FUNCTIONS

Chapter 6

Polynomials

6

SYLLABUS 1.11 2.12 HL only

Many quantities in nature and in engineering change smoothly, without jumps or sharp corners. A thrown ball rises and falls along an unbroken arc. The cable of a suspension bridge, carrying the even weight of the road, hangs in a parabola. A railway track does not turn suddenly: it eases into a bend along a curve that engineers often design as a cubic, so that passengers are not pushed sideways all at once. Curves like these are described by **polynomials**.

A polynomial uses one variable and a few constants, combined by addition and multiplication only: no division by the variable, no square root, no trigonometry. So it is defined for every real number, and its graph is a smooth, unbroken curve with no gaps, no asymptotes and no corners. Distance at a constant speed is linear, and the volume of a sphere grows with the cube of its radius. Over a short enough interval almost any smooth curve can be matched closely by a polynomial, which is why measurements are often fitted with one. See Ch. 16.

By the end of this chapter you will be able to sketch a polynomial from its factorised form and find its equation from a graph. You will divide one polynomial by another, and use the remainder and factor theorems to test a value and to factorise. You will solve polynomial equations and inequalities. You will also find the sum and the product of the roots from the coefficients alone, a result that still holds when the roots are complex (Ch. 17).

One idea connects most of the chapter: a polynomial's coefficients and its zeros describe the same object, and most techniques here move between the two. The chapter closes with **partial fractions**, which splits a fraction by factorising its denominator and giving each factor a numerator of its own. It makes many rational functions straightforward to integrate later. See Ch. 15.

© 2026 Hasan Khan Free for personal and classroom use www.bettermath.org

6.1 Polynomial Functions and their Graphs

The end behaviour, the zeros and the turning points of a polynomial graph can be described before a single point is plotted, because they can be read from the degree and the coefficients. Reading them from the equation is faster than building a table of values, and they are what a sketch has to show.

DEFINITION 6.1 **Polynomial function**

A **polynomial function** of **degree** n has the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0, \qquad a_n \neq 0,$$

where the coefficients are constants and n is a non-negative integer.

6.1.1 Degree, Coefficients and End Behaviour

Polynomials are named by their degree. Degree 1 is **linear**, degree 2 is quadratic, degree 3 is **cubic**, degree 4 is **quartic**, and degree 5 is **quintic**. The degree also limits the shape: how many times the curve can meet the x-axis, and how often it can change direction.

The constants $a_n, a_{n-1}, \ldots, a_1, a_0$ are the coefficients of the powers of x. The coefficient of the highest power is a_n, called the **leading coefficient**, and the term $a_n x^n$ that contains it is the **leading term**.

For a factorised polynomial, neither the degree nor the leading coefficient needs the brackets multiplied out. The degree is the sum of the degrees of the factors, which for linear factors is simply how many there are. The leading coefficient is the product of the leading coefficients of the factors.

The leading term is the term with the highest power, which need not be the term written first. Take each factor's leading coefficient with its sign: the leading coefficient of $(1 - 2x)$ is -2, not 1 or 2.

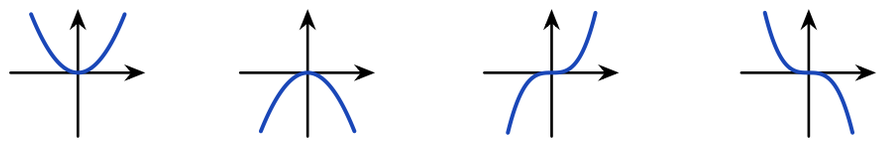
A coefficient may take any value, with one restriction: the leading coefficient cannot be zero. If a_n were zero, the term $a_n x^n$ would be zero, and what remained would have degree at most $n - 1$. A non-zero constant is a polynomial of degree 0; the zero polynomial is given no degree.

The sign of the leading coefficient determines which way the ends of the curve point. It does not determine the sign of the function elsewhere: $y = x^2 - 4$ has a positive leading coefficient, yet y is negative for $-2 < x < 2$.

The **end behaviour**, meaning the direction of the curve as $x \to \pm\infty$, is determined entirely by the leading term. If n is odd, the two ends point in opposite directions. If n is even, they

point the same way, and the sign of a_n determines whether that direction is upward or downward.

So the ends of any polynomial of even degree behave like those of the parabola $y = a_n x^2$, and the ends of any polynomial of odd degree behave like those of the cubic $y = a_n x^3$. Sketch the ends first, then fill in the middle from the zeros.



n even, $a_n > 0$	n even, $a_n < 0$	n odd, $a_n > 0$	n odd, $a_n < 0$
both ends up	both ends down	down left, up right	up left, down right

Figure 6.1 The four possible end behaviours: even degree like a parabola, odd degree like a cubic.

To see why the leading term determines the end behaviour, take it out as a factor:

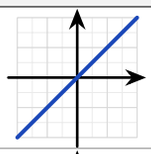
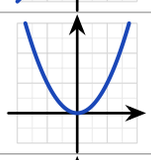
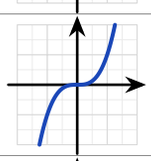
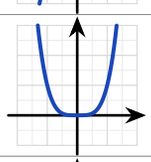
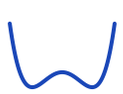
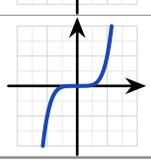
$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n}\right).$$

For large $|x|$ every fraction in the bracket is close to zero. The bracket is then close to 1, so the polynomial has the same sign as $a_n x^n$ and, for $n \geq 1$, its size increases without bound as $|x|$ increases.

The zeros of f are the values of x for which $f(x) = 0$, and they are the roots of the equation $f(x) = 0$ (§4.1.3); this chapter uses both words. The real zeros are the x-coordinates of the x-intercepts of the graph, and the number of distinct real zeros can never exceed the degree. A cubic meets the axis at most three times, a quartic at most four. The reason rests on the factor theorem of §6.3.2: each distinct zero gives its own linear factor, and the degrees of the factors cannot add to more than n.

A turning point (§4.1.3) is a place where the curve stops rising and begins to fall, or stops falling and begins to rise. A polynomial of degree $n \geq 1$ has at most $n - 1$ turning points; a non-zero constant has none. The reason uses the derivative, a polynomial of degree $n - 1$ that is zero at every turning point. See §14.1. The possible numbers then decrease in steps of two: $n - 1$, $n - 3$, and so on. The number changes by two rather than by one because the degree fixes the direction of the two ends. Removing a maximum without also removing a minimum would reverse one end, so the number of turning points is always odd when n is even, and even when n is odd.

The table sets out the end behaviour, the zeros and the turning points for the degrees you meet most often.

n	$y = x^n$	Shape when $a_n > 0$	Shape when $a_n < 0$	Times it meets the x-axis	Turning points
1				1	0
2				0, 1 or 2	1
3				1, 2 or 3	0 or 2
4				0, 1, 2, 3 or 4	1 or 3
5				1, 2, 3, 4 or 5	0, 2 or 4

EXAMPLE 6.1

State the degree, the leading coefficient and the two end behaviours of
(a) $p(x) = 5 - 4x + 6x^3 - x^2$ and (b) $q(x) = (1 - 2x)(x + 3)^2(x - 4)$.

SOLUTION

Work from the leading term in each case.
(a) The terms are not written in order of power, so find the highest power rather than reading the first term. It is $6x^3$, so the degree is 3 and the leading coefficient is 6. The degree is odd, so the two ends point in opposite directions. The leading coefficient is positive, so the right-hand end points upward:

$$p(x) \to +\infty \text{ as } x \to +\infty, \qquad p(x) \to -\infty \text{ as } x \to -\infty$$

(b) Nothing needs to be multiplied out. Count the linear factors: $(1 - 2x)$, two copies of $(x + 3)$, and $(x - 4)$, so the degree is 4. The leading coefficient is the product of the four leading coefficients,

$$(-2) \times 1 \times 1 \times 1 = -2.$$

The degree is even, so both ends point the same way. The leading coefficient is negative, so both ends point downward: $q(x) \to -\infty$ as $x \to \pm\infty$

The answer to (b) depends on the sign of the -2 from $(1-2x)$. A leading coefficient of $+2$ would reverse both ends.

6.1.2 Zeros and Multiplicity

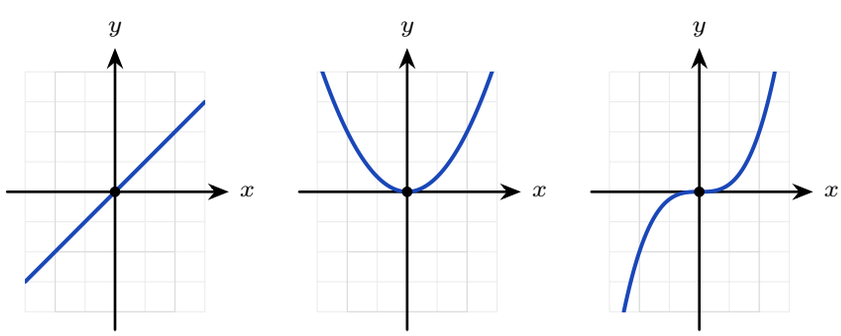
The table gives the shape in outline. More detail comes from what the curve does at each zero, and that depends on whether a factor is repeated.

The **multiplicity** of a zero is the number of times its factor appears in the factorised polynomial. In $(x-1)(x-2)^2$ the zero $x=1$ has multiplicity one, and the zero $x=2$ has multiplicity two. A zero of multiplicity one is called a **simple zero**, and one of multiplicity two a **double zero**. The equation $f(x)=0$ then has a simple root or a **double root** there.

KEY 6.1 **Zeros and multiplicity**

At a zero of **multiplicity one** the curve crosses the axis at a slant (not flat). At a zero of **even** multiplicity it touches the axis and turns back. At a zero of **odd** multiplicity greater than one it crosses the axis and is flat where it crosses.

The factorised form shows why. Near a zero r of multiplicity m, write $f(x)=(x-r)^m\,g(x)$, where $g(x)$ is the product of the other factors and $g(r)\neq 0$. Close to r the value $g(x)$ stays close to $g(r)$, so it keeps one sign, and the sign of f is determined by $(x-r)^m$. An odd power changes sign as x passes r, so the curve crosses. An even power is never negative, so the curve stays on one side of the axis and touches it. When $m\geq 2$, the factor $(x-r)^m$ is much smaller in size than $x-r$ close to the zero, which is why the curve is flat there.



