Chapter 5

Functions & Graphs

5

SYLLABUS 2.5 2.8–2.11 2.13–2.16

Hold a card in front of a lamp and look at its shadow on the wall. Slide the card and the shadow slides. Move it closer to the lamp and the shadow grows. Turn it over and the shadow flips. The card itself never changes. Only what you do to it changes.

Graphs work the same way, and that makes this part of the course more manageable. There are only a few shapes to memorise: a line, a parabola, a square root, a reciprocal, an exponential and a logarithm. There is also a short list of moves that slide them, stretch them and reflect them. Many other curves you will meet are one of those few shapes with some of those moves applied to it.

The chapter has two main ideas. The first is **composition** and the list of moves: how the output of one function becomes the input of the next, and what each move does to an equation. The second is the small set of shapes those moves are applied to, in particular the ones whose graphs approach a line, called an **asymptote**. Reading that behaviour from a formula is the second skill this chapter builds.

By the end of this chapter you will be able to:

- compose two functions, find the domain of the result, and find an inverse;
- apply translations, stretches and reflections, test a function for symmetry, and sketch $|f(x)|$, $f(|x|)$, $[f(x)]^2$ and $1/f(x)$;
- find the asymptotes of a rational function, and solve rational and modulus inequalities;
- use exponential and logarithmic graphs to model real situations, and solve equations from a graph when algebra cannot.

Every constant in these formulas is something you can see: change the number and the graph moves in a way you can predict, just as the shadow follows the card. A formula and a picture are two views of one function, and you will learn to read each from the other.

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5.1 Composite Functions

To calculate $(2x)^2 + 1$ you do two things, in order. Double the input. Then square the result and add 1. Many formulas in this course are built like that, from a few simple steps joined end to end. Reading a formula as a chain of steps is what lets you undo it, and later what lets you differentiate it. Composition is the notation for that chain.

Two functions can be joined end to end. Send a number into the first function. Take the number that comes out. Send that number into the second function. The single rule that does both steps at once is called a **composite** function. With $g(x) = 2x$ and $f(x) = x^2 + 1$, the chain above is g followed by f.

DEFINITION 5.1 **Composite function**

The **composite** $f \circ g$ is the function $(f \circ g)(x) = f(g(x))$: apply g first, then f.

The symbol $\circ$ is read "of", so $f \circ g$ is "f of g". It is also read "f after g", which gives the order in which the two functions act. It is not multiplication. Read the notation from right to left: the function written closest to x acts on x first.

5.1.1 Substituting a Whole Function

Composition looks unfamiliar because a function is being written where a number normally goes. The rule for f does not change at all. Only what you write in the slot changes.

The x in $f(x)$ is a **placeholder** (a marker showing where the input goes). It is not a fixed value, and it does not have to be a single letter.

Read f as a rule with an empty slot. If $f(x) = x^2 + 1$, the rule says: square the input, then add one.

$$f(\square) = \square^2 + 1 \qquad \Longrightarrow \qquad \begin{aligned} f(3) &= 3^2 + 1, \\ f(2x) &= (2x)^2 + 1, \\ f(g(x)) &= (g(x))^2 + 1. \end{aligned}$$

The last one is composition. You are substituting a whole function exactly as you would substitute a number.

To build $(f \circ g)(x)$:

1. Write the formula for $f(x)$.
2. Substitute the expression $g(x)$ for **every** x in that formula.
3. Simplify.

5.1.2 Order of Composition

Putting on socks and then shoes is not the same as putting on shoes and then socks. Composition behaves in the same way, so $f \circ g$ and $g \circ f$ are usually two different functions. Deciding which function acts first is therefore a good first step in a composition question.

Take $f(x) = x^2 + 1$ and $g(x) = 2x$. One input makes the difference concrete: send $x = 3$ through both orders and follow it step by step.

Composite	Input	After the first function	After the second function
$f \circ g$	3	$g(3) = 6$	$f(6) = 37$
$g \circ f$	3	$f(3) = 10$	$g(10) = 20$

EXAMPLE 5.1

For $f(x) = x^2 + 1$ and $g(x) = 2x$, find $(f \circ g)(x)$ and $(g \circ f)(x)$.

