le \pi$, giving R and α exactly. [3]

(b) A student writes $\alpha = \arctan\left(\frac{\sqrt{3}}{-1}\right) = -\frac{\pi}{3}$. Explain why this value of α is wrong. [2]

(c) Hence solve $f(x) = 1$ for $0 \le x \le 2\pi$. [4]

33. Find the exact value of each. Each inverse function here is applied to a negative number.

(a) $\cos\left(\arcsin\left(-\frac{3}{5}\right)\right)$ [2]

(b) $\tan\left(\arccos\left(-\frac{5}{13}\right)\right)$ [3]

(c) $\sin\left(2\arctan\left(-\frac{1}{2}\right)\right)$ [3]

34. GDC In this question the intervals are in degrees, so give every answer in degrees.

(a) Solve $2\sin^2 x - 5\sin x - 3 = 0$ for $0^\circ \le x \le 360^\circ$. [4]

(b) Solve $5\cos x = 2$ for $-180^\circ \le x \le 180^\circ$, giving your answers to one decimal place. [3]

Challenge Questions

C1. GDC **When is a sum of waves periodic?** A function f is *periodic* with period $T > 0$ if $f(x + T) = f(x)$ for every x. Every sine wave is periodic, but a sum of two sine waves need not be. This investigation finds when the sum repeats, what it looks like when the two periods are close, and what happens when it never repeats.

(a) State the periods of $\sin 2x$ and $\sin 3x$. Show that $f(x) = \sin 2x + \sin 3x$ satisfies $f(x + 2\pi) = f(x)$, and by evaluating $f(\frac{\pi}{2})$ and $f(\frac{3\pi}{2})$ show that π is not a period of f. [3]

(b) Let p and q be positive numbers with $\dfrac{p}{q} = \dfrac{m}{n}$, where m and n are positive integers. Show that $\sin px + \sin qx$ is periodic, with period $\dfrac{2\pi m}{p}$, and check this against part (a). [3]

(c) By writing $A = u + v$ and $B = u - v$ and expanding $\cos(u+v) + \cos(u-v)$, show that

$$\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right).$$

[3]

(d) Two tuning forks sound at 440 Hz and 442 Hz together. The combined air pressure is modelled by $y = \cos(880\pi t) + \cos(884\pi t)$, with t in seconds. Write y as a product, and use it to describe the graph of y. Hence find how many times each second the sound is loudest, and state the general rule for forks of frequencies f_1 and f_2 Hz that are close together. [4]

(e) Let $g(x) = \cos x + \cos(\sqrt{2}\,x)$. Show that $g(x) = 2$ only when $x = 0$, and deduce that g is not periodic. You may assume that $\sqrt{2}$ is irrational. [5]

(f) Use technology to find the greatest value of $g(x)$ for $5 \le x \le 100$, and the value of x at which it occurs. Explain why this x is close to a multiple of 2π, by relating it to a fraction close to $\sqrt{2}$. Suggest how this could be investigated further. [3]

C2. GDC **Machin's formula for π.** In 1706 John Machin used the compound-angle formula for the tangent to write π in terms of two small arctangents, and so found 100 decimal places of π by hand.

(a) Use the formula for $\tan(A+B)$ to show that $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \frac{\pi}{4}$. Explain why the sum on the left must lie between 0 and $\frac{\pi}{2}$. [3]

(b) Let $\alpha = \arctan\frac{1}{5}$. Show that $\tan 2\alpha = \frac{5}{12}$ and $\tan 4\alpha = \frac{120}{119}$. [3]

(c) Show that $\tan\left(4\alpha - \frac{\pi}{4}\right) = \frac{1}{239}$. [2]

(d) Show that $0 < 4\alpha - \frac{\pi}{4} < \frac{\pi}{2}$, and hence prove Machin's formula

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}.$$

(Hint: use part (b) to show first that $2\alpha < \frac{\pi}{4}$) [4]