(a) $y=x$ crosses (b) $y=x^2$ touches (c) $y=x^3$ crosses, flat

Figure 6.2 The three behaviours at a zero $x=0$: multiplicity one crosses, two touches and turns back, three is flat where it crosses.

KEY 6.2 **Sketching a factorised polynomial**

1. At each x-intercept, read the behaviour from the multiplicity of the zero.
2. Find the y-intercept by substituting $x=0$.
3. Find the two ends from the leading term.

EXAMPLE 6.2

Sketch $y = (x+1)(x-2)^2$, marking the points where it meets the axes.

SOLUTION

Read the zeros and their multiplicities from the factors. The factor $(x+1)$ is single, so the graph crosses the axis at $x = -1$. The factor $(x-2)^2$ is squared, so the graph touches the axis at $x = 2$ and turns back. Find the y-intercept. Substitute $x = 0$: $y = (1)(-2)^2 = 4$, the point $(0, 4)$. Fix the ends from the leading term. Multiplying out leads with x^3: an odd degree with a positive coefficient, so the curve runs from $-\infty$ at the left to $+\infty$ at the right. These three facts are enough for the sketch.

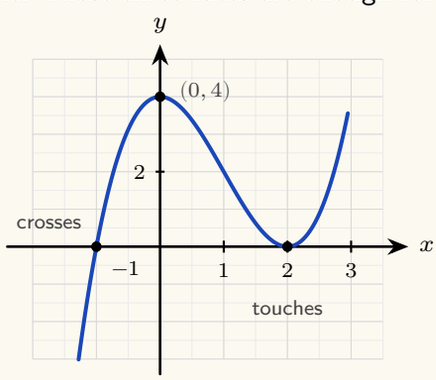


Figure 6.3 $y = (x+1)(x-2)^2$ crosses the axis at the simple zero $x = -1$, touches it at the double zero $x = 2$, and meets the y-axis at $(0, 4)$.

When the polynomial is written out in full rather than factorised, the y-intercept is even quicker to read. Every term except the constant contains x, so substituting $x = 0$ makes all of them zero. The constant term a_0 is what remains.

KEY 6.3 **The y-intercept**

The graph of $y = a_n x^n + \cdots + a_1 x + a_0$ crosses the y-axis at $(0, a_0)$.

6.1.3 From the Graph Back to the Equation

The reverse problem is also examined. Given a sketch, you are asked for the polynomial that produced it. Most of the information you need comes from the intercepts.

Each x-intercept gives a factor, and the behaviour of the curve there gives its multiplicity. A crossing at a slant gives a single factor. A touch gives a factor to an even power, usually squared. A flat crossing gives a factor to an odd power of at least three, usually cubed. When the degree and the graph together determine every factor and its multiplicity, the polynomial is a constant times their product, and any known point that is not an x-intercept finds the constant. Check that the factors account for the whole degree of the curve. If they do not, an intercept has been missed, a multiplicity is higher than it looks, or the polynomial has a factor with no real zeros, such as $x^2 + 1$.

KEY 6.4 **Finding a polynomial from its graph**

Write one factor for each x-intercept, with the multiplicity the graph shows at that point. Then substitute the coordinates of any known point that is not an x-intercept to find the leading coefficient a_n.

Use this only when the degree and the graph together fix every factor and its multiplicity. A factor with no real zeros, such as $x^2 + 1$, does not show on the axis.

EXAMPLE 6.3

Find the equation of the cubic in the graph below, which touches the x-axis at $x = -1$, crosses it at $x = 2$, and passes through $(0, -1)$.

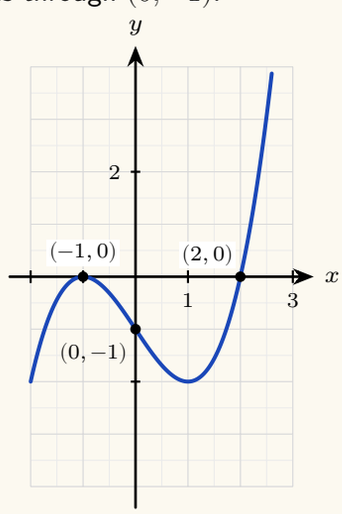


Figure 6.4 The curve touches the x-axis at $x = -1$, crosses it at $x = 2$, and passes through $(0, -1)$. These three facts determine the equation.

SOLUTION

Use the three features the graph shows. Write the factors from the x-intercepts. The curve touches at $x = -1$, so that zero is repeated and contributes $(x + 1)^2$. It crosses at $x = 2$, so that zero is single and contributes $(x - 2)$. Those factors already account for degree three, so

$$y = a(x + 1)^2(x - 2)$$

for some constant a.

Find a from the remaining point. Substitute $x = 0$ and $y = -1$:

$$-1 = a(1)^2(-2) = -2a,$$

so $a = \frac{1}{2}$ and

$$\boxed{y = \tfrac{1}{2}(x + 1)^2(x - 2)}$$

Check the answer against the graph. The leading coefficient $\frac{1}{2}$ is positive, so the curve rises to the right, and the sketch agrees.

YOUR TURN 6.1 Polynomial Functions and their Graphs

Degree, leading coefficient and end behaviour

1. State the degree and the leading coefficient of each polynomial.

(a) $f(x) = 3x^4 - 2x + 7$
(b) $f(x) = 5 - x^3$
(c) $f(x) = (x-1)(x+2)(x-3)$
(d) $f(x) = -2x^5 + x^2$
(e) $f(x) = (2x-1)(x+3)^2$
(f) $f(x) = (3-x)^2(2x+1)^3$

2. Describe the end behaviour of each graph as $x \to +\infty$ and as $x \to -\infty$.

(a) $y = x^3$
(b) $y = -x^4$
(c) $y = 2x^5 - x$
(d) $y = 3 - x^2$

3. Answer each question about degree and shape.

(a) State the maximum number of real zeros of a degree-5 polynomial.
(b) State the maximum number of turning points of a quartic.
(c) List every possible number of turning points a cubic can have.
(d) Explain why a cubic must have at least one real zero.

Zeros and multiplicity

4. State the zeros of each polynomial, and say at each one whether the graph crosses or touches the x-axis.

(a) $y = (x-2)(x+3)(x-5)$
(b) $y = (x-1)^2(x+4)$
(c) $y = x(x-3)^2$
(d) $y = (x+1)^2(x-2)^2$
(e) $y = (x-2)^3(x+1)$
(f) $y = x^3(x-4)^2$
(g) Sketch $y = (x-2)(x+1)^2$, showing the intercepts and whether the graph crosses or touches the x-axis at each zero.
(h) Sketch $y = -(x+2)(x-1)^2(x-3)$, showing the intercepts and whether the graph crosses or touches the x-axis at each zero.

5. Consider $y = (x+1)(x-3)^2$.

(a) State the zeros and their multiplicities.
(b) Find the y-intercept.
(c) State the end behaviour.
(d) Sketch the curve.

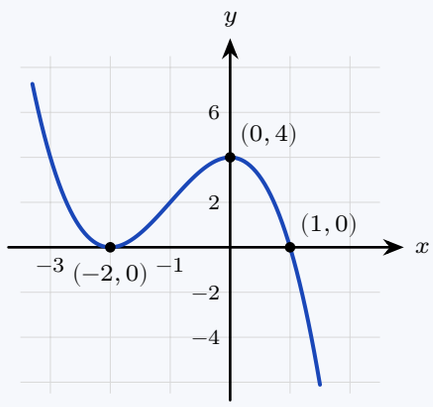
From the graph to the equation

6. Find the equation of each polynomial in factorised form.

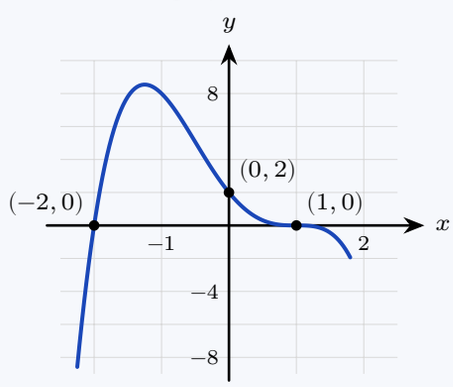
(a) A cubic crosses the x-axis at $x = -2$, $x = 1$ and $x = 3$, and passes through $(2, -8)$.

(b) A quartic touches the x-axis at $x = 1$, crosses it at $x = -1$ and $x = 3$, and passes through $(0, 6)$.

(c) The cubic shown in the graph.



(d) The quartic (degree 4) shown in the graph.



Concept check

7. A student says: "*the graph of* $y = (x+1)^3(x-4)$ *touches the* x*-axis at* $x = -1$*, because the factor* $(x+1)$ *is repeated.*" Explain why this is false.

6.2 Polynomial Division

You can already add, subtract and multiply polynomials. Addition and subtraction collect like terms, and multiplication expands every product before collecting. Division is the operation that still needs a method. Factorising depends on it: once one factor of a polynomial is known, the other is found by dividing, and §6.3 and §6.4 are built on that step.

6.2.1 The Division Algorithm

The answer to a polynomial division has the same form as in whole-number division: a quotient and a remainder. When you divide 47 by 5 you write

$$47 = 5 \times 9 + 2,$$

which records a divisor, a quotient, and a remainder smaller than the divisor. Polynomial division works the same way, with "smaller" measured by degree instead of size.

KEY 6.5 **The division algorithm**

For polynomials $f(x)$ and $d(x)$, with $d(x)$ not the zero polynomial, there are unique polynomials $q(x)$ and $r(x)$ such that

$$f(x) = d(x)\,q(x) + r(x),$$

where r is zero or has degree less than the degree of d.

Dividend equals divisor times quotient, plus remainder.

Fix the four names now, because the methods that follow use them. The polynomial being divided, f, is the **dividend**. The polynomial you divide by, d, is the **divisor**. The result q is the **quotient**, and the part that remains, r, is the **remainder**. When $r = 0$ the division is exact, and d is a factor of f.