SOLUTION

Replace the 3 in the table by x. For $f \circ g$, apply g first. Its output is $2x$, so substitute $2x$ for every x in $f(x) = x^2 + 1$, then simplify:

$$(f \circ g)(x) = f(2x) = (2x)^2 + 1 = \boxed{4x^2 + 1}$$

For $g \circ f$, apply f first. Its output is $x^2 + 1$, so substitute $x^2 + 1$ for every x in $g(x) = 2x$, then simplify:

$$(g \circ f)(x) = g(x^2 + 1) = 2(x^2 + 1) = \boxed{2x^2 + 2}$$

5.1.3 The Domain of a Composite

A number can be allowed in g and still not be allowed in $f \circ g$. The reason is that f is never applied to x. It is applied to $g(x)$, and $g(x)$ may be a value that f does not accept. So x must be in the domain of g, and $g(x)$ must also be in the domain of f. The second condition is easy to miss.

KEY 5.1 **Domain of a composite**

The **domain** of $f \circ g$ is the set of values x for which

1. x is in the domain of g, **and**
2. $g(x)$ is in the domain of f.

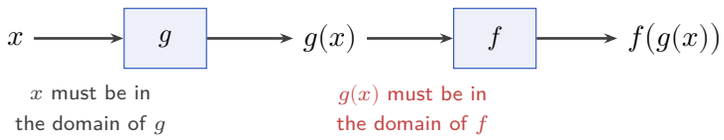


Figure 5.1 A composite works like two machines in a row. Each machine accepts only inputs in its own domain, so the domain of $f \circ g$ has two conditions.

- Find the domain from the two original functions, **before simplifying**. Simplification is an algebraic step, and a simplified formula may be defined at values where the composite is not.

EXAMPLE 5.2

For $f(x) = x^2$ and $g(x) = \sqrt{x-1}$, find $(f \circ g)(x)$ and state its domain.

SOLUTION

Apply g first, then square its output:

$$(f \circ g)(x) = f(\sqrt{x-1}) = \left(\sqrt{x-1}\right)^2 = \boxed{x-1}$$

Now take the two conditions in order.

Condition 1: the domain of g requires $x - 1 \geq 0$, so $x \geq 1$.

Condition 2: the domain of f is all real numbers, so this adds nothing here.

The domain of $f \circ g$ is therefore $\boxed{x \geq 1}$

The simplified formula $x - 1$ is defined for every x. The restriction from g still applies after simplifying, so the composite is the line $y = x - 1$ with everything to the left of $x = 1$ removed.

EXAMPLE 5.3

For $f(x) = \dfrac{1}{x-2}$ and $g(x) = \sqrt{x}$, find $(f \circ g)(x)$ and state its domain.

SOLUTION

Substitute $g(x)$ into f:

$$(f \circ g)(x) = f(\sqrt{x}) = \boxed{\frac{1}{\sqrt{x}-2}}$$

Condition 1: the domain of g requires $x \geq 0$. This removes every negative x.

Condition 2: 2 is not in the domain of f, so the output $g(x) = \sqrt{x}$ must not equal 2. This removes $x = 4$.

Both conditions must hold, so the domain of $f \circ g$ is $\boxed{x \geq 0,\ x \neq 4}$

Each condition removes a different value. $x = -1$ fails at the first machine, because $\sqrt{-1}$ is not real. $x = 4$ passes g, but $g(4) = 2$ is an input that f does not accept.

5.1.4 Composition and Inverses

Some pairs of functions undo each other. Composition is how that relationship is written down, and composing the two is how you check that an inverse is correct.

The inverse of f, written f^{-1}, is the function that reverses f. If f sends 3 to 10, then f^{-1} sends 10 back to 3. Apply f and then apply f^{-1}, and you arrive back at the number you started with. The composite is therefore the **identity function** $I(x) = x$, which returns every input unchanged.

KEY 5.2 **Composition with an inverse**

For a function f with inverse f^{-1},

$$(f^{-1} \circ f)(x) = x \qquad \text{and} \qquad (f \circ f^{-1})(x) = x.$$

An inverse exists exactly when f is **one-to-one**: no two inputs share an output.

$(f^{-1} \circ f)(x) = x$ for x in the domain of f; $(f \circ f^{-1})(x) = x$ for x in the domain of f^{-1}, which is the range of f.

The condition is needed because f^{-1} must return one input for each output. If two inputs shared an output, f^{-1} would have no single value to return there.

To find an inverse, use the method of Ch. 4, where inverses were first met:

1. Write $y = f(x)$.
2. Exchange x and y. An inverse exchanges the roles of input and output, so the letters swap too.
3. Solve the new equation for y. The result is $f^{-1}(x)$.
4. State the domain of f^{-1}, which is the range of f.

