GEOMETRY & TRIGONOMETRY

Chapter 7

# Shapes & Measures

7

SYLLABUS 3.1 3.2 3.3 3.4 3.5

You cannot pace out the whole Earth. So how did anyone find its size? The answer came from something much smaller: an angle.

You can see the idea on any hill. On flat ground the horizon is level with your eye. Climb a hill and it drops slightly below eye level; the higher you climb, the further it drops. That drop is the **dip**: about half a degree from a hill a few hundred metres high, the width of the Moon.

Your line of sight to the horizon is a tangent to the Earth: it touches the surface at one point, and there it meets the radius at a right angle. So the hilltop, the horizon and the centre of the Earth form one right-angled triangle. Its angle at the centre equals the dip $\theta$, because the two lines that make the dip are perpendicular to the two radii. With hill height $h$,

$$\cos\theta = \frac{R}{R+h}, \qquad \text{so} \qquad R = \frac{h\cos\theta}{1-\cos\theta}.$$

Measure the hill, measure the dip, and the formula gives a radius close to the modern $6371$ km, from a single hill. Al-Bīrūnī did this a thousand years ago at Nandana, in modern Pakistan.

Size, not only shape, affects how much surface an object has for its volume. Double every length and the surface grows fourfold while the volume grows eightfold, so bigger things have less surface for what they hold. That is why crushed ice melts faster than a block, and one reason insects do not grow to the size of a horse.

Geometry and trigonometry let you measure distances, angles, areas and volumes in two and three dimensions, even where no ruler can reach. By the end of this chapter you will be able to:

- find lengths and angles inside a solid, including the angle between a line and a plane;
- find the volume and surface area of pyramids, cones, spheres and solids built from them;
- solve triangles with no right angle by the sine and cosine rules, including the ambiguous case;
- turn heights, distances and bearings given in words into a labelled triangle, and solve it;
- measure angles in radians, in which arc length is $r\theta$ and sector area is $\frac{1}{2}r^2\theta$.

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> A calculator is assumed throughout this chapter. The EXACT badge marks the questions whose answer must be left exact.

# 7.1 Three-Dimensional Geometry

The measurements you need from a solid are often not the ones marked on its faces. The diagonal of a room, from one floor corner to the opposite ceiling corner, runs through the inside of the room. Neither its length nor the angle it makes with the floor is marked on any wall. Right-angled triangles drawn inside the solid, and sometimes coordinates, give you both.

First, the names. Examination papers name points, segments, lines and angles with a fixed notation, and they do not explain it.

| Notation | Meaning |
|---|---|
| $A(x, y, z)$ | the point $A$ with coordinates $x$, $y$ and $z$ |
| $[AB]$ | the line segment with endpoints $A$ and $B$ |
| $AB$ | the length of $[AB]$ |
| $(AB)$ | the line through $A$ and $B$, continuing both ways |
| $A\hat{B}C$ | the angle at $B$, between $[BA]$ and $[BC]$ |
| $\triangle ABC$ | the triangle with vertices $A$, $B$ and $C$ |

The middle letter names the vertex. So $A\hat{B}C$ and $C\hat{B}A$ are the same angle, while $B\hat{A}C$ is a different one. Write the length as $AB$, not $|AB|$. The vertical bars $|\ |$ are kept for the magnitude of a vector, which Chapter 9 introduces. In running text these notes write $AB$ for the segment as well, where no confusion can arise.

Coordinates in the plane extend to space by adding a third axis, $z$. The distance formula extends with them. In two dimensions it is Pythagoras' theorem. In three dimensions it is Pythagoras used *twice*, as shown below the formula.

**FORMULA 7.1** **Distance and midpoint in 3D** FORMULA BOOKLET · 3.1

The distance between $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}.$$

The midpoint is the average of the coordinates, $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$.

Picture a box with the two points at opposite corners, as in the figure below. Across the floor of the box, Pythagoras gives

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

Treat that as one side of a new right triangle. Its other side is the vertical gap, of length $|z_1 - z_2|$. Pythagoras on this second triangle gives $d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2$.

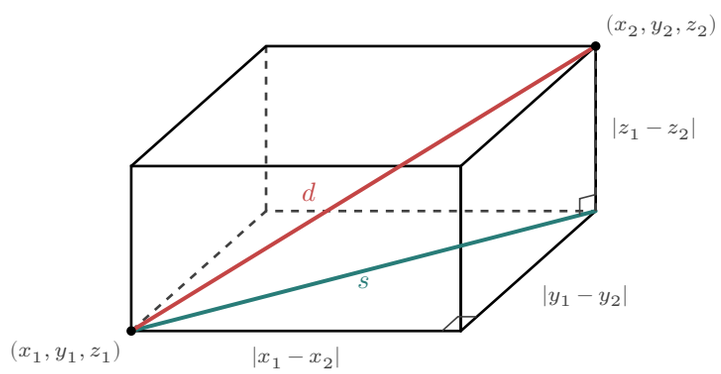



**Figure 7.1** Pythagoras in the floor gives the teal diagonal, $s = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Pythagoras again, in the upright triangle with sides $s$ and $|z_1 - z_2|$, gives $d$.

At the midpoint of the segment, each coordinate has changed by half of its total change, so $x = x_1 + \frac{1}{2}(x_2 - x_1) = \frac{x_1 + x_2}{2}$. The same holds for $y$ and $z$.

The equation of a sphere comes from the same formula. A sphere is the set of points at a fixed distance $r$ from a centre $(a, b, c)$, so its equation is $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$. It is beyond the syllabus.

## 7.1.1 The angle between a line and a plane

A line that is not parallel to a plane meets it at an angle. Many angles can be measured at that meeting point, so one of them has to be chosen, and the convention is to choose the smallest. Drop a perpendicular (draw the line that meets the plane at a right angle) from each point of the line to the plane. The feet of those perpendiculars form a line in the plane, called the **projection** of the line onto the plane (its 'shadow': what a light directly overhead would cast). The angle between the line and its projection is the smallest angle available. This course uses that fact without proving it.

**DEFINITION 7.1** **Angle between a line and a plane**

The **angle between a line and a plane** is the angle between the line and its projection on the plane. To find it, drop a perpendicular from a point on the line down to the plane. The line, its projection and that perpendicular form a right triangle, and you solve it with right-angled trigonometry.

In practice the steps come in this order. Find the length of the projection in the plane, then the perpendicular height, then use $\tan\theta = \dfrac{\text{height}}{\text{projection}}$.

A line perpendicular to the plane makes an angle of $90°$ with it, and a line parallel to the plane, or lying in it, makes $0°$. Neither case needs a triangle.

**EXAMPLE 7.1**

Find the angle between the space diagonal $AG$ and the base of a cuboid with a $4 \times 3$ base and height $2$.

**SOLUTION**

Start with the diagonal of the base: $AC = \sqrt{4^2 + 3^2} = 5$. This is the projection of $AG$ onto the base, and $CG = 2$ is vertical, meeting the base at a right angle. In the right triangle $ACG$,

$$\tan\theta = \frac{CG}{AC} = \frac{2}{5},$$

so $\theta = \arctan(0.4)$ (the angle whose tangent is $0.4$; §8.5), giving $\theta \approx 21.8°$

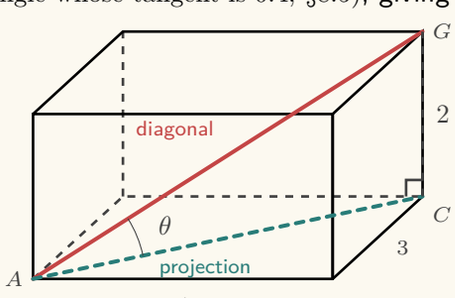



**Figure 7.2** The projection $AC$ of the diagonal $AG$; $\theta$ lies between them.

## 7.1.2 The angle between two intersecting lines

Two lines that meet also make an angle, and inside a solid those two lines are often edges from the same corner. Take the two edges, then join their far ends. The three segments form a triangle, and the angle you want is at the shared corner. Measure the three sides with the distance formula, then find the angle with the cosine rule §7.3, later in this chapter. When the triangle is isosceles, as in the next example, halving it gives a right triangle instead.

**EXAMPLE 7.2**

A square-based pyramid has base corners $A(0,0,0)$, $B(6,0,0)$, $C(6,6,0)$, $D(0,6,0)$ and apex $V(3,3,4)$. Find the angle between the slant edges $VA$ and $VB$.

**SOLUTION**

The two edges meet at $V$ and span the triangle $VAB$, so the angle you want is $A\hat{V}B$. First measure the triangle. By the distance formula,

$$VA = \sqrt{(3-0)^2 + (3-0)^2 + (4-0)^2} = \sqrt{34},$$

and $VB = \sqrt{34}$ by the same calculation, while $AB = 6$ along the base.

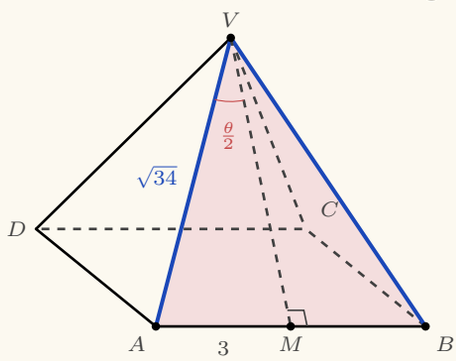



**Figure 7.3** The triangle $VAB$ is isosceles, so $VM$ cuts it into two right triangles.

The triangle is isosceles, so cut it in half. The midpoint of $AB$ is $M(3,0,0)$, and $VM$ meets $AB$ at a right angle. In the right triangle $VMA$, the side $AM = 3$ is opposite the half-angle at $V$, and the hypotenuse is $VA = \sqrt{34}$:

$$\sin\frac{\theta}{2} = \frac{3}{\sqrt{34}} \implies \frac{\theta}{2} \approx 31.0^\circ \implies \boxed{\theta \approx 61.9^\circ}$$

This uses the two formulas of this section: the distance formula for the three sides, the midpoint to cut the triangle in half, then right-angled trigonometry for the angle.

## YOUR TURN 7.1 Three-Dimensional Geometry

### Distance and midpoint

**1.** Find the distance between each pair of points.

(a) $(0,0,0)$ and $(2,3,6)$

(b) $(2,-1,0)$ and $(-2,3,7)$

(c) $(-3,4,1)$ and $(3,1,-1)$

(d) $(1.5,0,-2)$ and $(-0.5,4,2)$

**2.** Answer each part.

(a) Find the midpoint of $(1,2,3)$ and $(5,4,1)$.

(b) Find the midpoint of $(2, -1, 4)$ and $(6, 3, 0)$.

(c) $M(2, -1, 5)$ is the midpoint of $AB$, where $A$ is $(-1, 3, 4)$. Find the coordinates of $B$.

### Angle between a line and a plane

**3.** GDC A cuboid has base $3 \times 4$ and height $12$.

(a) Find the length of the diagonal of the base.

(b) Find the length of the space diagonal.

(c) Find the angle between the space diagonal and the base.