Long division, in §6.2.2, produces a quotient and a remainder. They are also the only possible pair. If $d\,q_1 + r_1$ and $d\,q_2 + r_2$ both equal f, then $d\,(q_1 - q_2) = r_2 - r_1$. The right side has degree less than the degree of d, and the left side does not unless $q_1 = q_2$. Then $r_1 = r_2$ as well.

6.2.2 Long Division

Long division works for a divisor of any degree. It is three steps repeated until the remainder has lower degree than the divisor.

KEY 6.6 **Polynomial long division**

1. **Divide** the leading term of the dividend by the leading term of the divisor. The result is the next term of the quotient.
2. **Multiply** the whole divisor by that term.
3. **Subtract** the product from the dividend, bring down the next term, and repeat on what is left.

To **bring down** a term is to copy the next unused term of the dividend beside the result of the subtraction, so that what is left always holds every term still to be divided. Set the work out in columns, one for each power of x. The divisor goes to the left of the bracket and the dividend inside it. The quotient is written on the top line and the remainder on the bottom

line. Write the result as $f(x) = d(x)\,q(x) + r(x)$ when you finish, because multiplying that right-hand side out checks the whole division.

EXAMPLE 6.4

Divide $2x^3 - 3x^2 + x - 5$ by $(x - 2)$.

SOLUTION

Set the division out in columns:

$$\begin{array}{rl} & \quad 2x^2 \;+x\; +3 \quad \text{quotient} \\ \hline x-2\;) & \; 2x^3 -3x^2 \;+x\; -5 \quad \text{dividend} \\ & \; 2x^3 -4x^2 \\ \hline & \quad\; x^2 \;+x \\ & \quad\; x^2 -2x \\ \hline & \qquad\; 3x \;-5 \\ & \qquad\; 3x \;-6 \\ \hline & \qquad\qquad\; 1 \quad \text{remainder} \end{array}$$

Stage 1. Divide the leading term of the dividend by the leading term of the divisor: $2x^3 \div x = 2x^2$. Write $2x^2$ in the quotient. Multiply the divisor by it: $2x^2(x - 2) = 2x^3 - 4x^2$. Subtract that from the dividend, which leaves $x^2 + x$.
Stage 2. Repeat on $x^2 + x$. Divide: $x^2 \div x = x$. Multiply: $x(x - 2) = x^2 - 2x$. Subtract, which leaves $3x - 5$.
Stage 3. Repeat on $3x - 5$. Divide: $3x \div x = 3$. Multiply: $3(x - 2) = 3x - 6$. Subtract, which leaves 1.
The degree of 1 is lower than the degree of $x - 2$, so the division stops.
The quotient is $2x^2 + x + 3$ and the remainder is 1:

$$2x^3 - 3x^2 + x - 5 = (x - 2)(2x^2 + x + 3) + 1$$

CAUTION Write the dividend in descending powers and insert a zero term for every missing power before dividing. Divide $x^3 - 13x + 12$ as $x^3 + 0x^2 - 13x + 12$. Without that zero term the columns shift, and the subtractions that follow go wrong.

EXT A divisor of higher degree changes none of the three steps. It changes only where the division stops. The remainder must have degree lower than the divisor, so dividing by a quadratic can leave a remainder with an x in it.

EXAMPLE 6.5

Divide $x^4 + 2x^2 - 3x + 1$ by $(x^2 + x - 1)$.

SOLUTION

There is no x^3 term, so write the dividend as $x^4 + 0x^3 + 2x^2 - 3x + 1$ and give that power a column of its own.

$$
\begin{array}{r|l}
 & x^2 \quad -x \quad +4 \\
\hline
x^2 + x - 1 \;) & x^4 \;+0x^3 \;+2x^2 \;-3x \;+1 \\
 & \underline{x^4 \;+x^3 \;-x^2} \\
 & \quad -x^3 \;+3x^2 \;-3x \\
 & \quad \underline{-x^3 \;-x^2 \;+x} \\
 & \qquad\quad 4x^2 \;-4x \;+1 \\
 & \qquad\quad \underline{4x^2 \;+4x \;-4} \\
 & \qquad\quad\quad -8x \;+5
\end{array}
$$

Stage 1. Divide the leading terms: $x^4 \div x^2 = x^2$. Multiply the divisor by it: $x^2(x^2 + x - 1) = x^4 + x^3 - x^2$. Subtract, which leaves $-x^3 + 3x^2 - 3x$.
Stage 2. Repeat. Divide: $-x^3 \div x^2 = -x$. Multiply: $-x(x^2 + x - 1) = -x^3 - x^2 + x$. Subtract, which leaves $4x^2 - 4x + 1$.
Stage 3. Repeat. Divide: $4x^2 \div x^2 = 4$. Multiply: $4(x^2 + x - 1) = 4x^2 + 4x - 4$. Subtract, which leaves $-8x + 5$.
The remainder $-8x + 5$ has degree one, the divisor has degree two, so the division stops:

$$x^4 + 2x^2 - 3x + 1 = (x^2 + x - 1)(x^2 - x + 4) - 8x + 5$$

YOUR TURN 6.2 Polynomial Division

Dividing by a linear factor

8. Divide, and state the quotient and the remainder.

(a) $x^2 + 5x + 6$ by $(x + 2)$
(b) $x^3 - 3x^2 + 2x$ by $(x - 1)$
(c) $x^3 + 2x^2 - x + 1$ by $(x - 1)$
(d) $x^3 - 8$ by $(x - 2)$

9. Divide, and state the quotient and the remainder. Insert a zero term for each missing power first.

(a) $2x^3 - x^2 - 3$ by $(x + 1)$
(b) $x^3 + 3x^2 - 4$ by $(x + 2)$
(c) $2x^4 - 3x^3 + x - 6$ by $(x - 2)$

Dividing by a quadratic

10. Divide by the quadratic divisor, and state the quotient and the remainder.

(a) $x^4 - 1$ by $(x^2 - 1)$ (b) $x^4 - 1$ by $(x^2 + 1)$

(c) $x^3 + 2x^2 - 3x + 1$ by $(x^2 + x - 1)$ (d) $2x^4 - x^3 + 3x - 2$ by $(x^2 - 2x + 2)$

The division algorithm

11. Use $f(x) = d(x)\,q(x) + r(x)$ in each part.

(a) When $f(x)$ is divided by $x^2 - 1$, the quotient is $2x + 3$ and the remainder is $x - 4$. Find $f(x)$.

(b) Divide $x^4 + x^3 - 2x + 5$ by $x^2 + 2$, and write the result in the form $f(x) = d(x)\,q(x) + r(x)$.

(c) When $x^3 + ax^2 + bx + 3$ is divided by $x^2 + 1$, the remainder is $2x + 1$. Find a and b.

Concept check

12. A student divides $x^3 + 2x^2 + x - 1$ by $x^2 + 1$ and writes: "*quotient* x*, remainder* $2x^2 - 1$." Explain why this cannot be the final answer, and find the correct quotient and remainder.

6.3 The Remainder and Factor Theorems

6.3.1 The Remainder Theorem

Evaluating f at 2 and *dividing* f by $(x - 2)$ appear to be unrelated. Evaluating gives the height of the graph above one point, and dividing is the long calculation of the previous section. The remainder theorem states that the two produce the same number.

Take the division algorithm with a linear divisor $(x - a)$:

$$f(x) = (x - a)\,q(x) + r,$$

where the remainder r must be a *constant*, because it is zero or has degree less than the degree of $x - a$, and less than one means zero.

Now substitute $x = a$ into that identity. The factor $(x - a)$ becomes zero, so the whole first term is zero, leaving $f(a) = r$. **The height of the graph at a *is* the remainder**, because the rest of $f(x)$, the term $(x - a)\,q(x)$, is zero at $x = a$.

KEY 6.7 **Remainder theorem**

When a polynomial $f(x)$ is divided by $(x-a)$, the remainder is $f(a)$.

One substitution replaces the division.

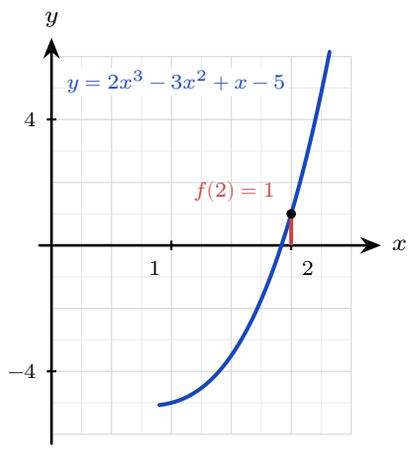


Figure 6.5 Dividing $2x^3 - 3x^2 + x - 5$ by $(x-2)$ leaves remainder 1 (§6.2.2), and 1 is the height of the curve at $x = 2$.

The theorem holds for any linear divisor, not only for one written as $(x-a)$. Divide by $(bx-c)$ and the remainder is f evaluated at the **zero of the divisor**, meaning the value of x that makes the divisor equal to zero. For $(2x-1)$ that value is $\frac{1}{2}$, never 1 or 2. The proof is the same substitution: in $f(x) = (bx-c)\,q(x) + r$, substituting $x = \frac{c}{b}$ makes the first term zero and leaves $f(\frac{c}{b}) = r$.

EXAMPLE 6.6

Find the remainder when $f(x) = 8x^3 + 4x^2 - 6x + 5$ is divided by $(2x-1)$.

SOLUTION

Evaluate at the zero of the divisor. Here $2x - 1 = 0$ when $x = \frac{1}{2}$, so the remainder is

$$f\left(\tfrac{1}{2}\right) = 8 \cdot \tfrac{1}{8} + 4 \cdot \tfrac{1}{4} - 3 + 5 = 1 + 1 - 3 + 5 = \boxed{4}$$

Read the theorem in either direction. If you know the coefficients, it gives the remainder. If you know two remainders, it gives two linear equations in the unknown coefficients.

EXAMPLE 6.7

The polynomial $f(x) = x^3 + ax + b$ leaves remainder 7 when divided by $(x-2)$ and remainder -5 when divided by $(x+1)$. Find a and b.