Composition is also how you check the answer. Take $f(x) = 2x + 1$, whose inverse is $f^{-1}(x) = \dfrac{x - 1}{2}$. Substituting one into the other gives

$$f^{-1}(f(x)) = \frac{(2x + 1) - 1}{2} = \frac{2x}{2} = x,$$

which confirms the pair. An important pair of inverse functions in this chapter is the exponential and the logarithm; see §5.5.

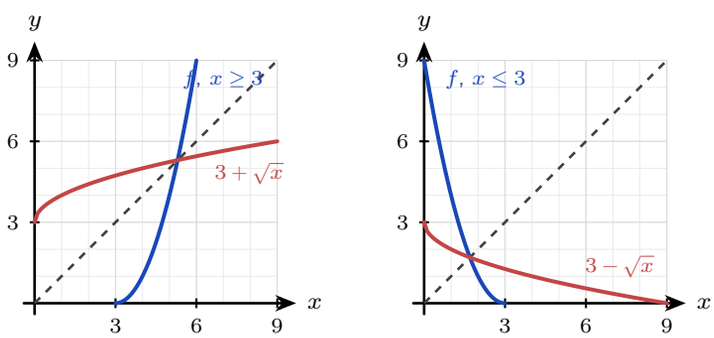
A quotient of two linear expressions needs one extra move at step 3. The variable appears twice, once in the numerator and once in the denominator, so it has to be collected into a single term before it can be isolated. For $f(x) = \dfrac{3x - 1}{x - 1}$, exchange the letters to get $x = \dfrac{3y - 1}{y - 1}$, then multiply both sides by $y - 1$: $xy - x = 3y - 1$. Collect the y-terms on the left, $xy - 3y = x - 1$, and factorise: $y(x - 3) = x - 1$. So $f^{-1}(x) = \dfrac{x - 1}{x - 3}$, $x \neq 3$. The range of such a function, which is the domain of its inverse, is worked out in §5.4.1.

5.1.5 Restricting the Domain to Get an Inverse HL

Many familiar functions are not one-to-one on their full domain. On a smaller domain they can be one-to-one.

For example, $f(x) = x^2$ is not one-to-one on all of $\mathbb{R}$, since $f(2) = f(-2)$, so it has no inverse there. Restricting the domain to $x \geq 0$ makes it one-to-one, and the inverse is $f^{-1}(x) = \sqrt{x}$.

Restricting a domain is a common examination task, and the inverse you get depends on the restriction you choose, because each restriction keeps a different half of the curve. For $f(x) = (x-3)^2$, the restriction $x \geq 3$ gives $f^{-1}(x) = 3 + \sqrt{x}$, while $x \leq 3$ gives $f^{-1}(x) = 3 - \sqrt{x}$. Two facts hold for any inverse. The domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.



(a) keep $x \geq 3$ (b) keep $x \leq 3$

Figure 5.2 Two restrictions of $f(x) = (x-3)^2$ and their inverses, each the reflection of the kept half in $y = x$ (dashed).

YOUR TURN 5.1 Composite Functions

Forming a composite

1. For $f(x) = x + 3$ and $g(x) = 2x$, find each composite.

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$
(c) $(f \circ f)(x)$ (d) $(g \circ g)(x)$

2. For $f(x) = x^2$ and $g(x) = x - 1$, find each of the following.

(a) $(f \circ g)(x)$ (b) $(g \circ f)(x)$
(c) $(f \circ g)(3)$ (d) $(g \circ f)(3)$

3. Let $f(x) = x^2$ and $g(x) = x + 2$.

(a) Find $(f \circ g)(x)$ and $(g \circ f)(x)$, and show that they are not the same.
(b) Find the value of x for which the two composites are equal.

The domain of a composite

4. For $f(x) = \dfrac{1}{x}$ and $g(x) = x + 1$, find each composite and state its domain.

(a) $(f \circ g)(x)$
(b) the domain of $f \circ g$
(c) $(g \circ f)(x)$
(d) the domain of $g \circ f$

5. For each pair of functions, find $(f \circ g)(x)$ and state the domain of $f \circ g$. Find the domain from f and g before you simplify.