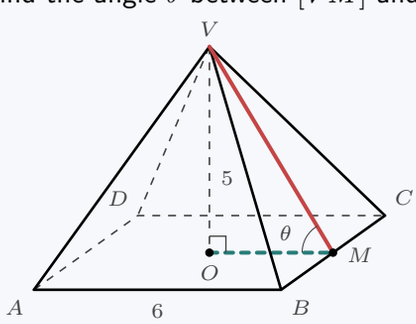
(d) Draw a labelled diagram of the cuboid. Label the base $ABCD$, with $AB = 4$ and $BC = 3$, and label $E$, $F$, $G$, $H$ the vertices above $A$, $B$, $C$, $D$. Draw the space diagonal $[AG]$ and its projection on the base, and mark the angle you found in part (c).

**4.** GDC A square-based pyramid has base side $6$, and its apex is $5$ vertically above the centre of the base.

(a) Find the distance from a corner of the base to the centre of the base.

(b) Find the angle between a slant edge and the base.

(c) The diagram shows the pyramid $VABCD$. $O$ is the centre of the base and $M$ is the midpoint of $[BC]$. Find the angle $\theta$ between $[VM]$ and the base.




### Angle between two intersecting lines

**5.** GDC The points $O(0, 0, 0)$, $P(4, 3, 0)$ and $Q(0, 3, 2)$ are three corners of a cuboid.

(a) Find the lengths $OP$, $OQ$ and $PQ$.

(b) Use the cosine rule to find the angle $P\hat{O}Q$ between the lines $(OP)$ and $(OQ)$. *(Hint: the cosine rule is set out in §7.3.3)*

**6.** GDC A pyramid has square base $A(0, 0, 0)$, $B(4, 0, 0)$, $C(4, 4, 0)$, $D(0, 4, 0)$ and apex $V(2, 2, 6)$.

(a) Show that $VA = VB = 2\sqrt{11}$.

(b) Find the angle $A\hat{V}B$ between the slant edges $VA$ and $VB$. *(Hint: the midpoint of $AB$ cuts triangle $VAB$ into two right triangles)*

**Concept check**

**7.** GDC A cuboid has a base $6$ by $8$ and height $5$. To find the angle between a space diagonal and the base, a student writes $\tan\theta = \frac{5}{8}$, so $\theta \approx 32.0°$.

(a) Identify the error.

(b) Find the correct angle.

# 7.2 Volumes and Surface Areas of Solids

You already know the cuboid, the prism and the cylinder. The formula booklet adds the sphere, the cone and the pyramid. Knowing where each formula comes from makes it harder to misapply.

**FORMULA 7.2** **Volume and surface area of solids** FORMULA BOOKLET · 3.1

| | |
|---|---|
| Sphere, radius $r$ | $V = \frac{4}{3}\pi r^3, \quad A = 4\pi r^2$ |
| Cone, radius $r$, height $h$ | $V = \frac{1}{3}\pi r^2 h, \quad$ curved $A = \pi r l$ |
| Pyramid, base area $A$, height $h$ | $V = \frac{1}{3}Ah$ |
| Cylinder, radius $r$, height $h$ | $V = \pi r^2 h, \quad$ curved $A = 2\pi r h$ |

In the cone, $l$ is the slant height. In the pyramid, $A$ is the area of the base.

The cylinder is listed in the *Prior learning* table at the front of the booklet rather than beside the other three.

Each formula has a short reason behind it.

- **Cylinder.** The volume is the base area $\pi r^2$ times the height $h$, as for any prism. For the area, cut the curved side along one vertical line and unroll it into a rectangle of height $h$ and width $2\pi r$, the base circumference. Its area is $2\pi r h$.
- **Cone.** The same cut unrolls the curved side into a **sector** (the region between two radii and the arc joining them). Its radius is the **slant height** $l$, measured from the edge of the base to the **apex** (the top point), and its arc is the base circumference $2\pi r$. A sector's area is half its arc times its radius, which follows from the two formulas of §7.5, so the curved area is $\frac{1}{2}(2\pi r)(l) = \pi r l$.
- **The $\frac{1}{3}$.** A cube can be cut into three identical square pyramids. Each one stands on a face of the cube, and all three tips meet at the opposite corner, as in the cube figure below. Each is therefore a third of the cube, and the same fraction holds for every pyramid and every cone: $V = \frac{1}{3}Ah$, where $A$ is the area of the base. The cube proves

this for one pyramid only. For other pyramids and for cones the proof needs integration. For a cone, rotate a straight line about the axis with the method of §15.7.

- **Sphere.** Put a sphere inside the smallest cylinder that holds it, of radius $r$ and height $2r$. Archimedes proved the sphere is two thirds of that cylinder. One proof uses integration: rotate a semicircle about its diameter with the method of §15.7. The cylinder holds $\pi r^2(2r) = 2\pi r^3$, so the sphere holds $\frac{4}{3}\pi r^3$. Its surface, $4\pi r^2$, is the area of four **great circles** (a circle round the sphere at its widest).

The surface area also follows from the volume, by an informal argument. Cut the sphere into many thin pyramids, with their tips at the centre and their bases on the surface. The thinner the pyramids, the closer each height is to $r$. Their total volume therefore approaches $\frac{1}{3}Ar$, where $A$ is the surface area. So $V = \frac{1}{3}Ar$, and $A = \dfrac{3V}{r} = 4\pi r^2$.

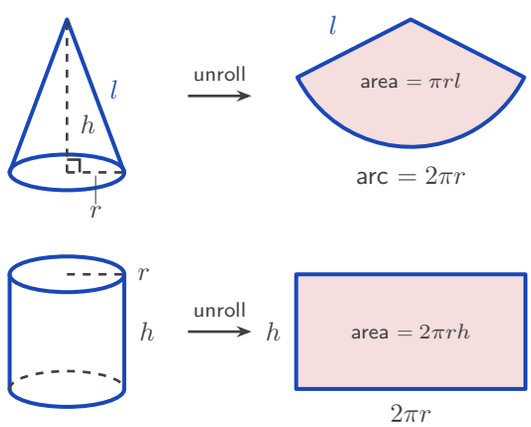



**Figure 7.4** Cut along a straight line down its side, a cone opens into a sector of radius $l$ and a cylinder into a rectangle. Each curved area is the area of that flat shape.

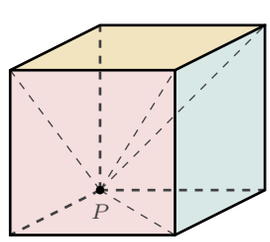



**Figure 7.5** Each tinted face is the base of a pyramid with its tip at $P$. The three pyramids fill the cube.

Two mistakes are common. First, the cone's curved surface uses the slant height $l$, not the vertical height $h$. Pythagoras links them by $l = \sqrt{r^2 + h^2}$. Second, "total surface area" means every face. A solid hemisphere has a curved part $2\pi r^2$ *and* a flat base $\pi r^2$, giving $3\pi r^2$. The same hemisphere as an open bowl has only $2\pi r^2$. For a solid made of two parts, count only the exposed surface: the faces where the parts join are hidden inside and do not count.

**EXAMPLE 7.3**

A solid is made of a cone of base radius $8$ and height $6$ sitting on top of a hemisphere of radius $8$. Find its total surface area.

**SOLUTION**

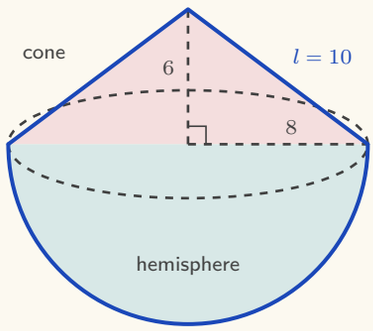



**Figure 7.6** Only the two curved surfaces are exposed. The circle where they join is hidden.

The exposed surface is the cone's curved part plus the hemisphere's curved part. The two flat circles where they join are hidden inside, so neither counts. The cone has slant height

$$l = \sqrt{8^2 + 6^2} = 10,$$

so its curved area is $\pi r l = 80\pi$. The hemisphere's curved area is $2\pi r^2 = 128\pi$, and the total surface area is

$$80\pi + 128\pi = \boxed{208\pi \approx 653}$$

EXT **Similar solids.** Two solids are **similar** when they have the same shape and all corresponding lengths are in the same ratio, the **linear scale factor** $k$. Every area formula above is a constant times a product of two lengths, and every volume formula a constant times a product of three, so areas scale by $k^2$ and volumes by $k^3$. A cut parallel to the base of a cone leaves a small cone above it with the same apex and slant, so the small cone is similar to the whole cone. For example, a cone of radius $9$ cm and height $12$ cm holds $\frac{1}{3}\pi(9)^2(12) = 324\pi$ cm$^3$. Cut $4$ cm below the apex, $k = \frac{4}{12} = \frac{1}{3}$, so the cut has radius $\frac{1}{3}(9) = 3$ cm and the small cone holds $\left(\frac{1}{3}\right)^3(324\pi) = 12\pi$ cm$^3$. The piece below the cut, a **frustum** (a cone with its top cut off), **is not similar to either cone**, so no scale factor applies to it. Find its volume as a difference, here $324\pi - 12\pi = 312\pi$ cm$^3$.

## YOUR TURN 7.2 Volumes and Surface Areas of Solids

### Volume and surface area

**8.** EXACT Find the volume of each solid. Leave your answers in terms of $\pi$ where appropriate.

(a) sphere of radius $3$

(b) cone of radius $3$, height $4$

(c) pyramid with base area $20$, height $9$

(d) cylinder of radius $4$, height $10$

**9.** EXACT Find the surface area asked for in each case.

(a) the surface area of a sphere of radius $5$

(b) the curved surface area of a cone of radius $5$, slant height $13$

(c) the total surface area of a solid hemisphere of radius $4$

(d) the curved surface area of a cylinder of radius $4$, height $10$

(e) the surface area of an open hemispherical bowl of radius $6$ (the outside surface only)

**10.** EXACT A cone has radius $7$ and height $24$.

(a) Find its slant height.

(b) Hence find its curved surface area.

### Composite solids

**11.** EXACT A solid is a cylinder of radius $3$ and height $10$, with a hemisphere of radius $3$ fixed to its top face.

(a) Explain which surfaces are exposed.

(b) Find the total surface area, in terms of $\pi$.

(c) Find the total volume, in terms of $\pi$.

### Finding a missing dimension

**12.** EXACT Work backwards from the given measurement.

(a) A sphere has volume $288\pi$. Find its radius.

(b) A cone of radius $6$ has volume $96\pi$. Find its height, then its curved surface area.

(c) A cylinder of height $10$ has curved surface area $120\pi$. Find its radius, then its volume.

### Similar solids

**13.** Two similar cones have heights $6$ cm and $9$ cm. The smaller cone has volume $40$ cm$^3$ and total surface area $24$ cm$^2$.

(a) Find the volume of the larger cone.

(b) Find the total surface area of the larger cone.

**14.** Two similar jugs hold $250$ cm$^3$ and $2000$ cm$^3$.

(a) Find the ratio of their heights.