SOLUTION

Each remainder condition is one evaluation, so two conditions give two equations:

$$f(2) = 8 + 2a + b = 7 \implies 2a + b = -1,$$

$$f(-1) = -1 - a + b = -5 \implies -a + b = -4.$$

Subtracting, $3a = 3$, so

$$\boxed{a = 1, \quad b = -3}$$

With the remainder theorem, two divisions became a pair of linear equations. Questions of this kind, which work from the remainders back to the coefficients, are common.

EXT The theorem is stated for a linear divisor, but the same idea works for other divisors. Divide by a quadratic and the remainder has degree at most one, so it is $rx + s$: two unknowns instead of one. The division algorithm still records the whole calculation as an identity, and each zero of the divisor makes the divisor term equal to zero, leaving one equation in r and s. Two distinct zeros give two equations, which is enough to find two unknowns. The method needs the two zeros to be distinct. With a repeated factor such as $(x-1)^2$, both zeros are the same, so only one substitution is available. Find the remainder by long division instead (§6.2.2).

EXAMPLE 6.8

Find the remainder when $f(x) = x^3 + 2x^2 - 5x + 3$ is divided by $(x-1)(x-2)$.

SOLUTION

The divisor has degree two, so the remainder has degree at most one. Write it as $rx + s$ and record the division as an identity:

$$f(x) = (x-1)(x-2)\,q(x) + rx + s.$$

This holds for every value of x, so choose the two values that make the product term disappear. Substituting $x = 1$,

$$f(1) = 1 + 2 - 5 + 3 = 1 \implies r + s = 1,$$

and substituting $x = 2$,

$$f(2) = 8 + 8 - 10 + 3 = 9 \implies 2r + s = 9.$$

Subtracting the first equation from the second gives $r = 8$, and then $s = -7$. The remainder is

$$\boxed{8x - 7}$$

Long division confirms it: the quotient is $x+5$, and $(x-1)(x-2)(x+5)+8x-7$ returns $f(x)$.

6.3.2 The Factor Theorem

When the remainder is zero, $(x-a)$ divides $f(x)$ exactly: it is a factor. That is what turns a test for a zero into a method for factorising. The reverse direction is shorter still: if $f(x)=(x-a)\,q(x)$, then substituting $x=a$ gives $f(a)=0$.

KEY 6.8 **Factor theorem**

$(x-a)$ is a factor of $f(x)$ if and only if $f(a)=0$.

The proofs of both theorems use only the division algorithm and one substitution. So both theorems still hold when a is a complex number, which Ch. 17 uses to factorise polynomials with complex zeros.

To factorise an unfamiliar cubic, search for a zero by testing small values. The factors of the constant term are the natural candidates, and the reason is short. If k is an integer zero of $a_nx^n+\cdots+a_1x+a_0$ with integer coefficients, then substituting and moving a_0 across gives

$$k\left(a_nk^{n-1}+a_{n-1}k^{n-2}+\cdots+a_1\right)=-a_0,$$

so an integer zero k other than 0 divides a_0. When the leading coefficient is not 1, the candidates are the fractions $\frac{p}{q}$ with p a factor of a_0 and q a factor of a_n. The reason is the same argument applied to $\frac{p}{q}$ in lowest terms, after multiplying through by q^n. It is not required in this course and is not written out here. Once one zero k is found, $(x-k)$ is a factor, and dividing it out leaves a polynomial one degree lower; for a cubic, that is a quadratic you can finish by hand. For a zero $\frac{p}{q}$ the factor is $(qx-p)$.

That division need not be long division. To **compare coefficients**, write the quotient with unknown coefficients. Its degree is the degree of the dividend minus the degree of the divisor, so it is one lower for a linear divisor and two lower for a quadratic one. Multiply out and equate the coefficients of matching powers. Each unknown comes from one comparison, and any coefficient left over is a check.

EXAMPLE 6.9

Show that $(x-1)$ is a factor of $f(x)=x^3-7x^2+14x-8$, and hence factorise $f(x)$ completely.

SOLUTION

Test $x=1$:

$$f(1)=1-7+14-8=0,$$

so by the factor theorem $(x-1)$ is a factor.

Now divide it out. Instead of long division, *compare coefficients*. The quotient must be a quadratic, so write

$$x^3 - 7x^2 + 14x - 8 = (x-1)(x^2 + bx + c),$$

then compare the coefficients of matching powers on the two sides. Compare the constant terms: $-c = -8$, so $c = 8$. Compare the coefficients of x^2: $b - 1 = -7$, so $b = -6$. The coefficient of x has not been used, so use it as a check: on the right it is $c - b = 8 + 6 = 14$, which matches the left. ✓ Hence

$$f(x) = (x-1)(x^2 - 6x + 8) = \boxed{(x-1)(x-2)(x-4)}$$

Had the factor not been given, the constant term -8 would give the candidates $\pm 1, \pm 2, \pm 4, \pm 8$, and $x = 1$ is the first to test.

TIP Comparing coefficients is often quicker than long division for a linear divisor. Long division is equally acceptable if you prefer it.

YOUR TURN 6.3 The Remainder and Factor Theorems

The remainder theorem

13. Find the remainder in each case.

(a) $x^3 + 2x - 1$ divided by $(x-1)$

(b) $x^3 - 4x^2 + x + 6$ divided by $(x-2)$

(c) $2x^3 - 3x + 1$ divided by $(x+1)$

(d) $x^3 + x^2 + x + 1$ divided by $(x+2)$

14. Find the remainder in each case. The divisor is not of the form $(x-a)$.

(a) $4x^3 - 4x^2 + 5x - 1$ divided by $(2x-1)$

(b) $2x^3 + x^2 - 2$ divided by $(2x+1)$

15. Use the remainder and factor theorems to answer each part.

(a) The remainder when $f(x)$ is divided by $(x-3)$ is 0. State what this tells you about f.

(b) The remainder when $f(x)$ is divided by $(x-2)$ is 7. State $f(2)$.

(c) $f(-1) = 0$. Write down a factor of $f(x)$.

(d) $x^4 - 1$ is divided by $(x-1)$. Find the remainder without dividing.

16. The polynomial $f(x) = x^3 + ax^2 + bx + 6$ has $(x+1)$ as a factor, and leaves remainder 12 when divided by $(x-2)$.

(a) Write down two equations in a and b.

(b) Solve them to find a and b.

The remainder on division by a quadratic

17. The remainder on division by a quadratic $(x-a)(x-b)$ has the form $rx+s$. Find it in each case.

(a) x^3-2x^2+4x-1 divided by $(x-1)(x-3)$
(b) x^4+x-3 divided by $(x+1)(x-2)$
(c) x^3+2x divided by $(x-1)^2$, by long division

The factor theorem

18. Decide whether the given linear expression is a factor. Show your working.

(a) Is $(x-2)$ a factor of x^3-5x^2+8x-4?
(b) Is $(x+3)$ a factor of x^3+x^2-5x+3?
(c) Is $(x-1)$ a factor of x^3+x^2+x+1?
(d) Is $(x+2)$ a factor of x^3+8?

19. Factorise each polynomial completely.

(a) x^3-x
(b) x^3-4x
(c) $x^3-3x^2-4x+12$
(d) $2x^3+3x^2-11x-6$

20. In each part, $(x-3)$ is a factor. Find the quadratic factor by comparing coefficients, then factorise completely.

(a) x^3-2x^2-5x+6
(b) $2x^3-5x^2-4x+3$

21. Use the factor theorem in each part.

(a) One factor of x^2-5x+6 is $(x-2)$. State the other.
(b) Show that x^3-3x^2+4 has a repeated factor.

Concept check

22. Each student below has made an error in using the remainder theorem. Identify the error and find the correct remainder.

(a) To find the remainder when $f(x)=x^3-2x+5$ is divided by $(x+3)$, a student calculates $f(3)=26$.
(b) To find the remainder when $g(x)=4x^3-2x+3$ is divided by $(2x-1)$, a student calculates $g(1)=5$.

6.4 Solving Polynomial Equations

For degrees three and four, general formulas exist but are far too long to use by hand, and from degree five upward there is no general formula that uses only arithmetic and roots. When the equation has a rational root, the factor theorem is used instead.

KEY 6.9 **Solving a polynomial equation**

To solve a polynomial equation of degree 3 or more:

1. For integer coefficients, test the factors of the constant term, or the fractions $\frac{p}{q}$ of §6.3.2 if the leading coefficient is not 1, until one value a gives $f(a) = 0$.
2. Divide $f(x)$ by $(x - a)$, or by $(qx - p)$ when $a = \frac{p}{q}$, either by long division or by comparing coefficients.
3. Repeat on the quotient until a quadratic remains, then solve that in the usual way.
4. If no candidate is a root, find the real roots with a GDC. These are decimal approximations, not exact values.

Every real root is an x-intercept of the graph, so a sketch or a GDC graph shows how many real roots to look for.

EXAMPLE 6.10

Solve $x^3 - 3x^2 - x + 3 = 0$.

SOLUTION

Test the factors of 3. At $x = 1$: $1 - 3 - 1 + 3 = 0$, so $(x - 1)$ is a factor. Dividing,

$$x^3 - 3x^2 - x + 3 = (x - 1)(x^2 - 2x - 3) = (x - 1)(x - 3)(x + 1).$$

The roots are $x = -1,\ 1$ or 3

When the leading coefficient is not 1, a rational root need not be an integer.

EXAMPLE 6.11

Solve $2x^3 - x^2 - 6x + 3 = 0$.

SOLUTION

Let $f(x) = 2x^3 - x^2 - 6x + 3$ and test the integer candidates first, the factors of 3:

$$f(1) = -2, \qquad f(-1) = 6, \qquad f(3) = 30, \qquad f(-3) = -42.$$

None is zero. The leading coefficient is 2, so the remaining candidates are $\pm\frac{1}{2}$ and $\pm\frac{3}{2}$. Test $x = \frac{1}{2}$:

$$f\left(\tfrac{1}{2}\right) = \tfrac{1}{4} - \tfrac{1}{4} - 3 + 3 = 0.$$

So $\frac{1}{2}$ is a zero of f. Use the factor $(2x - 1)$ in place of $\left(x - \frac{1}{2}\right)$, so that the coefficients stay integers. Compare coefficients in

$$2x^3 - x^2 - 6x + 3 = (2x - 1)(x^2 + bx + c).$$

The constant terms give $-c = 3$, so $c = -3$. The coefficients of x^2 give $2b - 1 = -1$, so $b = 0$. Check with the coefficient of x: $2c - b = -6$, which matches. Hence

$$(2x - 1)(x^2 - 3) = 0 \implies \boxed{x = \tfrac{1}{2} \text{ or } x = \pm\sqrt{3}}$$

A cubic with real coefficients always has at least one real root. One end goes to $+\infty$ and the other to $-\infty$, so the curve must cross the axis. Ch. 17 gives a second reason, from conjugate pairs. If no integer or fraction candidate is a root, a cubic with integer coefficients has no rational roots, so each of its real roots is irrational. Step 4 of the card then applies.