(e) For small x, $\arctan x \approx x - \dfrac{x^3}{3} + \dfrac{x^5}{5}$. Use these three terms for $\arctan\frac{1}{5}$, and only the first term for $\arctan\frac{1}{239}$, to estimate π, and compare it with π. Explain why Machin's formula gives π far faster than the formula of part (a). [4]

(f) Show that for every positive integer n,

$$\arctan\frac{1}{n} - \arctan\frac{1}{n+1} = \arctan\frac{1}{n^2+n+1},$$

and check that $n = 1$ gives part (a). Suggest how this could lead to other formulas of Machin's kind. [3]

C3. $\cos n\theta$ **as a polynomial in** $\cos\theta$**.** You have already met $\cos 2\theta = 2\cos^2\theta - 1$ and, in the end-of-chapter problems, $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. Each is a polynomial in $\cos\theta$ alone. This investigation shows that this always happens, and finds the pattern.

Write $c = \cos\theta$ throughout.

(a) Use the compound-angle formulas to show that

$$\cos\big((n+1)\theta\big) = 2\cos\theta\,\cos(n\theta) - \cos\big((n-1)\theta\big).$$

(Hint: expand $\cos(n\theta+\theta)$ *and* $\cos(n\theta-\theta)$*, then add)* [5]

(b) Starting from $\cos 0 = 1$ and $\cos\theta = c$, use the recurrence to obtain $\cos 2\theta$ and $\cos 3\theta$ in terms of c. [3]

(c) Use it again to find $\cos 4\theta$ and $\cos 5\theta$ in terms of c. [4]

(d) For $n \geq 1$, state the degree of the polynomial for $\cos n\theta$, and the coefficient of its highest power. Justify both from the recurrence. [4]

(e) Explain why no such polynomial in $\cos\theta$ alone exists for $\sin n\theta$, for any $n \geq 1$. *(Hint: compare* $\sin(-\theta)$ *with* $\sin\theta$*)* [3]

(f) Find the three roots of $4c^3 - 3c = 0$ by solving $\cos 3\theta = 0$ for $0 \leq \theta \leq \pi$. Generalise: state the n zeros of the polynomial for $\cos n\theta$, explain why there are no others, and suggest how this could be investigated further. [3]

C4. GDC **The window of safe passage.** At a harbour mouth the depth of water is modelled by

$$d(t) = 6 + 2\sin\left(\frac{\pi t}{6}\right) \text{ metres},$$

where t is the number of hours after midnight. A ferry can enter only while the depth is at least the figure its hull requires. This investigation finds the window of time available, and then finds it in general.

(a) State the greatest and least depths, the period of the tide, and the time of the first high tide. [4]

(b) The ferry needs at least 7 m of water. Solve $d(t) \geq 7$ on $0 \leq t \leq 12$, and state the window as a pair of clock times. [5]

(c) Show that the window is exactly 4 hours long, and explain why the same window opens once in every tidal cycle. [3]

(d) Now take a general tide $d(t) = m + a\sin\left(\dfrac{2\pi t}{T}\right)$ and a required depth k with $m < k < m + a$. Show that the time available in each cycle is

$$\frac{T}{\pi}\arccos\left(\frac{k-m}{a}\right).$$

(Hint: solve for the two ends in terms of $\arcsin$*, subtract, then use* $\frac{\pi}{2} - \arcsin c = \arccos c$*)* [6]

(e) A larger ferry needs 7.5 m. Use the formula to find its window, to the nearest minute. Then explain, from the shape of the arccosine graph, why the last half-metre of clearance costs far more time than the first. [4]

(f) Real tides are closer to a sum of waves. The two largest have periods of about 12.42 hours (from the Moon) and 12.00 hours (from the Sun), so a better model is

$$d(t) = 6 + 1.5\sin\left(\frac{2\pi t}{12.42}\right) + 0.5\sin\left(\frac{2\pi t}{12}\right).$$

Explain why the range between high and low water grows and shrinks, state the largest and smallest ranges, and find, in days, how long the range takes to return to its largest value. Suggest how the window of part (d) could be studied for this model. [3]