(b) The smaller jug has surface area $150$ cm$^2$. Find the surface area of the larger jug.

**15.** EXACT A cone has base radius $6$ cm and height $12$ cm. It is cut parallel to its base, $8$ cm below the apex, and the small cone above the cut is removed. The piece left below the cut is called a *frustum*.

(a) Find the radius of the circular cut.
(b) Find the volume of the small cone as a fraction of the volume of the whole cone.
(c) Find the volume of the frustum, in terms of $\pi$.

**Concept check**

**16.** EXACT A cone has base radius $5$ and slant height $13$. A student writes: "$V = \frac{1}{3}\pi(5^2)(13) = \frac{325\pi}{3}$."
(a) Identify the error.
(b) Find the correct volume, in terms of $\pi$.

**17.** Two jugs are similar, and one is twice as tall as the other. A student says: "*the larger jug is twice as tall, so it holds twice as much.*" Explain the error, and state the correct factor for the capacity.

# 7.3 The Sine and Cosine Rules

Right-angled triangles are solved with $\sin$, $\cos$ and $\tan$. Name the sides opposite, adjacent and hypotenuse relative to an angle, and $\sin\theta = \frac{\text{opp}}{\text{hyp}}$, $\cos\theta = \frac{\text{adj}}{\text{hyp}}$, $\tan\theta = \frac{\text{opp}}{\text{adj}}$. Most triangles have no right angle. The sine rule and the cosine rule extend trigonometry to all triangles. With them you can solve a triangle from three known parts, provided at least one of them is a side. One arrangement, two sides and an angle not between them, can give two triangles or none (§7.3.2).

Both rules label each side with the lower-case letter of the angle *opposite* it, so side $a$ faces angle $A$.

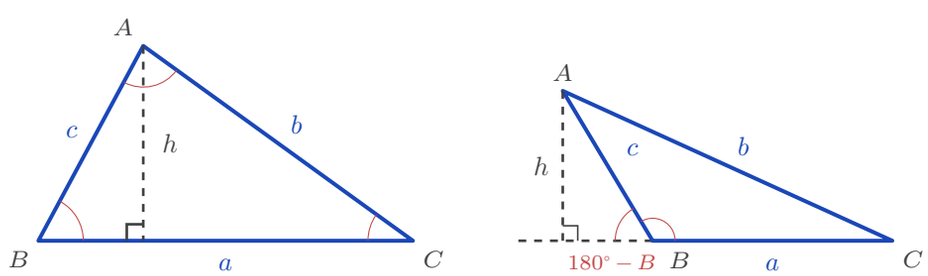



**Figure 7.7** **Left:** the dashed altitude $h$ (the perpendicular height) splits the triangle into two right triangles, which is where the sine rule comes from. **Right:** when $B$ is obtuse the altitude falls outside, and $h = c\sin(180^\circ - B)$.

## 7.3.1 The sine rule

Drop a perpendicular of height $h$ from $A$ onto side $a$, as in the figure above. It cuts the triangle into two right triangles, and in each one $h$ is the side opposite a known angle. On

the left, $h = c\sin B$. On the right, $h = b\sin C$. Both expressions measure the same segment, so $c\sin B = b\sin C$, which rearranges to $\frac{b}{\sin B} = \frac{c}{\sin C}$. Dropping a perpendicular from a different corner gives the third ratio. If the triangle has an obtuse angle, the perpendicular can fall outside the triangle, as on the right of the figure above. The result still holds, because an obtuse angle and its supplement $(\text{the angle that, added to it, gives } 180°)$ have the same sine. The sine and cosine of an obtuse angle are defined in §8.1.

**FORMULA 7.3** **Sine rule** FORMULA BOOKLET · 3.2

In any triangle,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

To find an angle, use the rule upside down, so that the unknown is on top:
$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

The sine rule pairs a side with the angle facing it. Use it whenever you know one complete pair, plus one more side or angle. The standard shorthand writes S for a known side and A for a known angle, in the order they occur round the triangle. These are the cases AAS, ASA and SSA; in ASA, find the third angle first.

In any triangle the largest angle faces the longest side, and the smallest angle faces the shortest. The sine rule shows why for acute angles: a larger side needs a larger sine. Use this to check every answer.

**EXAMPLE 7.4**

In triangle $ABC$, $A = 50°$, $B = 72°$ and $a = 8$. Find $b$.

**SOLUTION**

The pair $a$ and $A$ is complete, and $B$ is known, so use the sine rule to find $b$:

$$\frac{b}{\sin 72°} = \frac{8}{\sin 50°} \implies \boxed{b = \frac{8\sin 72°}{\sin 50°} \approx 9.93}$$

As expected, $B > A$ and $b > a$. Finding a side this way gives one value only.

## 7.3.2 The ambiguous case

When you use the sine rule to find an *angle*, also test the supplement of the angle your calculator gives, because both angles have the same sine.

Suppose you know two sides and an angle that is *not* between them. This is the SSA case. The sine rule gives an equation of the form $\sin B = k$. If $k > 1$ there is no solution and no triangle; if $k = 1$ there is one solution, $B = 90°$. If $0 < k < 1$, the equation has two solutions between $0°$ and $180°$: the angle $B$ and its supplement $180° - B$. The figure below shows why both can fit. Swing side $a$ from $C$: it meets the base at two points, $B_1$ and $B_2$,

and $CB_2B_1$ is isosceles because both sides from $C$ have length $a$. Its base angles are equal, so the angle of the triangle at $B_2$ is $180°$ minus the angle at $B_1$. Both triangles have the same height from $C$, so both angles have the same sine, $\frac{\text{height}}{a}$, and the equation cannot separate them. §8.1.2 proves $\sin(180° - B) = \sin B$ for every angle.

- Your calculator **gives only the acute solution**, between $0°$ and $90°$. Test **each** candidate, the calculator's angle as well as its supplement, by adding it to the angle you were given. If the total is below $180°$, a positive third angle remains and that triangle exists; if not, reject it. Two surviving candidates mean two triangles. None means no triangle, which can happen even when $k \leq 1$ if the given angle is obtuse.

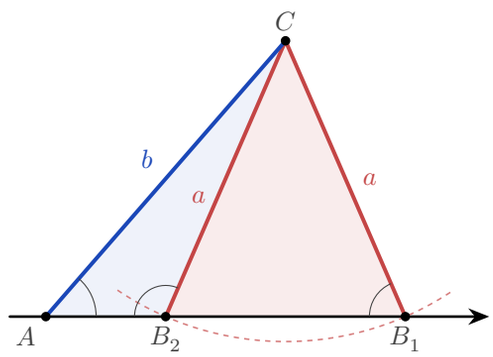



**Figure 7.8** Swinging side $a$ from $C$ meets the base twice, at $B_1$ and $B_2$: two triangles with the same $a$, $b$ and $A$. Their angles at $B$ are supplementary, so both satisfy the sine rule.

**EXAMPLE 7.5**

In triangle $ABC$, $a = 9$, $b = 11$ and $A = 40°$. Find angle $B$.

**SOLUTION**

Apply the sine rule: $\dfrac{\sin B}{11} = \dfrac{\sin 40°}{9}$, so

$$\sin B = \frac{11 \sin 40°}{9} \approx 0.786.$$

The calculator gives $B \approx 51.8°$, and $40° + 51.8° = 91.8°$ is below $180°$, so that triangle exists. The supplement is $180° - 51.8° = 128.2°$, so test it:
$A + B = 40° + 128.2° = 168.2°$, which is below $180°$. A third angle of $11.8°$ remains, so that triangle exists too.
Both values of $B$ are correct. There are two triangles, and a complete answer reports both:

$$B \approx 51.8° \quad \text{or} \quad B \approx 128.2°$$

The other two outcomes come from the same first step. Keep $b = 11$ and $A = 40°$ and change only $a$. With $a = 5$,

$$\sin B = \frac{11 \sin 40°}{5} \approx 1.41,$$

and no angle has a sine greater than $1$, so no triangle exists. With $a = 13$,

$$\sin B = \frac{11 \sin 40^\circ}{13} \approx 0.544 \implies B \approx 33.0^\circ \quad \text{or} \quad 180^\circ - 33.0^\circ = 147.0^\circ.$$

Test the supplement: $40^\circ + 147.0^\circ = 187.0^\circ$, which is more than $180^\circ$, so it is rejected. There is one triangle, with $B \approx 33.0^\circ$. This happens whenever the side opposite the given angle is at least as long as the other given side. When it is shorter and the given angle is acute, there are two triangles, one right-angled triangle, or none, as the table below sets out.

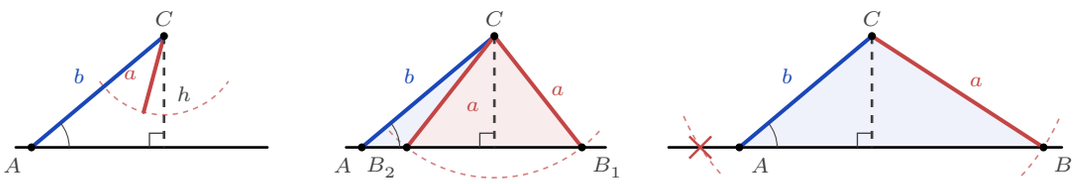



**Figure 7.9** The same $b$ and $A$, three lengths of $a$. The height $h = b \sin A$ decides whether side $a$ reaches the base.

For an acute given angle $A$, the table counts the triangles.

| Side opposite $A$ | Number of triangles |
|---|---|
| $a < b \sin A$ | none |
| $a = b \sin A$ | one, with a right angle at $B$ |
| $b \sin A < a < b$ | two |
| $a \geq b$ | one |

If the given angle $A$ is obtuse, there is one triangle when $a > b$ and none otherwise.

CAUTION The supplement check is needed when the sine rule gives an *angle*. A side found from the sine rule has one value. The cosine rule gives one angle between $0^\circ$ and $180^\circ$, because cos is positive for acute angles and negative for obtuse ones §8.1.2. The SSA case is still ambiguous with the cosine rule. Written about the given angle, it becomes a quadratic in the unknown side. With $a = 9$, $b = 11$ and $A = 40^\circ$ it gives $c^2 - (22 \cos 40^\circ)c + 40 = 0$, so $c \approx 14.0$ or $c \approx 2.86$. Two positive roots mean two triangles, and a negative root is rejected.

## 7.3.3 The cosine rule

When the known angle sits *between* the two known sides, no side is paired with the angle opposite it, so you cannot start with the sine rule. Use the cosine rule instead. Read it as Pythagoras' theorem with a correction term for the angle not being $90^\circ$.

A proof with coordinates is direct. Put the corner $C$ at the origin with side $a$ along the $x$-axis, so $B$ sits at $(a, 0)$. The point $A$ lies a distance $b$ from the origin, at angle $C$ to the

positive $x$-axis. By the definitions of cosine and sine, its coordinates are $(b\cos C,\ b\sin C)$. For an obtuse angle $C$ these are the definitions of §8.1. The distance formula now measures the third side:

$$c^2 = (a - b\cos C)^2 + (b\sin C)^2 = a^2 - 2ab\cos C + b^2\cos^2 C + b^2\sin^2 C,$$

and $\cos^2 C + \sin^2 C = 1$, so the last two terms add to $b^2$. Therefore $c^2 = a^2 + b^2 - 2ab\cos C$. For an acute angle that identity is Pythagoras' theorem in a right triangle with hypotenuse $1$. §8.2 extends it to every angle.