6.4.1 Polynomial Inequalities

A polynomial changes sign only at its zeros, so the zeros cut the x-axis into intervals on which the sign stays the same. To solve a polynomial inequality, collect every term on one side, factorise, and build a sign table with one row for each factor (the method of §5.4.5).

EXAMPLE 6.12

Solve $x^3 - 3x^2 - x + 3 > 0$.

SOLUTION

Use the factorisation found earlier in this section:

$$(x + 1)(x - 1)(x - 3) > 0.$$

The zeros -1, 1 and 3 cut the axis into four intervals.

Interval	$x < -1$	$-1 < x < 1$	$1 < x < 3$	$x > 3$
$x + 1$	$-$	$+$	$+$	$+$
$x - 1$	$-$	$-$	$+$	$+$
$x - 3$	$-$	$-$	$-$	$+$
product	$-$	$+$	$-$	$+$

The product is positive on the second and fourth intervals, so

$$\boxed{-1 < x < 1 \quad \text{or} \quad x > 3}$$

The inequality is strict, so the zeros themselves are left out. For $x^3 - 3x^2 - x + 3 \leq 0$ the same table gives $x \leq -1$ or $1 \leq x \leq 3$, with the zeros included. A sketch agrees: the cubic rises from the lower left, so it is above the axis between its first two zeros and to the right of the third.

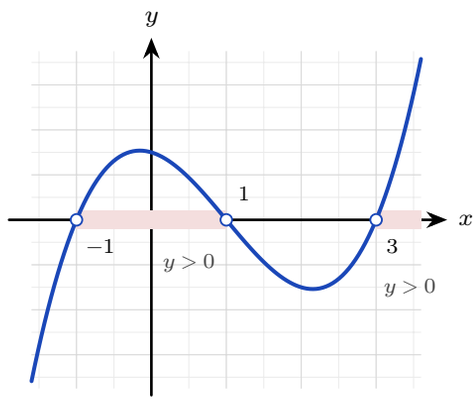


Figure 6.6 $y = (x+1)(x-1)(x-3)$ lies above the axis on $-1 < x < 1$ and on $x > 3$ (gold), the intervals the sign table gives. The open circles mark the zeros, which the strict inequality leaves out.

A squared factor needs care. $(x-2)^2$ is never negative, so its row has $+$ on both sides of 2 and the sign does not change there. With $\leq$ or $\geq$, the value $x = 2$ **is still a solution**, because the polynomial is zero there, even when the intervals on both sides of it are not. The figure of $y = (x+1)(x-2)^2$ in §6.1.2 shows this. The curve touches the axis at $x = 2$ without changing sign, so $(x+1)(x-2)^2 \leq 0$ gives $x \leq -1$ or $x = 2$.

YOUR TURN 6.4 Solving Polynomial Equations

Equations and inequalities that factorise at sight

23. Solve each equation or inequality. Each is already factorised or factorises at sight.

(a) $(x-1)(x+2)(x-4) = 0$

(b) $(x-3)^2(x+1) = 0$

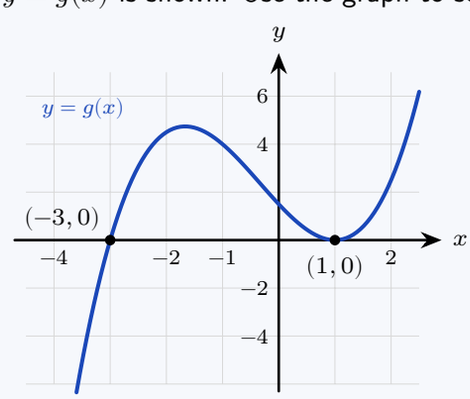
(c) $x^3 = 4x$

(d) $x^3 + x^2 - 2x = 0$

(e) $(x+2)(x-1)(x-4) \geq 0$

(f) $(x-1)^2(x+3) < 0$

(g) The graph of a cubic $y = g(x)$ is shown. Use the graph to solve $g(x) \leq 0$.



Using the factor theorem

24. Solve each equation, using the factor theorem to find the first root.

(a) $x^3 + 2x^2 - 5x - 6 = 0$

(b) $x^3 - 3x^2 - 10x + 24 = 0$

(c) $2x^3 - 3x^2 - 3x + 2 = 0$

25. Solve $x^3 - 1 = 0$, giving only the real root, and explain why there is exactly one.

Rational roots

26. The leading coefficient is not 1, so test the candidates $\frac{p}{q}$. Find one root, then solve the equation.

(a) $2x^3 - x^2 + 4x - 2 = 0$
(b) $12x^3 - 16x^2 - 5x + 3 = 0$

Quartic equations

27. Solve each equation.

(a) $x^4 - 5x^2 + 4 = 0$ *(Hint: substitute $u = x^2$)*
(b) $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$ *(Hint: use the factor theorem twice)*

When no candidate is a root

28. GDC For each equation, show that none of the candidate integer roots is a root. Then use a GDC to find every real root, correct to three significant figures.

(a) $x^3 - 4x + 1 = 0$
(b) $x^3 + 2x^2 - 2 = 0$

Concept check

29. To solve $x^3 + 6x^2 + 11x + 6 = 0$, a student tests $x = 1, 2, 3$ and 6, finds that none is a root, and concludes: "*there are no integer roots, so a GDC is needed.*" Identify the error, and solve the equation.

6.5 Sum and Product of Roots

The coefficients of a polynomial determine the sum and the product of its roots. You can therefore find those two values *without solving the equation at all*. This matters whenever the roots are irrational, or when the equation is too hard to solve, because the sum and the product remain easy to obtain.

6.5.1 The Quadratic Case

§4.6.2 showed that if $ax^2 + bx + c = 0$ has roots α and β, then comparing $ax^2 + bx + c$ with $a(x - \alpha)(x - \beta)$ gives

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

The minus sign on the sum comes from the brackets, one $-\alpha$ and one $-\beta$.

6.5.2 The General Case

The same comparison works for every degree, using the counting argument of the binomial theorem (§2.9). Expanding

$$a_n(x-r_1)(x-r_2)\cdots(x-r_n)$$

means choosing, from each bracket, either the x or the root, exactly as in the binomial theorem. To build the x^{n-1} term, take x from all brackets but one. Each root then appears exactly once, with a single minus sign. So the coefficient of x^{n-1} is $-a_n(r_1+r_2+\cdots+r_n)$, which is $-a_n$ times the *sum*. To build the constant term, take the root from *every* bracket. This multiplies all n roots together with n minus signs, so the constant term is $(-1)^n a_n$ times the *product*. Dividing each by the leading coefficient a_n gives the sum and the product of the roots. The argument assumes the polynomial has n roots $r_1,\dots,r_n$, which is true once complex roots are included and each root is counted with its multiplicity. See Ch. 17.

FORMULA 6.10 **Sum and product of roots, any degree** FORMULA BOOKLET · 2.12

For the equation $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0=0$,

$$\text{sum of roots}=-\frac{a_{n-1}}{a_n},\qquad \text{product of roots}=(-1)^n\frac{a_0}{a_n}.$$

Divide by the leading coefficient. The sum is the next coefficient with its sign changed; the product is the constant term, with sign $(-1)^n$. The roots are all n of them, complex ones included, each counted as often as its multiplicity.

These two are the first and last of a family of relations called **Vieta's formulas**. The syllabus requires only the sum and the product. They are used to check answers, to build equations with given roots, and to find an unknown coefficient.

A third relation gives the sum of the roots multiplied together in pairs. Building the x^{n-2} term means taking the root from exactly two brackets and x from the rest, so that coefficient is a_n times this pairwise sum. Dividing by a_n gives that pairwise sum. For a cubic $a_3x^3+a_2x^2+a_1x+a_0=0$ with roots α,β,γ, the three relations together are

$$\alpha+\beta+\gamma=-\frac{a_2}{a_3},\qquad \alpha\beta+\beta\gamma+\gamma\alpha=\frac{a_1}{a_3},\qquad \alpha\beta\gamma=-\frac{a_0}{a_3}.$$

The middle relation is a short extension, and it is needed for expressions such as $\alpha^2+\beta^2+\gamma^2$, which involve the roots in pairs. Squaring the sum gives the rearrangement:

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha).$$

For $x^3-2x^2-9x+18=0$ below, this is $2^2-2(-9)=22$.

EXAMPLE 6.13

The cubic equation $x^3 - 2x^2 - 9x + 18 = 0$ has roots α, β, γ. Find $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

SOLUTION

Here $a_3 = 1$, $a_2 = -2$, $a_0 = 18$ and the degree is $n = 3$, so

$$\alpha + \beta + \gamma = -\frac{-2}{1} = \boxed{2}, \qquad \alpha\beta\gamma = (-1)^3 \frac{18}{1} = \boxed{-18}$$

(The roots are -3, 2 and 3, with sum 2 and product -18, which checks both answers.)

6.5.3 Symmetric Combinations of the Roots

Many other expressions in the roots can be rebuilt from the sum and the product, so these two values are enough for a wide range of questions.

A **symmetric combination** of the roots is an expression whose value does not change when the roots are exchanged. Both $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$ are symmetric; $\alpha - \beta$ is not. Every symmetric combination of two roots can be written in terms of $\alpha + \beta$ and $\alpha\beta$. The proof of that general statement is beyond this course. You met $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ in §4.6.2. This rearrangement and one for the sum of reciprocals cover many of the expressions you will meet. A sum of reciprocals is simplified by putting it over a common denominator, $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}$.