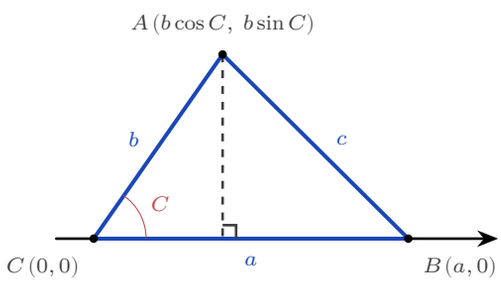



**Figure 7.10** Placing $C$ at the origin with $a$ along the axis fixes all three corners. Only $A$ needs trigonometry, and the distance from $A$ to $B$ is then the cosine rule.

**FORMULA 7.4** **Cosine rule** FORMULA BOOKLET · 3.2

In any triangle,

$$c^2 = a^2 + b^2 - 2ab\cos C, \qquad \text{equivalently} \qquad \cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

The letters rotate. Written about a different corner the same rule reads $a^2 = b^2 + c^2 - 2bc\cos A$, and the pattern is the same in each version: the side opposite the named angle is alone on the left.

When $C = 90°$, $\cos C = 0$ and the rule becomes $c^2 = a^2 + b^2$, which is Pythagoras' theorem. Use the first form to find a third side from two sides and the angle between them (SAS). Use the rearranged form to find an angle from all three sides (SSS).

**EXAMPLE 7.6**

A triangle has sides $a = 7$, $b = 9$ and $c = 12$. Find its largest angle.

**SOLUTION**

The largest angle faces the longest side, so it is $C$. The rearranged form gives

$$\cos C = \frac{7^2 + 9^2 - 12^2}{2(7)(9)} = \frac{-14}{126} \implies \boxed{C = \arccos\left(-\tfrac{1}{9}\right) \approx 96.4°}$$

The negative cosine shows that the angle is obtuse.

## 7.3.4 The area of a triangle

The familiar area $\frac{1}{2}$ base $\times$ height becomes a trigonometric formula as soon as the height is written in terms of a side. Take $b$ as the base. The perpendicular height from $B$ down to it is the opposite side of the right triangle at $C$, so $h = a\sin C$ and

$$\text{Area} = \tfrac{1}{2}\, b\,(a\sin C).$$

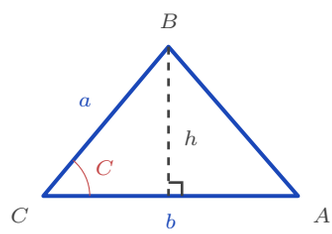



**Figure 7.11** The height on base $b$ is $h = a\sin C$.

**FORMULA 7.5** **Area of a triangle** FORMULA BOOKLET · 3.2

$$A = \frac{1}{2}ab\sin C$$

Here $a$ and $b$ are two sides and $C$ is the angle between them. The $A$ on the left is the area, not an angle.

This formula uses the same three parts as the SAS form of the cosine rule: two sides and the angle between them. So one sketch gives you the data for both.

**EXAMPLE 7.7**

A triangular field has $AB = 90$ m, $AC = 70$ m, and the angle at $A$ is $110°$. Find the length $BC$ and the area of the field.

**SOLUTION**

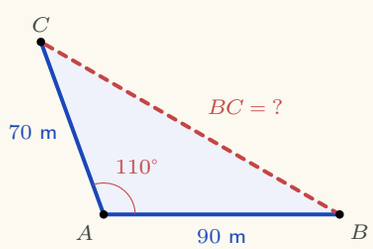



**Figure 7.12** Two sides and the angle between them (SAS).

The known angle lies between the two known sides, so use the cosine rule to find the third side $BC$. Rotate the letters to put $A$ in the named position:

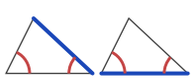
$$BC^2 = 90^2 + 70^2 - 2(90)(70)\cos 110° \approx 17\,309 \implies \boxed{BC \approx 132 \text{ m}}$$

The area of the field needs the same three pieces of information, so no further measurement is required:

$$\text{Area} = \tfrac{1}{2}(90)(70)\sin 110^\circ \approx 2960 \text{ m}^2$$

Two sides and the angle between them fix a triangle completely. Once you know these three parts, you can find every other part of the triangle.

The arrangement of the known parts decides the rule. In the pictures, the known sides are thick and the known angles are marked.

| Known | What you are given | What to use |
|---|---|---|
|  | Two angles and any side (AAS, ASA) | sine rule |
|  | Two sides and an angle opposite one (SSA) | sine rule, plus the ambiguous case |
|  | Two sides and the angle between them (SAS) | cosine rule, for the third side |
|  | All three sides (SSS) | cosine rule rearranged, for an angle |
|  | The same (SAS), when the area is wanted | area $= \tfrac{1}{2}ab\sin C$ |

Whichever rule you use, finish with two checks. When the sine rule gives an angle, test both candidates, the angle and its supplement. Then confirm that the triangle exists: the three angles add to $180^\circ$, and every one is positive.

Read the table by counting what you know. Three parts, at least one of them a side, determine the triangle, provided it exists: the two shorter sides must add to more than the longest, and the known angles must add to less than $180^\circ$. SSA is the exception: it may allow two triangles, or none.

## YOUR TURN 7.3 The Sine and Cosine Rules

### The sine rule

**18.** GDC Use the sine rule to find the required side or angle.

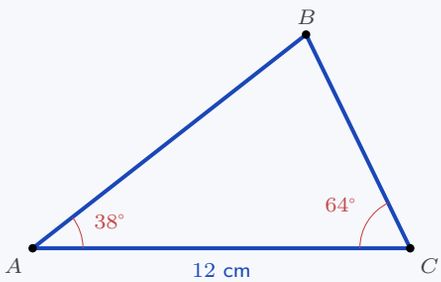
(a) $a = 7$, $A = 40^\circ$, $B = 60^\circ$: find $b$

(b) $a = 9$, $A = 35^\circ$, $B = 55^\circ$: find $b$

(c) $A = 45^\circ$, $B = 60^\circ$, $a = 10$: find $C$ and $b$

(d) $a = 12$, $b = 10$, $A = 65°$: find $B$, and explain why there is only one possible value

(e) The diagram shows triangle $ABC$. Find the length $BC$.




### The ambiguous case

**19.** GDC For each triangle $ABC$, use the sine rule to find $\sin B$. State how many triangles fit the data, and give every possible value of $B$.

(a) $a = 6$, $b = 10$, $A = 45°$

(b) $a = 4$, $b = 8$, $A = 30°$

(c) $a = 7$, $b = 10$, $A = 35°$

(d) $a = 9$, $b = 10$, $A = 120°$

**20.** GDC In triangle $ABC$, $a = 7$, $b = 9$ and $A = 45°$.

(a) Find the two possible values of angle $B$.

(b) Show that both give a valid triangle.

(c) Explain why this situation cannot arise when the given angle is included between the two given sides.

(d) For each triangle, find the length of side $c$.

(e) Check part (d) by writing the cosine rule about $A$ as a quadratic in $c$.

(f) Draw one labelled diagram that shows both triangles. Mark the two positions of $B$ as $B_1$ and $B_2$, and label the given sides, the given angle and both values of angle $B$.

### The cosine rule

**21.** GDC Use the cosine rule.

(a) $b = 5$, $c = 8$, $A = 60°$: find $a$

(b) $a = 6$, $b = 9$, $C = 110°$: find $c$

(c) $a = 6$, $b = 7$, $c = 8$: find angle $A$

(d) sides $5$, $6$, $7$: find the largest angle

### The area of a triangle

**22.** GDC Find the area of each triangle.

(a) sides $5$ and $8$, with an angle of $30°$ between them

(b) sides $9$ and $6$, with an angle of $125°$ between them

(c) sides $7$ and $9$, with an angle of $40°$ between them

(d) sides $5$, $7$ and $8$: find the angle opposite the side $7$ first

**23.** GDC In triangle $PQR$, $PQ = 11$ cm, $PR = 7$ cm and $Q\hat{P}R = 52°$.

(a) Find the length $QR$.

(b) Find the area of the triangle.

**Concept check**

**24.** To find $a$ when $b = 5$, $c = 8$ and $A = 60°$, a student writes

$$a^2 = 5^2 + 8^2 - 2(5)(8)\cos 60° = 9\cos 60° = 4.5, \quad \text{so } a \approx 2.12.$$

(a) Identify the error and find the correct value of $a$.

(b) A second student says that sides of length $3$, $4$ and $8$ make a triangle. Explain why no triangle has these sides.

# 7.4 Elevation, Depression and Bearings

Applied questions give heights, distances and directions in words, not a labelled triangle. They ask how far away a boat is, or in which direction a ship must sail to return to port. Two conventions, angles of elevation and depression and bearings, let you draw the triangle from those words. Draw it first, and label every length and angle you are given.

An **angle of elevation** is measured upward from the horizontal to a line of sight. An **angle of depression** is measured downward from it. The observer's horizontal and the ground are parallel lines. So the angle of depression from a cliff top down to a boat equals the angle of elevation from the boat up to the cliff top. They are **alternate angles** (the equal angles on opposite sides of a line that crosses two parallel lines), and either one can be used in the triangle.

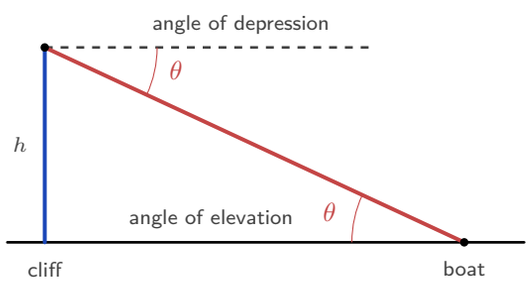



**Figure 7.13** The angles of depression and elevation are alternate angles, so they are equal.

When the distance to the object is unknown, one sighting is not enough to find a height: two unknowns, one equation. A second sighting, from a point a known distance nearer, gives the missing equation. The height is the same from both positions, so writing it twice and equating the two expressions eliminates the distance.

**EXAMPLE 7.8**

From a point on level ground the angle of elevation of the top of a tower is $25^\circ$. From a point $30$ m nearer the tower, on the same straight line, the angle of elevation is $40^\circ$. Find the height of the tower.