KEY 6.11 **Symmetric expressions in the roots**

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta, \qquad (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta, \qquad \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta}.$$

None is a new formula. The first two are the expansion of $(\alpha \pm \beta)^2$ rearranged; the third is a common denominator.

EXAMPLE 6.14

The equation $x^2 - 3x + 1 = 0$ has roots α and β. Without solving the equation, find $\alpha^2 + \beta^2$ and $\frac{1}{\alpha} + \frac{1}{\beta}$.

SOLUTION

Work from the sum and the product. The roots are irrational, but their sum and product are not: $\alpha + \beta = 3$ and $\alpha\beta = 1$. For the sum of squares, use the first rearrangement above:

$$(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \implies \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 2 = \boxed{7}$$

For the sum of reciprocals, combine over a common denominator:

$$\frac{1}{\alpha}+\frac{1}{\beta}=\frac{\alpha+\beta}{\alpha\beta}=\frac{3}{1}=\boxed{3}$$

Reading the relations backwards builds an equation from its roots. One quadratic whose roots have sum S and product P is

$$x^2-Sx+P=0,$$

so an equation whose roots are a transformation of α and β is found by transforming the sum and the product. The roots themselves are never needed.

EXAMPLE 6.15

Let α and β be the roots of $x^2-3x+1=0$, so that $\alpha+\beta=3$ and $\alpha\beta=1$. Find a quadratic equation with integer coefficients whose roots are 2α and 2β.

SOLUTION

The new sum and product follow from the old ones:

$$2\alpha+2\beta=2(\alpha+\beta)=6, \qquad (2\alpha)(2\beta)=4\alpha\beta=4.$$

So the required equation is

$$\boxed{x^2-6x+4=0}$$

EXAMPLE 6.16

The roots of $x^2+kx+12=0$ differ by 1. Find the possible values of k.

SOLUTION

Let the roots be α and β with $\alpha-\beta=1$. The coefficients give $\alpha+\beta=-k$ and $\alpha\beta=12$. Connect the difference to these totals by squaring it:

$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta \implies 1=k^2-48,$$

so $k^2=49$ and $\boxed{k=\pm 7}$

Check with $k=7$: the equation $x^2+7x+12=0$ has roots -3 and -4, which indeed differ by 1. A condition on the roots has become an equation in the coefficient.

YOUR TURN 6.5 Sum and Product of Roots

Sum and product from the coefficients

30. State the sum and the product of the roots of each equation.

(a) $x^2 - 7x + 10 = 0$ (b) $2x^2 + 3x - 5 = 0$

(c) $x^2 + 4x + 4 = 0$ (d) $3x^2 - 2x - 8 = 0$

31. For each cubic or quartic equation, state the sum and the product of the roots.

(a) $x^3 - 6x^2 + 11x - 6 = 0$ (b) $2x^3 - 4x^2 + x - 3 = 0$

(c) $x^4 - 3x^3 + 2x - 5 = 0$

Symmetric expressions in the roots

32. The roots of $2x^2 - 7x + 6 = 0$ are α and β. Without solving the equation, find each value.

(a) $\alpha + \beta$ and $\alpha\beta$ (b) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$

(c) $\alpha^2 + \beta^2$ (d) $(\alpha - \beta)^2$

33. The roots of $x^3 - 4x^2 + x + 6 = 0$ are α, β and γ.

(a) Write down $\alpha + \beta + \gamma$ and $\alpha\beta\gamma$.

(b) Write down $\alpha\beta + \beta\gamma + \gamma\alpha$.

(c) Hence find $\alpha^2 + \beta^2 + \gamma^2$.

Building an equation from its roots

34. Build the required equation.

(a) Write a quadratic equation with leading coefficient 1 whose roots sum to 5 and multiply to 6.

(b) Write a cubic equation with leading coefficient 1 whose roots are -1, 2 and 4.

35. The roots of $x^2 + 5x + 3 = 0$ are α and β.

(a) Write down $\alpha + \beta$ and $\alpha\beta$.

(b) Find the sum and product of $\alpha + 2$ and $\beta + 2$.

(c) Hence find a quadratic equation with integer coefficients whose roots are $\alpha + 2$ and $\beta + 2$.

Conditions on the roots

36. In each part, a condition on the roots fixes an unknown coefficient.

(a) The roots of $x^2 + bx + 12 = 0$ are 3 and one other. Find the other root and b.

(b) The roots of $x^2 - 7x + k = 0$ differ by 3. Find k.

(c) One root of $x^2 + kx + 18 = 0$ is twice the other. Find the possible values of k.

Concept check

37. A student writes: "*the product of the roots of* $2x^3 - 5x^2 + x + 6 = 0$ *is* $\frac{6}{2} = 3$*, as for a quadratic.*" Identify the error and state the correct product.

6.6 Partial Fractions

Adding two simple fractions produces one more complicated fraction. Partial fractions reverse that process: they split a single fraction back into a sum of simpler ones, one for each factor of the denominator. The split form is easier to use, and it makes a rational function easier to integrate. See Ch. 15.

The syllabus limits the problem to a denominator of **at most two distinct linear factors**, with the numerator of lower degree than the denominator (a **proper fraction**). That is the case treated below. The larger theory is sketched at the end of the section for reference.

- A fraction whose numerator has degree equal to or higher than its denominator's is **improper**. **Divide first, then split only the proper part**: $\frac{x^2}{x^2-1} = 1 + \frac{1}{x^2-1}$, and only the second term is split. Try it in Your Turn Q42. You will need the same first step when integrating in Ch. 15.

Each linear factor contributes one term with a constant numerator.

KEY 6.12 **Partial fractions**

For a proper fraction over two distinct linear factors,

$$\frac{px+q}{(x-\alpha)(x-\beta)} = \frac{A}{x-\alpha} + \frac{B}{x-\beta}.$$

Constants A and B of this kind always exist. Adding the two fractions on the right gives the numerator $A(x-\beta) + B(x-\alpha) = (A+B)x - (A\beta + B\alpha)$. Matching it to $px+q$ gives two linear equations, $A + B = p$ and $A\beta + B\alpha = -q$. They have exactly one solution because $\alpha \neq \beta$.

The same form holds when a factor is written as $(ax+b)$, for example $\frac{A}{2x-1}$; only the value of x that makes that factor zero changes.

To find A and B, factorise the denominator if it is not already factorised, multiply through by it, then substitute the values of x that make one factor zero. Each substitution isolates one unknown. The original equation is not defined at those values. After multiplying through, both sides are polynomials, and they agree for every other value of x, which is

infinitely many values. Two polynomials that agree at infinitely many values are identical, so they also agree at the values that make a factor zero.

EXAMPLE 6.17

Express $\dfrac{3x+5}{(x-1)(x+2)}$ in partial fractions.

SOLUTION

Write

$$\frac{3x+5}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2},$$

and multiply through by $(x-1)(x+2)$:

$$3x+5 = A(x+2) + B(x-1).$$

Setting $x=1$ isolates A: $8 = 3A$, so $A = \frac{8}{3}$. Setting $x=-2$ isolates B: $-1 = -3B$, so $B = \frac{1}{3}$. Therefore

$$\frac{3x+5}{(x-1)(x+2)} = \boxed{\frac{8}{3(x-1)} + \frac{1}{3(x+2)}}$$

Choosing the two values that each make one factor zero is what makes the method quick. It turns a pair of simultaneous equations into two one-line calculations.

EXT **Beyond the syllabus.** The following two cases extend the partial fractions method beyond AA HL.

- **Repeated linear factors.** Include a term for every power of the factor. For example, $(x-\alpha)^2$ requires
$$\frac{A}{x-\alpha} + \frac{B}{(x-\alpha)^2}.$$
- **Irreducible quadratic factors.** A quadratic with a negative discriminant cannot be factorised into real linear factors. Use a numerator of the form $Bx+C$:
$$\frac{Bx+C}{x^2+bx+c}.$$

The general rule. For each distinct linear or irreducible quadratic factor, use a numerator whose degree is lower than the factor's degree. If a factor is repeated, include a term for each power.

Why these forms? A proper fraction with a denominator of degree n has room for n numerator coefficients. The partial fraction form provides exactly n unknown constants to match them. This count helps explain the choice of form, though proving that it always works is beyond this course.

YOUR TURN 6.6 Partial Fractions

Two linear factors

38. Express each fraction in partial fractions.

(a) $\dfrac{1}{(x-1)(x-2)}$ (b) $\dfrac{5}{(x-2)(x+3)}$

(c) $\dfrac{3}{(x+1)(x+4)}$ (d) $\dfrac{x}{(x-1)(x+1)}$

39. Express each fraction in partial fractions.

(a) $\dfrac{2x+1}{x(x+1)}$ (b) $\dfrac{x+4}{x(x-2)}$

(c) $\dfrac{2x}{(x-3)(x+1)}$ (d) $\dfrac{5x-1}{(x-1)(x+2)}$

Factorising the denominator first

40. Factorise the denominator first, then express each fraction in partial fractions.

(a) $\dfrac{4x}{x^2-4}$ (b) $\dfrac{x+7}{x^2+x-2}$

A factor of the form $ax+b$

41. Express each fraction in partial fractions. Factorise the denominator first where it is not already factorised.

(a) $\dfrac{5}{(2x+1)(x-2)}$ (b) $\dfrac{5x+1}{(3x-1)(x+1)}$

(c) $\dfrac{4x-7}{2x^2-5x+3}$

An improper fraction

42. EXT Consider $\dfrac{x^2}{x^2-1}$.

(a) Explain why this fraction is improper.

(b) Divide to write it as $1+\dfrac{1}{x^2-1}$.

(c) Hence express the whole fraction in partial fractions.

Concept check

43. EXT A student writes $\dfrac{2x^2+1}{(x-1)(x+2)} = \dfrac{A}{x-1}+\dfrac{B}{x+2}$, substitutes $x=1$ and $x=-2$, and obtains $A=1$ and $B=-3$. Explain why the result is wrong, and give the correct partial fractions.

Chapter 6 · Polynomials

Reflections

TOK Abel and Galois proved that the roots of a general quintic *exist* but cannot be written using only addition, subtraction, multiplication, division and roots. The equation $x^5 - x - 1 = 0$ has a real solution. That number is completely determined, and a computer can calculate it to a thousand decimal places. Even so, no formula of that kind can express it exactly. What does it mean, then, to "know" a number? Is such a root any less real than one you can write down?

IM The history behind this chapter reads like a thriller. In sixteenth-century Italy, mathematicians fought public problem-solving duels, and a loser could lose his university post, so methods were kept secret. Scipione del Ferro solved one type of cubic first. Niccolò Tartaglia rediscovered the method and won a famous contest with it. Gerolamo Cardano persuaded him to share it and promised to keep it secret. Cardano later learned that del Ferro had found the method first, and then published it in his great book on algebra, crediting both men, together with his student Ferrari's solution of the quartic. Almost three hundred years later Niels Abel proved that no such formula exists for the general quintic, and Évariste Galois, killed in a duel at twenty, explained exactly which equations a formula can solve. To do so he created *group theory*, the mathematical study of symmetry, which is now used in particle physics and cryptography.

RW Polynomials are used widely in computing, because they need only multiplication and addition, which processors carry out very quickly. The curved letters on this page and the smooth motion in animated films are drawn with *Bézier curves*: polynomial curves whose shape is set by a few chosen points. *Error-correcting codes* are built from polynomials too. They add extra information to a message so that it can still be read after part of it is damaged, which is how a scratched disc still plays and how a faint signal from a spacecraft billions of kilometres away is recovered.

EXT The factor theorem leads to a question that it cannot answer. How many zeros does a polynomial really have? Over the real numbers, $x^2 + 1$ has none. The answer is the **Fundamental Theorem of Algebra**. Over the *complex* numbers of Chapter 17, every polynomial of degree $n \geq 1$ has exactly n zeros, counted with multiplicity. Its proof is beyond this course. So the sum and product formulas hold for every polynomial of degree 1 or more, because the roots they describe always exist.

End-of-Chapter Problems

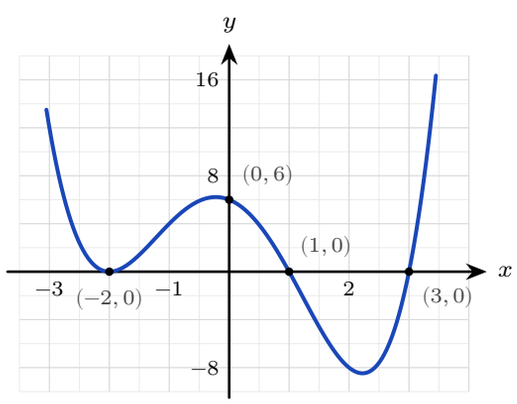
1. Sketch each graph, showing all intercepts and describing the behaviour at each x-intercept.

(a) $y = (x+1)(x-1)(x-3)$ [3]

(b) $y = x^2(x-3)$ [3]

(c) $y = -(x+1)^3(x-2)$ [3]

2. The graph of a polynomial $y = f(x)$ of degree four is shown. It touches the x-axis at $x = -2$, crosses it at $x = 1$ and $x = 3$, and meets the y-axis at $(0, 6)$.



(a) Find an expression for $f(x)$ in factorised form. [3]

(b) Hence solve the inequality $f(x) \leq 0$. [2]

3. Let $f(x) = 2x^4 - 3x^3 + 3x^2 + 8x - 1$.

(a) Find the quotient and the remainder when $f(x)$ is divided by $x^2 - 2x + 3$. [4]

(b) Hence find the values of the constants a and b for which $f(x) + ax + b$ is exactly divisible by $x^2 - 2x + 3$. [2]

4. The polynomial $x^3 - 3x^2 + kx - 8$ has $(x-2)$ as a factor. Find k. [3]

5. When a polynomial $p(x)$ is divided by $(x+1)$ the remainder is 7, and when it is divided by $(x-3)$ the remainder is -1. Find the remainder when $p(x)$ is divided by $x^2 - 2x - 3$. [3]

6. Find the unknown coefficients in each polynomial.

(a) The polynomial $x^3 + ax^2 + bx - 2$ has $(x-1)$ and $(x-2)$ as factors. Find a and b. [3]

(b) When $4x^3 + cx + d$ is divided by $(2x-1)$ the remainder is 6, and $(x+1)$ is a factor. Find c and d. [4]

7. When $f(x) = x^3 + ax^2 + bx + c$ is divided by $x^2 - 1$ the remainder is $2x - 1$, and $(x - 2)$ is a factor of $f(x)$.

(a) Find the values of a, b and c. [4]
(b) Hence solve $f(x) = 0$, giving your answers in exact form. [3]

8. Solve $3x^3 - 4x^2 - 5x + 2 = 0$. [4]

9. Let $f(x) = 2x^3 - x^2 - 13x - 6$.

(a) Show that $(x + 2)$ is a factor of $f(x)$. [1]
(b) Factorise $f(x)$ completely. [3]
(c) Hence solve $2e^{3y} - e^{2y} - 13e^y - 6 = 0$. [2]

10. The polynomial $x^3 - kx + 2$ has $(x - 2)$ as a factor.

(a) Find k. [2]
(b) Find the other two zeros, in exact form. [4]

11. Consider $f(x) = x^4 - x^2 - 12$.

(a) Use the substitution $u = x^2$ to find the real solutions of $f(x) = 0$. [3]
(b) Explain why $f(x) = 0$ has no other real solutions. [2]
(c) Factorise $f(x)$ as far as possible over the real numbers. [2]

12. Consider $f(x) = 12x^3 - 16x^2 - 7x + 6$.

(a) Any rational root of $f(x) = 0$ has the form $\frac{p}{q}$ in lowest terms. State the conditions on p and q. [2]
(b) Show that $x = \frac{1}{2}$ is a zero of f. [2]
(c) Hence factorise $f(x)$ completely. [3]
(d) Solve $f(x) = 0$. [1]

13. Consider $f(x) = x^3 - 3x + 2$.

(a) Show that $(x - 1)$ is a factor, and factorise $f(x)$ completely. [4]
(b) Sketch $y = f(x)$, showing all intercepts and the behaviour at each x-intercept. [3]
(c) Hence solve the inequality $f(x) > 0$. [2]

14. GDC An open box is made from a rectangular sheet of card 30 cm by 20 cm by cutting a square of side x cm from each corner and folding up the sides.

(a) Show that the volume of the box is $V(x) = 4x^3 - 100x^2 + 600x$ cm^3. [2]
(b) State the set of possible values of x. [1]

(c) Find the maximum volume of the box, and the value of x that gives it. [2]
(d) Find the values of x for which the volume is 800 cm^3. [3]

Give your answers to parts (c) and (d) to three significant figures.

15. Build the equation described in each part.

(a) A quadratic $x^2 + px + q$ has zeros 3 and -7. Find p and q. [2]
(b) The cubic $x^3 + px^2 + qx + r$ has zeros -1, 2 and 4. Find p, q and r. [3]
(c) The quadratic equation with roots α and β is $x^2 - 4x + 1 = 0$. Find a quadratic with leading coefficient 1 whose zeros are $\alpha + 1$ and $\beta + 1$. [4]

16. The roots of $x^2 + bx + 12 = 0$ differ by 1. Use the sum and product of the roots to find the possible values of b. [3]

17. Each equation has roots α and β. Answer without solving the equation.

(a) $3x^2 - x - 2 = 0$: find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$. [2]
(b) $x^2 - 6x + 7 = 0$: find $\alpha^2 + \beta^2$ and $(\alpha - \beta)^2$. [3]

18. The roots of $x^3 + 2x^2 + 3x - 4 = 0$ are α, β and γ.

(a) Find $\alpha^2 + \beta^2 + \gamma^2$ without solving the equation. [3]
(b) Hence show that the three roots are not all real. [1]

19. The equation $2x^3 - 3x^2 + kx + 6 = 0$ has roots α, $-\alpha$ and β. Find β, the value of k and the value of α^2. [5]

20. The cubic equation $x^3 + px^2 + qx + r = 0$ has roots α, β and γ.

(a) Write down $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$ in terms of p, q and r. [3]
(b) The equation $x^3 - 6x^2 + 3x + 10 = 0$ has roots α, β, γ. Find $\alpha^2 + \beta^2 + \gamma^2$ without solving the equation. [4]
(c) Find $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$. [3]
(d) Given that one root is -1, find the other two. [3]
(e) Write down a cubic equation with integer coefficients whose roots are 2α, 2β and 2γ. [3]

21. Express each fraction in partial fractions.

(a) $\dfrac{5x + 5}{x^2 - x - 6}$ [3]
(b) $\dfrac{7x + 1}{2x^2 + x - 1}$ [3]

22.

(a) Express $\dfrac{2}{(2r-1)(2r+1)}$ in partial fractions. [2]

(b) Hence show that $\displaystyle\sum_{r=1}^{n} \frac{2}{(2r-1)(2r+1)} = \frac{2n}{2n+1}$. [3]

(c) Find the least value of n for which the sum in part (b) is greater than 0.99. [2]

23. GDC Let $f(x) = x^3 - 5x^2 + 2x + 7$.

(a) Show that $f(x) = 0$ has no rational roots. [2]

(b) Use your GDC to find the three roots of $f(x) = 0$, giving your answers to three significant figures. [2]

(c) Hence solve the inequality $f(x) < 0$. [2]

(d) Verify that your answers to part (b) are consistent with the sum of the roots. [1]

24. The polynomial $p(x) = x^3 + ax^2 + bx + 8$ leaves a remainder of 20 when divided by $(x-3)$, and $(x+2)$ is a factor.

(a) Show that $3a + b = -5$ and $2a - b = 0$. [4]

(b) Hence find a and b. [2]

(c) Factorise $p(x)$ as far as possible over the real numbers. [3]

(d) State how many real roots $p(x) = 0$ has, giving a reason. [2]

(e) Find the remainder when $p(x)$ is divided by $(x-1)(x+2)$. *(Hint: the remainder has the form $rx + s$)* [3]

25. Consider $f(x) = 2x^3 - 3x^2 - 11x + 6$.

(a) Show that $(x-3)$ is a factor of $f(x)$. [2]

(b) Factorise $f(x)$ completely. [4]

(c) Solve $f(x) = 0$. [2]

(d) State the y-intercept, and describe the behaviour of $f(x)$ as $x \to \pm\infty$. [3]

(e) Sketch $y = f(x)$, showing all intercepts. [2]

(f) Hence solve the inequality $f(x) \leq 0$. [3]

26. GDC A section of a roller-coaster track is modelled by $h(x) = ax^3 + bx^2 + cx + d$, $0 \leq x \leq 30$, where h m is the height of the track above the ground at a horizontal distance of x m from the start of the section. The track is 20 m high at the start, 34 m high when $x = 10$, 18 m high when $x = 20$ and 26 m high at the end of the section.