**SOLUTION**

Let $h$ be the height and let $d$ be the distance from the *nearer* point to the foot of the tower. The far point is then $d + 30$ from the foot. Each sighting gives a right triangle with $h$ opposite the angle and the ground distance adjacent to it, so

$$h = d\tan 40^\circ \qquad \text{and} \qquad h = (d+30)\tan 25^\circ.$$

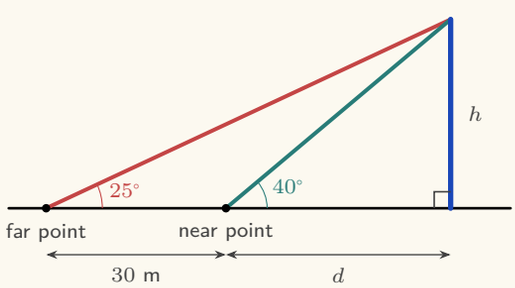



**Figure 7.14** Both sightings share the height $h$. Measure $d$ from the nearer point.

Both measure the same tower, so set them equal and collect the terms in $d$:

$$d\tan 40^\circ = (d+30)\tan 25^\circ \implies d\,(\tan 40^\circ - \tan 25^\circ) = 30\tan 25^\circ,$$

$$d = \frac{30\tan 25^\circ}{\tan 40^\circ - \tan 25^\circ} \approx \frac{13.99}{0.3728} \approx 37.5 \text{ m}.$$

The height follows from either equation:

$$h = 37.5\tan 40^\circ \approx 31.5 \text{ m}$$

Check it against the other sighting: the far point is $67.5$ m away, and $\arctan\left(\frac{31.5}{67.5}\right) \approx 25^\circ$, as given. Measure $d$ from the nearer point, so that the far distance is $d + 30$. If you measure $d$ from the far point, the near distance is $d - 30$. Mixing the two choices in one solution gives a negative distance.

A **bearing** gives a direction as an angle measured clockwise from north. It runs from $000^\circ$ up to, but not including, $360^\circ$, and is always written with three figures, so due east is $090^\circ$ and not $90^\circ$. Writing three figures prevents misreading on a chart. Due south is $180^\circ$, due west is $270^\circ$, and north-west is $315^\circ$.

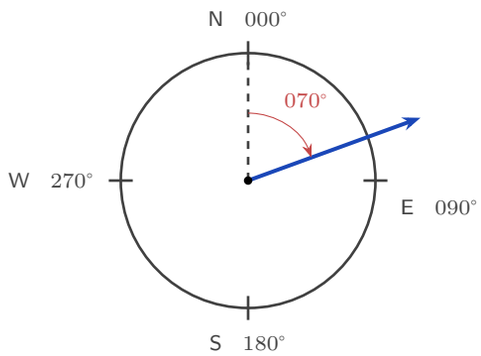



**Figure 7.15** The arrow here is on a bearing of $070°$.

The **back bearing** is the direction of the return journey, and it differs from the outward bearing by $180°$. Add $180°$ when the bearing is below $180°$, and subtract $180°$ when it is $180°$ or more. Either way the answer stays between $000°$ and $360°$.

**EXAMPLE 7.9**

A ship sails $60$ km from port $P$ on a bearing of $050°$ to a point $Q$, then $80$ km on a bearing of $140°$ to $R$. Find the distance $PR$.

**SOLUTION**

Sketch the journey first, with a north line at $P$ and at $Q$. The two bearings differ by $140° - 50° = 90°$, so the ship turns through $90°$ at $Q$. The interior angle is $180°$ minus the turn, so $P\hat{Q}R = 180° - 90° = 90°$.

The back-bearing method set out below agrees: the bearing from $Q$ back to $P$ is $050° + 180° = 230°$, and $230° - 140° = 90°$.

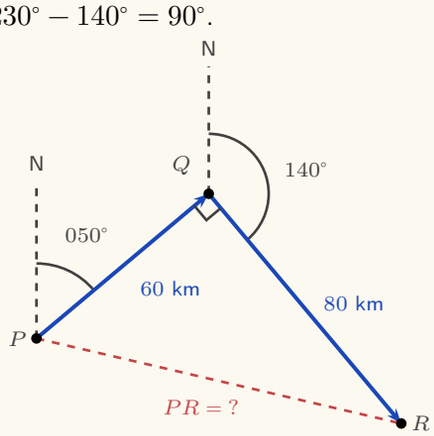



**Figure 7.16** Bearings are measured from the north lines at $P$ and $Q$.

Then $PR$ is a hypotenuse, and Pythagoras gives

$$PR = \sqrt{60^2 + 80^2} = \sqrt{10000} \implies \boxed{PR = 100 \text{ km}}$$

Interior angle $= 180° -$ turn works at any corner where the route turns by less than $180°$. The back-bearing method below gives the same angle, and it is safer when the sketch is hard to read.

When the corner angle is not $90°$, read the two distances and that angle off the same sketch. Find the missing distance with the cosine rule, then find the direction back with the sine rule. Many bearings questions have this shape. Solve them with the steps below, in this order.

**KEY 7.6** **Solving a bearings problem**

1. Sketch the journey, and draw a north line at **every** point where a bearing is given.
2. Mark each bearing as an angle clockwise from its own north line.
3. Find the interior angle of the triangle at the corner: the positive difference between the back bearing of the first leg $(\text{one stage of the journey})$ and the bearing of the second leg. If it is more than $180°$, subtract it from $360°$.
4. Use the cosine rule for the missing distance. For the missing angle, use the sine rule only if the angle is opposite the shorter leg, because that angle must be acute. Otherwise use the cosine rule. The angle may be obtuse, and the sine rule gives only the acute value.
5. Turn that angle back into a bearing: add it to the bearing of the first leg if the route turns right $(\text{clockwise})$ at the corner, and subtract it if the route turns left. The sketch shows which. If the result falls outside $000°$ to $360°$, add or subtract $360°$.

Do not skip step 1: without a north line drawn at the corner, step 3 is easy to get wrong.

**EXAMPLE 7.10**

A hiker walks $12$ km from $P$ on a bearing of $070°$ to $Q$, then $8$ km on a bearing of $150°$ to $R$. Find the distance $PR$ and the bearing of $R$ from $P$.

**SOLUTION**

First sketch the journey and construct the angle at the corner. The bearing back from $Q$ to $P$ is $070° + 180° = 250°$, and the onward bearing is $150°$, so the interior angle is

$$P\hat{Q}R = 250° - 150° = 100°.$$

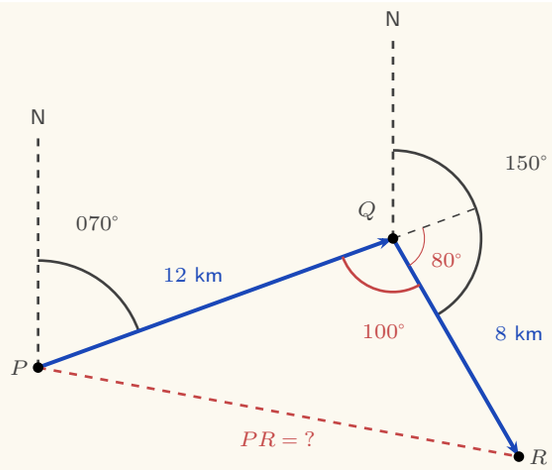



**Figure 7.17** The north line at $Q$ fixes the corner angle. The route turns $80°$, so the interior angle is $180° - 80° = 100°$.

The sketch agrees: the angle at the corner is obtuse. There is no right angle, so use the cosine rule to find $PR$, the third side that closes the triangle:

$$PR^2 = 12^2 + 8^2 - 2(12)(8)\cos 100° \approx 241.3 \implies \boxed{PR \approx 15.5 \text{ km}}$$

For the bearing of $R$ from $P$, find the angle $Q\hat{P}R$ by the sine rule:

$$\frac{\sin(Q\hat{P}R)}{8} = \frac{\sin 100°}{15.54} \implies \sin(Q\hat{P}R) \approx 0.507 \implies Q\hat{P}R \approx 30.47°.$$

This angle is opposite the shorter leg, $8$ km, so it is acute and the sine rule gives the correct angle. The route turns right at $Q$, so add. The bearing of $R$ from $P$ is $070° + 30.47° \approx 100.47°$, so the bearing of $R$ from $P$ is $\boxed{100°}$ to the nearest degree. Every step came from the labelled diagram: the bearings give the corner angle, the cosine rule the closing side, and the sine rule the final direction.

## YOUR TURN 7.4 Elevation, Depression and Bearings

### Elevation and depression

**25.** GDC Find the angle of elevation in each case.

(a) a $50$ m tower, from $120$ m away on level ground

(b) a $45$ m tower, from $90$ m away on level ground

**26.** GDC In each case, find the horizontal distance asked for.

(a) From the top of an $80$ m cliff, the angle of depression to a boat is $25°$. Find the distance from the foot of the cliff to the boat.

(b) From the top of a $30$ m building, the angle of depression to a car is $40°$. Find the distance from the foot of the building to the car.

(c) From the top of a $60$ m cliff, the angles of depression to two buoys are $20°$ and $35°$. The buoys and the foot of the cliff lie on one straight line, with both buoys on the same side of the cliff. Find the distance between the buoys.

### Two sightings

**27.** From a point $A$ on level ground, the angle of elevation of the top of a building is $30°$. From a point $B$, $50$ m nearer the building on the same straight line, the angle of elevation is $45°$. Find the height of the building. *(Hint: let $d$ be the distance from $B$ to the foot of the building, and write the height twice)*

**28.** GDC From a boat, the angle of elevation of the top of a lighthouse is $12°$. The boat sails $200$ m directly towards the lighthouse, and the angle of elevation is now $20°$.

(a) Find the height of the lighthouse.

(b) Find the distance from the boat to the foot of the lighthouse at the second sighting.

### Bearings

**29.** Answer each question about bearings.

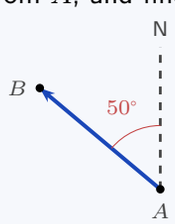
(a) Write the bearing of due west as a three-figure bearing.

(b) A ship travels on a bearing of $120°$. Find its back bearing.

(c) $B$ is on a bearing of $065°$ from $A$. Find the bearing of $A$ from $B$.

(d) Find the back bearing of $200°$, and explain why you subtract here rather than add.

(e) The diagram shows the north line at a point $A$ and the direction from $A$ to a point $B$. Write down the bearing of $B$ from $A$, and find the bearing of $A$ from $B$.




### Journeys with two legs

**30.** GDC A ship sails $15$ km from $P$ on a bearing of $040°$ to $Q$, then $20$ km on a bearing of $110°$ to $R$.

(a) Show that angle $P\hat{Q}R = 110°$.

(b) Find the distance $PR$.

(c) Find the bearing of $R$ from $P$.

(d) Draw a labelled diagram of the journey. Show the north lines at $P$ and $Q$, both bearings, both distances and the angle $P\hat{Q}R$.

**31.** GDC A hiker walks $25$ km from $P$ on a bearing of $080^\circ$ to $Q$, then $18$ km on a bearing of $330^\circ$ to $R$.

(a) Find the angle the route turns through at $Q$, and hence $P\hat{Q}R$.

(b) Find the distance $PR$.

(c) Find the bearing of $R$ from $P$.

(d) Find the bearing of $P$ from $R$.

**Concept check**

**32.** GDC From the top of a $50$ m cliff, the angle of depression to a boat is $20^\circ$. A student places the $20^\circ$ between the cliff face and the line of sight, and writes $d = 50\tan 20^\circ \approx 18.2$ m.