(a) Write down four equations in a, b, c and d. [2]

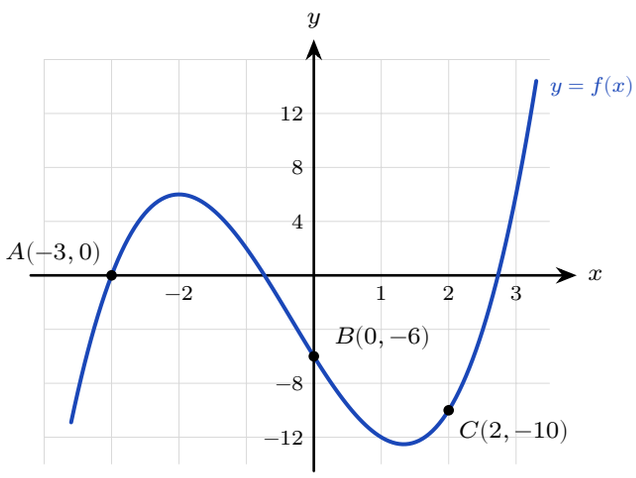
(b) Use your GDC to find the values of a, b, c and d. [2]

(c) Find the greatest and least heights of the track in this section. [3]

(d) Find the set of values of x for which the track is more than 25 m above the ground. [3]

Give your answers to parts (c) and (d) to three significant figures.

27. The graph of $y = f(x)$, where $f(x) = x^3 + ax^2 + bx + c$, is shown. The graph meets the x-axis at A, meets the y-axis at B and passes through C.



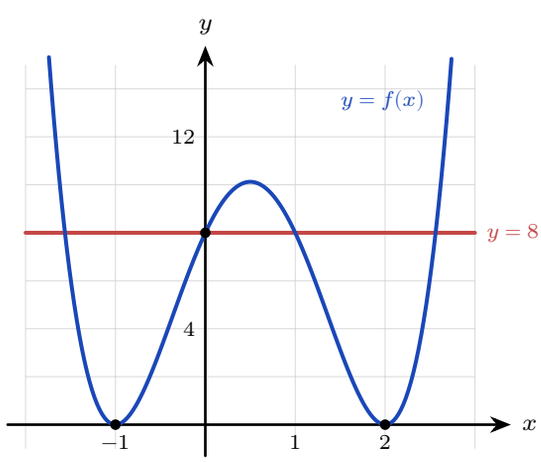
(a) Find the values of a, b and c. [4]

(b) Explain why $(x + 3)$ is a factor of $f(x)$. [1]

(c) Find the quotient when $f(x)$ is divided by $(x + 3)$. Hence find the exact x-coordinates of the other two points where the graph meets the x-axis. [4]

(d) Hence solve the inequality $f(x) > 0$. [2]

28. The graph of a quartic function $y = f(x)$ and the line $y = 8$ are shown. The graph of f meets the line at four points, one of them on the y-axis.



(a) Find an expression for $f(x)$ in factorised form. [3]

(b) Find the remainder when $f(x)$ is divided by $(x + 2)$. [2]

(c) Show that the x-coordinates of the points where the graph of f meets the line satisfy $(x^2 - x - 2)^2 = 4$. [2]

(d) Hence solve the inequality $f(x) \leq 8$. [5]

29. EXT Prove that if a polynomial equation with integer coefficients has a rational root $\frac{p}{q}$ in lowest terms, then p divides the constant term and q divides the leading coefficient. [6]

30. The cubic $x^3 + px + q = 0$ has a repeated root. Show that $4p^3 + 27q^2 = 0$. [6]

31. Find all real values of k for which $x^3 - 3x + k = 0$ has three distinct real roots. *(Hint: use Q30 to find the values of k that give a repeated root)* [6]

32. Let $f(x) = (2 - x)(x + 1)(x - 3)^2$.

(a) State the degree and the leading coefficient of $f(x)$. [2]
(b) Describe the behaviour of $f(x)$ as $x \to +\infty$ and as $x \to -\infty$. [2]
(c) Write down the y-intercept. [1]
(d) Solve the inequality $f(x) \geq 0$. [4]

33. Let $f(x) = x^4 - 2x^3 + 5$.

(a) Find the quotient and the remainder when $f(x)$ is divided by $(x - 1)^2$. [4]
(b) Hence find the values of the constants a and b for which $x^4 - 2x^3 + ax + b$ is exactly divisible by $(x - 1)^2$. [2]

Challenge Questions

C1. Power sums of the roots. The equation $x^2 - px + q = 0$ has roots α and β. For each integer $k \geq 0$ define

$$S_k = \alpha^k + \beta^k.$$

(a) Write down the values of S_0 and S_1. [2]
(b) Show that $S_2 = p^2 - 2q$. [2]
(c) Find S_3 in terms of p and q. [3]
(d) Explain why $\alpha^k = p\,\alpha^{k-1} - q\,\alpha^{k-2}$ for every $k \geq 2$, and deduce that $S_k = p\,S_{k-1} - q\,S_{k-2}$. [4]
(e) The equation $x^2 - 3x + 1 = 0$ has roots α and β. Use the recurrence to find S_2, S_3 and S_4. [4]
(f) The roots of $x^2 - 3x + 1 = 0$ are irrational. Explain why S_k is nevertheless an integer for every k. [3]
(g) For $x^2 - x - 1 = 0$ the numbers S_k are the *Lucas numbers* $2, 1, 3, 4, 7, \ldots$. Now let the cubic $x^3 - px^2 + qx - r = 0$ have roots α, β and γ, and let $S_k = \alpha^k + \beta^k + \gamma^k$. Find a recurrence for S_k in the style of part (d), and suggest how the investigation could continue. [3]

C2. Palindromic polynomials. A polynomial is *palindromic* if its list of coefficients reads the same forwards as backwards. For example $x^4 + x^3 - 4x^2 + x + 1$ has coefficients $1,\ 1,\ -4,\ 1,\ 1$.

(a) Explain why $x = 0$ is never a zero of a palindromic polynomial. [2]

(b) Let $P(x)$ be palindromic of degree n. Show that $x^n P\left(\frac{1}{x}\right) = P(x)$, and deduce that if α is a zero of P then so is $\frac{1}{\alpha}$. [4]

(c) Divide $x^4 + x^3 - 4x^2 + x + 1 = 0$ through by x^2 and substitute $u = x + \frac{1}{x}$. Show that the equation becomes $u^2 + u - 6 = 0$. [4]

(d) Hence find all four roots of $x^4 + x^3 - 4x^2 + x + 1 = 0$ in exact form. [5]

(e) Show that every palindromic polynomial of odd degree has $x = -1$ as a zero. [3]

(f) Show that $x^3 + \frac{1}{x^3} = u^3 - 3u$, where $u = x + \frac{1}{x}$. Explain why dividing a palindromic polynomial of degree $2m$ by x^m turns the equation $P(x) = 0$ into an equation of degree m in u. Suggest how the investigation could continue, for example with palindromic equations of degree 6 or with polynomials whose coefficients reverse with a change of sign. [3]

C3. Splitting a sum with partial fractions. Partial fractions are usually a step towards integration. This investigation uses them for something else: adding infinitely many terms exactly.

(a) Show that $\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ for every positive integer n. [2]

(b) Hence write out the sum $\sum_{n=1}^{N} \frac{1}{n(n+1)}$ term by term, and show that almost every term cancels. Deduce that the sum equals $\frac{N}{N+1}$. *(Hint: write the first three and the last two terms)* [4]

(c) State the value the sum approaches as $N \to \infty$, and explain what that value means for the graph of the partial sums. [2]

(d) Show that $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$, and hence find $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$. [4]

(e) Investigate $\sum_{n=1}^{\infty} \frac{1}{n(n+k)}$ for a positive integer k. Find its value for $k = 3$ and $k = 4$, then state a general formula and justify it. [6]

(f) Show that $\frac{1}{n(n+1)(n+2)} = \frac{1}{2}\left(\frac{1}{n(n+1)} - \frac{1}{(n+1)(n+2)}\right)$, and hence find $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$. Then use $\frac{1}{n^2} < \frac{1}{n(n-1)}$ for $n \geq 2$ to show that $\sum_{n=1}^{\infty} \frac{1}{n^2} < 2$, and suggest how the exact value of this sum could be investigated. [4]

C4. Dividing by a product of two linear factors. End-of-Chapter Questions 5 and 24 find a remainder on division by $(x - a)(x - b)$ from two given values. This investigation

finds that remainder in general, applies it to x^n, and then asks what happens when the two factors are the same. Throughout, $P(x)$ is a polynomial and a and b are real constants.

(a) Explain why the remainder when $P(x)$ is divided by $(x-a)(x-b)$ has the form $px+q$. Show that, when $a \neq b$, the remainder is

$$R(x) = P(a)\,\frac{x-b}{a-b} + P(b)\,\frac{x-a}{b-a}.$$

[4]

(b) Hence show that, for every positive integer n, the remainder when x^n is divided by x^2-3x+2 is $(2^n-1)x+2-2^n$. Find also the remainder when x^n is divided by x^2-1, treating even and odd n separately. [5]

(c) The formula in part (a) cannot be used when $a=b$. By long division or otherwise, find the remainder when x^n is divided by $(x-1)^2$ for $n=2$, 3 and 4, and conjecture the remainder for every positive integer n. [3]

(d) Prove your conjecture. *(Hint: $x^n-1=(x-1)(x^{n-1}+x^{n-2}+\cdots+x+1)$)* [5]

(e) Compare the line $y=nx+1-n$ with the graph of $y=x^n$ near $x=1$, and suggest what the remainder on division by $(x-a)^2$ tells you about the graph of any polynomial P. Suggest how the investigation could be extended to division by $(x-a)(x-b)(x-c)$. [3]