(a) Identify the error.

(b) Find the correct distance from the foot of the cliff to the boat.

# 7.5 Radians, Arc Length, and Sector Area

The degree is a human invention. The Babylonians cut the circle into $360$ parts, and the choice is still used today. Nothing about a circle requires $360$. The circle has a natural unit of its own: take the radius, bend it round the circumference, and measure the angle it makes at the centre. That angle is one **radian**. Because the circumference is $2\pi r$, exactly $2\pi$ radii fit around it, so a full turn is $2\pi$ radians and one radian is about $57.3^\circ$.

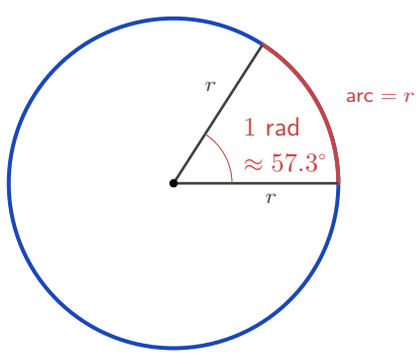



**Figure 7.18** One radian: the angle at the centre where the arc equals the radius. A full turn is $2\pi$ radians.

**KEY 7.7 Degrees and radians**

$2\pi$ radians $= 360^\circ$, so $\pi$ radians $= 180^\circ$. Convert by multiplying: degrees to radians $\times \dfrac{\pi}{180}$; radians to degrees $\times \dfrac{180}{\pi}$.

Learn these common angles in both units, so that you do not need to convert them.

| Degrees | 30° | 45° | 60° | 90° | 180° | 360° |
|---|---|---|---|---|---|---|
| Radians | $\frac{\pi}{6}$ | $\frac{\pi}{4}$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\pi$ | $2\pi$ |

Radians make arc length and sector area simple. Let an arc have length $l$; this $l$ is not the slant height of §7.2. The arc is the same fraction of the circumference as its angle $\theta$ is of a full turn, so $\dfrac{l}{2\pi r} = \dfrac{\theta}{2\pi}$. Multiply both sides by $2\pi r$ to get $l = r\theta$.

**FORMULA 7.8** **Arc length and sector area** FORMULA BOOKLET · 3.4

For a sector of radius $r$ and angle $\theta$ **in radians**,

$$\text{arc length } l = r\theta, \qquad \text{sector area } A = \tfrac{1}{2}r^2\theta.$$

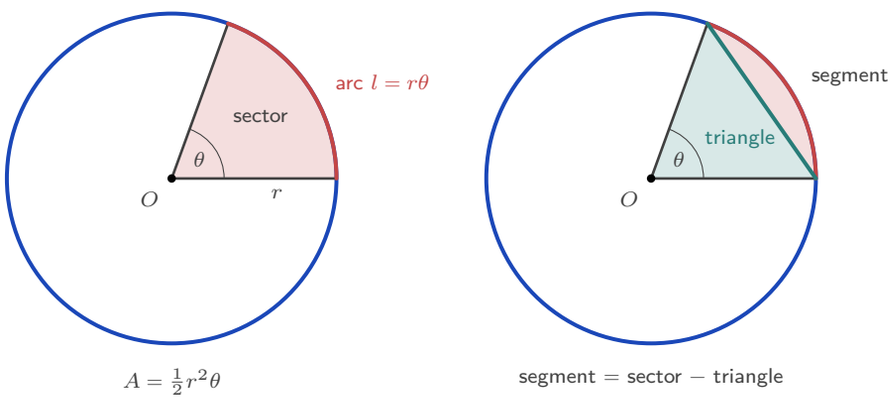



**Figure 7.19** **Left:** a *sector*, between two radii and an arc. **Right:** the chord splits it into a triangle and a *segment*, so segment = sector − triangle.

The area works the same way. The sector is the fraction $\dfrac{\theta}{2\pi}$ of the whole disc $\pi r^2$, so $A = \dfrac{\theta}{2\pi} \cdot \pi r^2 = \tfrac{1}{2}r^2\theta$. A **segment of a circle** is the region between a **chord** (a straight segment joining two points of the circle) and its arc. It is a sector with the triangle removed, and that triangle has two sides $r$ with the angle $\theta$ between them, so §7.3 gives its area as $\tfrac{1}{2}r^2 \sin\theta$:

$$\text{segment} = \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta = \tfrac{1}{2}r^2(\theta - \sin\theta).$$

The picture is for $\theta < \pi$. For a larger angle the segment is the sector *plus* the triangle, and the same formula still gives its area, because $\sin\theta$ is then negative §8.1.

CAUTION The arc, sector and segment formulas in this section need $\theta$ in **radians**. If you substitute an angle in degrees into $l = r\theta$ or $A = \tfrac{1}{2}r^2\theta$, the answer is too large by a factor of $\frac{180}{\pi}$. For the segment formula the error is not a fixed factor. The working looks correct, so the error is easy to miss. Check that your calculator is set to radians before you start.

**EXAMPLE 7.11**

A sector has radius $10$ cm and angle $\frac{2\pi}{5}$ radians. Find its arc length, its area, and the area of the segment cut off by its chord.

**SOLUTION**

The formulas give the arc length and area directly:

$$l = r\theta = 10 \cdot \frac{2\pi}{5} = \boxed{4\pi \approx 12.6 \text{ cm}}, \qquad A = \tfrac{1}{2}r^2\theta = \tfrac{1}{2}(100)\frac{2\pi}{5} = \boxed{20\pi \approx 62.8 \text{ cm}^2}$$

For the segment cut off by the chord, subtract the triangle from the sector:

$$\tfrac{1}{2}r^2(\theta - \sin\theta) = 50\left(\frac{2\pi}{5} - \sin\frac{2\pi}{5}\right) \approx 50(1.2566 - 0.9511) \approx \boxed{15.3 \text{ cm}^2}$$

The segment is far smaller than the sector, because most of the sector is triangle.

A sector's boundary is not the arc alone. Walk round the edge and you travel out along one radius, round the arc, and back along the other radius. The perimeter of a sector is therefore the arc plus two radii, and it has to be assembled from $l = r\theta$ rather than looked up, because the formula booklet does not print it.

**KEY 7.9** **Perimeter of a sector**

For a sector of radius $r$ and angle $\theta$ in radians, with $0 < \theta < 2\pi$:

$$P = 2r + r\theta = r(2 + \theta).$$

**EXAMPLE 7.12**

A sector has radius $7$ cm and angle $1.5$ radians. Find its perimeter.

**SOLUTION**

The arc is

$$l = r\theta = 7(1.5) = 10.5 \text{ cm},$$

and the two radii add $14$ cm, so

$$P = 2(7) + 10.5 = \boxed{24.5 \text{ cm}}$$

When you know the perimeter and the area, you cannot use either formula on its own, because each one contains both unknowns, $r$ and $\theta$. Treat them as simultaneous equations, and eliminate $\theta$. The resulting quadratic usually gives two values of $r$. Find $\theta$ for each, and reject any value with $\theta > 2\pi$, since no sector has an angle of more than a full turn.

**EXAMPLE 7.13**

A sector has perimeter $36$ cm and area $45$ cm$^2$. Find its radius and its angle.

**SOLUTION**

Write down both conditions:

$$2r + r\theta = 36, \qquad \tfrac{1}{2}r^2\theta = 45.$$

The second gives $r^2\theta = 90$, so $\theta = \dfrac{90}{r^2}$. Substituting that into the first,

$$2r + r \cdot \frac{90}{r^2} = 36 \implies 2r + \frac{90}{r} = 36.$$

Multiply through by $r$ and collect:

$$2r^2 - 36r + 90 = 0 \implies r^2 - 18r + 45 = 0 \implies (r-3)(r-15) = 0,$$

so $r = 3$ or $r = 15$. Test each against $\theta = \dfrac{90}{r^2}$. With $r = 3$ the angle is $\theta = 10$ radians, which is more than a full turn of $2\pi \approx 6.28$, so no such sector exists. With $r = 15$ the angle is $\theta = 0.4$ radians.
So

$$\boxed{r = 15 \text{ cm}, \quad \theta = 0.4 \text{ radians}}$$

Check: the arc is $15(0.4) = 6$ cm, giving a perimeter of $30 + 6 = 36$ cm, and the area is $\tfrac{1}{2}(225)(0.4) = 45$ cm$^2$. ✓

**EXAMPLE 7.14**

The opener measured the Earth from a hilltop. An older method uses the arc length formula $l = r\theta$. Eratosthenes measured the Sun's rays at $7.2^\circ$ to the vertical in Alexandria, while in Syene they were vertical. He took the arc between the towns to be $5000$ stadia. Find the Earth's radius and circumference in stadia.

**SOLUTION**

Convert the angle first: $7.2^\circ \times \frac{\pi}{180} = \frac{\pi}{25}$ radians. The Sun is so far away that its rays reach both towns along parallel lines. At Syene the ray points at the Earth's centre. At Alexandria the vertical also points at the centre, and the ray makes $7.2^\circ$ with it. The two verticals meet at the centre, and the angle between them is $7.2^\circ$, because the two angles are alternate angles.

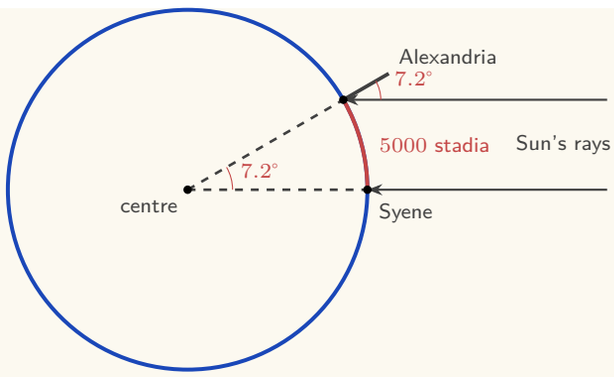



**Figure 7.20** Parallel rays make the angle at the centre equal to the shadow angle (not to scale).

The arc of the sector is the distance between the towns, so use $l = r\theta$ in reverse:

$$5000 = r \cdot \frac{\pi}{25} \implies \boxed{r = \frac{125\,000}{\pi} \approx 39\,800 \text{ stadia}},$$

and the circumference is $\boxed{2\pi r = 250\,000 \text{ stadia}}$. Eratosthenes went more directly: $7.2°$ is $\frac{1}{50}$ of a full turn, so the circumference is $50$ times the arc. At about $158$ m per stadion (a Greek unit of length, its exact size uncertain), this is near $39\,500$ km, close to the true $40\,075$ km.

## YOUR TURN 7.5 Radians, Arc Length, and Sector Area

### Degrees and radians

**33.** EXACT Convert to radians, in terms of $\pi$.

(a) $60°$ (b) $135°$ (c) $210°$ (d) $90°$

**34.** GDC Convert to degrees.

(a) $\dfrac{3\pi}{4}$ (b) $\dfrac{2\pi}{3}$

(c) $1$ radian, to three significant figures

### Arc length and sector area

**35.** Find the arc length and the area of each sector.

(a) radius $8$, angle $1.5$ radians (b) radius $6$, angle $\dfrac{\pi}{3}$ radians

(c) radius $9$, angle $2.4$ radians (d) radius $4$, angle $90°$

**36.** In each case the angle is unknown. Find it from the information given.

(a) A sector has radius $10$ and arc length $25$. Find its angle in radians.

(b) A sector has radius $6$ and area $54$. Find its angle in radians and its arc length.

### Segments and perimeters

**37.** GDC A chord subtends an angle of $2$ radians at the centre of a circle of radius $8$.

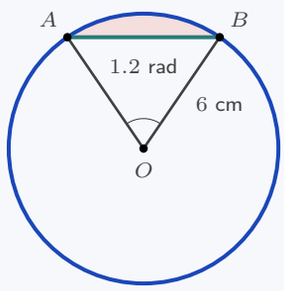
(a) Find the area of the sector.

(b) Find the area of the triangle formed by the chord and the two radii.

(c) Find the area of the minor segment (the smaller part cut off by the chord).

(d) Find the perimeter of the sector.

(e) The diagram shows a second circle, with centre $O$, and a chord $[AB]$. Find the perimeter of the shaded segment.




### Perimeter and area together

**38.** In each case, find the radius and angle of every sector that fits the information.

(a) The perimeter is $40$ cm and the area is $64$ cm$^2$.

(b) The perimeter is $20$ cm and the area is $24$ cm$^2$.

### Concept check

**39.** EXACT A student finds the arc length of a sector of radius $6$ and angle $60°$ as $l = r\theta = 6 \times 60 = 360$. Identify the error and find the correct arc length.

Chapter 7 · Shapes & Measures

# Reflections

TOK The formulas $l = r\theta$ and $A = \frac{1}{2}r^2\theta$ are this simple only when $\theta$ is in radians. Later you will meet another case: the derivative of $\sin x$ is exactly $\cos x$, again only in radians §13.5. The degree was chosen by people. The radian is 'chosen by the circle': it is the angle whose arc equals the radius, so it comes from the circle and not from a human choice. Does a unit that comes from the circle make the mathematics truer, or only simpler?

IM The word "sine" comes from a mistranslation. Over two thousand years ago the Greek astronomer Hipparchus made the first known table of chords. In India, Aryabhata replaced the chord with the half-chord and called it the *jyā*, which is our sine. The word was taken into Arabic as *jiba*, was later read as *jaib*, meaning "bay" or "fold", and was translated into Latin as *sinus*, which means the same. Islamic astronomers, al-Bīrūnī among them, extended the sine and cosine rules, partly to find the direction of Mecca from any city. Trigonometry was built by many people, in many languages, over many centuries.

RW Fixing an unknown position from measured angles is called **triangulation**. Surveyors once mapped whole countries by measuring chains of connected triangles. Astronomers use *parallax*: as the Earth moves round the Sun, a near star seems to shift against the far ones, and that tiny angle gives its distance. Your phone does the same job with distances instead of angles. A GPS receiver times the signal from each satellite, turns each time into a distance, and uses at least four satellites to find its position and correct its own clock. This is **trilateration**, the sphere equation of §7.1, written once for each satellite and solved simultaneously.

EXT The ambiguous case is really a statement about **congruence** (being identical in shape and size). Give a triangle by two angles and a side (ASA or AAS), by three sides (SSS), or by two sides and the angle between them (SAS), and only one triangle fits. Those are the congruence rules. Two sides and an angle that is *not* between them (SSA) is not one of them, so it can leave the triangle undecided. The sine rule's two answers are the two triangles that SSA allows. One SSA case is a congruence rule: RHS (right angle, hypotenuse, side). When the given angle is a right angle, the side opposite it is the longest, so only one triangle fits.

TOK A dog and a mathematician both see a triangle. Only one calls it isosceles. Do shapes have properties, or do we give them?

# End-of-Chapter Problems
# Chapter 7: Shapes & Measures

**1.** A cuboid has dimensions $6 \times 8 \times 10$.

(a) Find the length of the space diagonal. [2]

(b) Find the angle between the space diagonal and the $6 \times 8$ face. [3]

**2.** EXACT The points $A(2, -1, 3)$ and $B(4, 4, 7)$ are given, and $O$ is the origin.

(a) Find $AB$. [2]

(b) Find the midpoint of $AB$. [1]

(c) Show that $OB = 9$. [1]

**3.** EXACT A cone has radius $5$ and height $12$.

(a) Find the slant height. [1]

(b) Find the volume. [2]

(c) Find the total surface area. [2]

**4.** EXACT A solid consists of a cylinder of radius $3$ and height $10$ with a hemisphere of radius $3$ on top. Find its total surface area. [5]

**5.** EXACT A cone has volume $100\pi$ and radius $5$.

(a) Find the height. [2]

(b) Find the slant height. [2]

(c) Find the curved surface area. [1]

**6.** A square-based pyramid has base side $10$ and height $12$.

(a) Find the volume. [2]

(b) Find the slant height of a triangular face. [2]

(c) Find the total surface area. [2]

**7.** EXACT A container is a cone of base radius $6$ cm and height $15$ cm, held with its point downwards.

(a) Find the volume of the cone when full, in terms of $\pi$. [2]

(b) Water is poured in to a depth of $10$ cm. Using similar triangles, find the radius of the circular water surface. [3]

(c) Find the volume of water, in terms of $\pi$. [3]

(d) Find the fraction of the cone's volume that is filled, and explain why it is not $\frac{2}{3}$ even though the water reaches $\frac{2}{3}$ of the height. [4]

(e) Find the depth of water when the cone is exactly half full, giving your answer to three significant figures. [3]

**8.** GDC In triangle $ABC$, $a = 8$, $b = 11$, $c = 15$.

(a) Find the largest angle. [3]

(b) Find the area of the triangle. [2]

**9.** GDC In triangle $PQR$, $p = 10$, $q = 7$ and $R = 95°$.

(a) Find $r$. [2]

(b) Find the area of the triangle. [2]

**10.** GDC A triangle has two sides of length $9$ and $12$, and the angle between them is $40°$.

(a) Find the length of the third side. [2]

(b) Find the area of the triangle. [2]

**11.** GDC In triangle $ABC$, $A = 50°$, $a = 6$ and $b = 7$.

(a) Find the two possible values of angle $B$. [3]

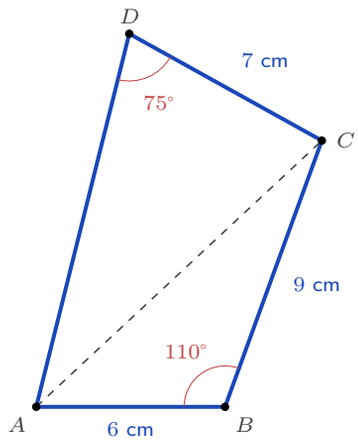
(b) Find the two possible areas of the triangle. [3]

**12.** EXACT A regular hexagon is inscribed in a circle of radius $8$. Find its area. [4]

**13.** GDC The area of a triangle is $24$, with $a = 8$ and $b = 10$. Find the two possible values of the angle $C$ between these sides. [3]

**14.** GDC In triangle $ABC$, $AB = 12$, $BC = 9$ and $B\hat{A}C = 35°$. Find the two possible values of the angle $B\hat{C}A$. [4]

**15.** GDC The diagram shows the quadrilateral $ABCD$ and its diagonal $[AC]$.




(a) Find the length $AC$. [3]

(b) Find the angle $C\hat{A}D$. [3]

(c) Find the area of the quadrilateral $ABCD$. [4]

**16.** GDC A boat sails $30$ km from $P$ on a bearing of $100°$ to $Q$, then $20$ km on a bearing of $040°$ to $R$.

(a) Show that $P\hat{Q}R = 120°$. [3]

(b) Find the distance $PR$. [3]

(c) Find the bearing of $R$ from $P$. [4]

**17.** GDC From the top of a vertical cliff $80$ m high, the angles of depression of two boats are $32°$ and $18°$. The boats and the foot of the cliff lie on one straight line, with both boats on the same side of the cliff. Find the distance between the two boats. [4]

**18.** GDC Two ships leave a port at the same time. One sails on a bearing of $040°$ at $12$ km/h, the other on a bearing of $100°$ at $16$ km/h. Find the distance between them after $2$ hours. [4]

**19.** GDC A yacht leaves harbour $H$ and sails $25$ km on a bearing of $048°$ to a buoy $A$. It then sails $18$ km on a bearing of $130°$ to a second buoy $B$.

(a) Show that $H\hat{A}B = 98°$. [3]

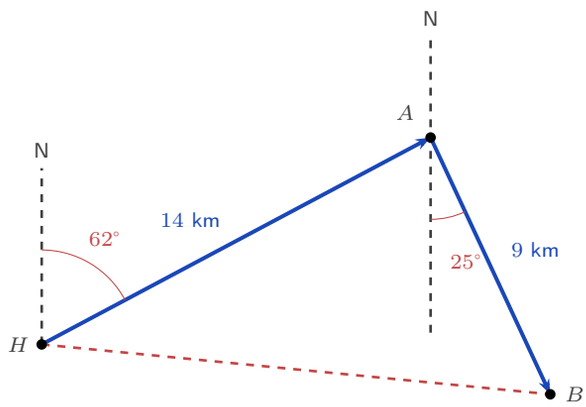
(b) Find the distance $HB$. [4]

(c) Find the bearing of $B$ from $H$. [4]

(d) Find the area of triangle $HAB$. [3]

(e) The yacht returns directly from $B$ to $H$. Find the total distance it has sailed. [2]

**20.** GDC A boat sails in a straight line from a harbour $H$ to a buoy $A$, then in a straight line from $A$ to a second buoy $B$. The diagram shows the two legs of the journey, with the north lines at $H$ and at $A$. At $A$ the north line is extended to the south.




(a) Write down the bearing of $A$ from $H$ and the bearing of $B$ from $A$. [2]
(b) Show that $H\hat{A}B = 87°$. [2]
(c) Find the distance $HB$. [3]
(d) Find the bearing of $B$ from $H$. [4]

**21.**

(a) Convert $\frac{5\pi}{6}$ radians to degrees. [1]
(b) Convert $\frac{7\pi}{4}$ radians to degrees. [1]
(c) A sector has radius $9$ cm and angle $\frac{5\pi}{6}$. Find its arc length, giving your answer exactly. [2]

**22.** A pendulum of length $0.8$ m swings through an angle of $0.5$ radians. Find the length of the arc traced by the bob. [2]

**23.** A sector has radius $12$ cm and angle $1.2$ radians.

(a) Find the arc length. [1]
(b) Find the area of the sector. [2]
(c) Find the perimeter of the sector. [2]

**24.** A sector has area $30$ cm$^2$ and radius $6$ cm.

(a) Find the angle of the sector in radians. [2]
(b) Find the arc length. [2]

**25.** GDC A chord subtends an angle of $1.5$ radians at the centre of a circle of radius $10$ cm.

(a) Find the area of the sector. [2]
(b) Find the area of the triangle formed by the chord and the two radii. [2]

(c) Find the area of the minor segment. [2]

**26.** A windscreen wiper arm $OA$ of length $50$ cm turns about $O$ through an angle of $1.8$ radians. The rubber blade covers the part of the arm between $20$ cm and $50$ cm from $O$.

(a) Find the length of the arc traced by the tip $A$. [2]
(b) Find the area of glass wiped by the blade. [4]
(c) Find the perimeter of the wiped region. [3]

**27.** GDC Consider the points $A(2, 0, 0)$, $B(0, 4, 0)$ and $C(0, 0, 4)$.

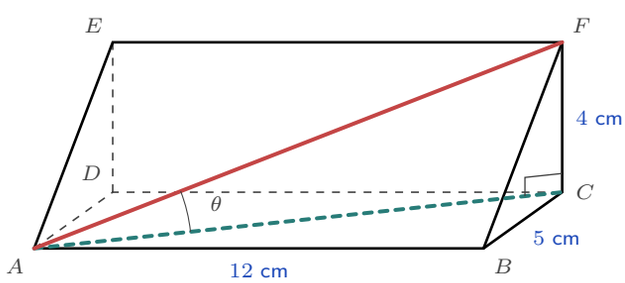
(a) Find the lengths $AB$, $AC$ and $BC$. [3]
(b) Find the angle $B\hat{A}C$. [3]

**28.** GDC A sector of a circle of radius $12$ and angle $1.5$ radians is rolled into a cone, so that the radius becomes the slant height and the arc becomes the base circumference. Find the radius and height of the cone. [6]

**29.** GDC A rectangular box $ABCDEFGH$ has a horizontal base $ABCD$ with $AB = 8$ cm and $BC = 6$ cm, and vertical edges of length $5$ cm, so that $G$ is the vertex above $C$.

(a) Find the length of the base diagonal $AC$. [2]
(b) Find the exact length of the space diagonal $AG$. [3]
(c) Find the angle that $AG$ makes with the base $ABCD$. [3]
(d) Find the angle $G\hat{A}B$ between the diagonal $AG$ and the edge $AB$. [4]
(e) Find the total surface area of the box. [3]

**30.** GDC A solid wedge $ABCDEF$ has a horizontal rectangular base $ABCD$. The edges $[DE]$ and $[CF]$ are vertical, so $CDEF$ is a vertical rectangle. The diagram shows the wedge and the angle $\theta$ between $[AF]$ and $[AC]$.




(a) Find the length $AC$. [2]
(b) Find the length $AF$. [2]
(c) Find the angle $\theta$ between $[AF]$ and the base $ABCD$. [3]
(d) Find the angle $B\hat{A}F$. [3]
(e) Find the volume of the wedge. [2]

**31.** GDC Two observers stand $200$ m apart on level ground, with a balloon in the vertical plane between them. From one the angle of elevation to the balloon is $40°$; from the other it is $55°$. Find the height of the balloon. [6]

**32.** A sector of a circle has perimeter $26$ cm and area $30$ cm$^2$. Find the radius $r$ and the angle $\theta$ of the sector. [6]

**33.** GDC A chord of a circle of radius $10$ has length $12$. Find the area of the minor segment it cuts off. [6]

**34.** GDC The area of a triangle is $84$, and two of its sides have lengths $13$ and $14$. Find both possible lengths of the third side. [6]

**35.** GDC In triangle $ABC$, $AC = 11$ cm, $BC = 8$ cm and $B\hat{A}C = 42°$.

(a) Find the possible values of $A\hat{B}C$. [3]
(b) Hence find the possible lengths of $AB$. [3]
(c) A second triangle has $AC = 11$ cm, $BC = 13$ cm and $B\hat{A}C = 42°$. Find $A\hat{B}C$, and justify that it has only one value. [3]

**36.** GDC A ship sails $6$ km from port $P$ on a bearing of $030°$ to a point $Q$, then $10$ km on a bearing of $175°$ to a point $R$.

(a) Show that $P\hat{Q}R = 35°$. [2]
(b) Find the distance $PR$. [2]
(c) Find the bearing of $R$ from $P$. [5]

**37.** GDC A circle has centre $O$ and radius $9$ cm. The chord $[AB]$ subtends an angle $A\hat{O}B = 140°$ at $O$.

(a) Find the exact length of the minor arc $AB$. [2]
(b) Find the area of the major segment cut off by $[AB]$. [4]

# Challenge Questions

**C1. How reliable was al-Bīrūnī's measurement?** The opener described his method. From a hill of height $h$ he measured the angle of dip $\theta$ to the horizon, and the radius of the Earth follows from one right-angled triangle. This investigation asks how much trust the answer deserves.

(a) The line of sight touches the Earth, so it meets the radius there at a right angle. Show that $\cos\theta = \dfrac{R}{R+h}$, and hence that $R = \dfrac{h\cos\theta}{1-\cos\theta}$. [3]

(b) Al-Bīrūnī recorded $h = 652$ cubits and $\theta = 34'$. Find $R$ in cubits. *(Hint: $34' = \frac{34}{60}$ of a degree)* [3]

(c) Repeat the calculation for $\theta = 33'$ and for $\theta = 35'$. Find the percentage change in $R$ in each case. [4]

(d) For small $\theta$ the approximation $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ holds, with $\theta$ in radians. Use it to show that $R \approx \dfrac{2h}{\theta^2}$. [3]

(e) Hence explain why an error of about $3\%$ in $\theta$ produces an error of about $6\%$ in $R$. Comment on what this means for how confidently his result can be quoted. [4]

(f) Suppose his instrument could read any angle to within about $\delta$ radians, however large the angle. Use part (d) to show that the percentage error in $R$ is then roughly proportional to $\dfrac{1}{\sqrt{h}}$, and state what this suggests about the choice of hill. Suggest one further effect that an extended investigation of the method should include. [3]

**C2.** GDC **A line's angles with three perpendicular faces.** A line leaves the corner $O$ of a cuboid. It makes some angle with each of the three faces that meet at $O$. This investigation finds the rule that links the three angles.

The cuboid has edges of lengths $p$, $q$ and $r$ along the $x$-, $y$- and $z$-axes, with $O$ at the origin, and $G(p, q, r)$ is the opposite corner. Let $\alpha$, $\beta$ and $\gamma$ be the angles that the diagonal $OG$ makes with the faces $x = 0$, $y = 0$ and $z = 0$ respectively.

(a) Take $p = 2$, $q = 3$ and $r = 6$. Find the length $OG$, and find $\alpha$, $\beta$ and $\gamma$ in degrees, correct to one decimal place. [4]

(b) Find $\sin^2\alpha + \sin^2\beta + \sin^2\gamma$ for the cuboid in part (a) and for the cuboid with $p = 1$, $q = 2$, $r = 2$. State a conjecture. [2]

(c) Prove your conjecture for every cuboid. *(Hint: the perpendicular from $G$ to the face $x = 0$ has length $p$)* [3]

(d) Explain why any line through $O$ that enters the region $x > 0$, $y > 0$, $z > 0$ is the diagonal of some cuboid with a corner at $O$. State what the rule becomes for a line that lies in the face $z = 0$, and what it then says about $\alpha + \beta$. [3]

(e) Show that no line can make an angle of $30^\circ$ with each of the three faces. Find the exact value of $\sin\alpha$ when the three angles are equal, and name the cuboid whose diagonal does this. [3]

(f) Find $\alpha+\beta+\gamma$ for the three lines met in parts (a), (b) and (e), and for a line in the face $z=0$. Conjecture the least and greatest possible values of $\alpha+\beta+\gamma$, and suggest how this investigation could be extended. [3]

**C3.** GDC **How many triangles fit the data?** In triangle $ABC$ you are given the side $b$, the angle $A$, and the side $a$ opposite $A$. This is the SSA case. This investigation finds exactly how many triangles fit, for every possible value of $a$, first from a diagram and then from a quadratic equation.

In parts (a) to (e), take $A$ to be acute, and let $h$ be the distance from $C$ to the line $AB$.

(a) Show that $h = b\sin A$. [2]

(b) Take $b=10$ and $A=40^\circ$. Find the number of triangles when $a=5$, when $a=8$, and when $a=12$. [3]

(c) Explain why no triangle exists when $a < b\sin A$. *(Hint: what is the shortest distance from $C$ to the line $AB$?)* [2]

(d) Explain why exactly one triangle exists when $a = b\sin A$, and state what is special about it. [2]

(e) Show that two triangles exist exactly when $b\sin A < a < b$, and that only one exists when $a > b$. Explain what happens at $a=b$. [4]

(f) Now let the side $c=AB$ be the unknown. Use the cosine rule to show that

$$c^2 - (2b\cos A)\,c + (b^2 - a^2) = 0,$$

and use this equation to find both values of $c$ when $b=10$, $A=40^\circ$ and $a=8$. [4]

(g) Each positive root $c$ of this equation gives one triangle. Use its discriminant, and the sum and product of its roots, to prove the counts of parts (c) to (e) again, including the case $a=b$. Then use the same method to find how many triangles exist when $A$ is obtuse, for $a \le b$ and for $a > b$. [5]

(h) Take $b=10$ and $A=40^\circ$ again. Find both values of $c$ when $a=6.44$ and when $a=6.5$. Comment on what this means for a surveyor whose measurement of $a$ carries a small error when $a$ is close to $b\sin A$, and suggest how this could be investigated further. [3]

**C4.** GDC **The best cone from a sector.** A sector of radius $R$ and angle $\theta$ radians is cut from a sheet of paper. The two straight edges are joined to make a cone. This investigation finds the angle that gives the largest cone.

Write $t = \dfrac{\theta}{2\pi}$, so that $0 < t < 1$.

(a) The arc of the sector becomes the circumference of the base. Show that the base radius is $r = Rt$. [2]

(b) Show that the height of the cone is $h = R\sqrt{1-t^2}$. [3]

(c) Hence show that the volume is $V = \frac{1}{3}\pi R^3 t^2\sqrt{1-t^2}$. [2]

(d) Taking $R = 1$, use technology to find the value of $t$ that makes $V$ largest, and hence find $\theta$ to the nearest degree. [3]

(e) Let $u = t^2$. Show that

$$\frac{4}{27} - u^2(1-u) = \left(u - \frac{2}{3}\right)^2\left(u + \frac{1}{3}\right),$$

and hence verify that $V$ is largest when $t^2 = \frac{2}{3}$. Find the exact values of $\frac{r}{R}$ and $\frac{h}{r}$ at this maximum. *(Hint: $V$ is largest where $V^2$ is largest, and $V^2$ is a multiple of $u^2(1-u)$)* [5]

(f) Now the whole disc of radius $R$ is cut into two sectors, a fraction $t$ and a fraction $1-t$ of the disc, and each is made into a cone. Write down the total volume of the two cones as a function of $t$, and use technology to find the value of $t$ that makes it largest. Compare the result with part (d), and suggest how this could be investigated further. [4]