(a) $f(x) = \sqrt{x}, \quad g(x) = 3 - x$
(b) $f(x) = x^2, \quad g(x) = \sqrt{x + 2}$
(c) $f(x) = \dfrac{1}{x - 2}, \quad g(x) = x^2 - 2$
(d) $f(x) = \sqrt{x - 1}, \quad g(x) = x^2$

Composites and inverses

6. Find the inverse of each function, and state the domain of the inverse.

(a) $f(x) = \dfrac{x + 3}{x - 2}$
(b) $f(x) = \dfrac{3x}{2x - 1}$

7. Let $f(x) = 2x + 1$.

(a) Find $f^{-1}(x)$.
(b) Show that $(f \circ f^{-1})(x) = x$.
(c) Show that $(f^{-1} \circ f)(x) = x$.
(d) State what these two results tell you about f^{-1}.

Restricting a domain HL

8. In each part, a domain is restricted so that an inverse exists.

(a) State a domain on which $f(x) = x^2$ has an inverse, and give that inverse.
(b) Do the same for $f(x) = (x - 1)^2$.
(c) State the domain of the inverse you found in part (b).
(d) Explain why $f(x) = x^3$ needs none.
(e) Let $f(x) = (x + 2)^2$. Find $f^{-1}(x)$ when the domain of f is restricted to (i) $x \geq -2$, (ii) $x \leq -2$. State the domain and range of each inverse, and explain which sign each restriction selects.

Concept check

9. Let $f(x) = x + 3$ and $g(x) = x^2$. A student writes: "$(f \circ g)(x) = (x + 3)^2$." Identify the error and find $(f \circ g)(x)$.

10. Let $f(x) = x^2$ and $g(x) = \sqrt{x + 2}$. A student writes: "$(f \circ g)(x) = \left(\sqrt{x + 2}\right)^2 = x + 2$, *so the domain of* $f \circ g$ *is* $x \in \mathbb{R}$." Identify the error and state the correct domain.

5.2 Transformations of Graphs

Plotting points is slow, and for most curves it is unnecessary. A small change to an equation moves the graph in one predictable way. Once you know which change produces which move, you can quickly sketch a new curve from a familiar one. You can also work the other way, and write down the equation of a curve you are shown.

There are three moves: a **translation** (a slide), a **reflection** (a flip) and a **stretch** (a scaling). Each one matches one change in the equation, and each one sends the point (p, q) to a new point you can write down.

Move	Equation	Acts on	(p, q) becomes	Effect
Translation	$y = f(x) + k$	output	$(p,\ q + k)$	k up
	$y = f(x - h)$	input	$(p + h,\ q)$	h right
Reflection	$y = -f(x)$	output	$(p,\ -q)$	reflect in the x-axis
	$y = f(-x)$	input	$(-p,\ q)$	reflect in the y-axis
Stretch	$y = a\,f(x)$	output	$(p,\ aq)$	vertical, scale factor a
	$y = f(bx)$	input	$(\frac{p}{b},\ q)$	horizontal, scale factor $\frac{1}{b}$

The table is three pairs, one pair for each move. The **Acts on** column gives the rule behind all six rows: a change outside the bracket acts on the output, and a change inside the bracket acts on the input.

> **KEY 5.3** **Reading a transformation**
>
> **Outside** the bracket, the change acts on the output: it is **vertical**, and it moves the graph in the direction of the sign: $+k$ moves it up by k. **Inside** the bracket, the change acts on the input: it is **horizontal**, and it moves the graph the **opposite** way: $x - h$ moves it right by h, and bx divides every x-coordinate by b.

The reversal inside the bracket has a reason. Take $y = f(x - h)$ and evaluate it at $x = p + h$:

$$f((p + h) - h) = f(p),$$

which is the value the original graph had at $x = p$. So the point that was at $x = p$ now appears at $x = p + h$. Every point moves h units to the *right*, not left. The same calculation works for $y = f(bx)$: at $x = \frac{p}{b}$ it returns $f(b \cdot \frac{p}{b}) = f(p)$, so the point at p moves to $\frac{p}{b}$.

That is a horizontal stretch of scale factor $\frac{1}{b}$, which brings every point closer to the y-axis whenever $b > 1$.

5.2.1 Translations

A translation slides the whole graph. Its shape and its size do not change, only its position.

Vertically, $y = f(x) + k$ adds k to every output, so every point moves up by k. A negative k moves the graph down.

Horizontally, $y = f(x - h)$ moves the graph right by h. It is easy to get this sign wrong. Use the reason above rather than memorising the result.

In examination questions a translation is usually written as a vector $(\text{a column of two numbers: } h \text{ across}, k \text{ up})$. The graph of $y = f(x - h) + k$ is the graph of $y = f(x)$ translated by $\binom{h}{k}$.

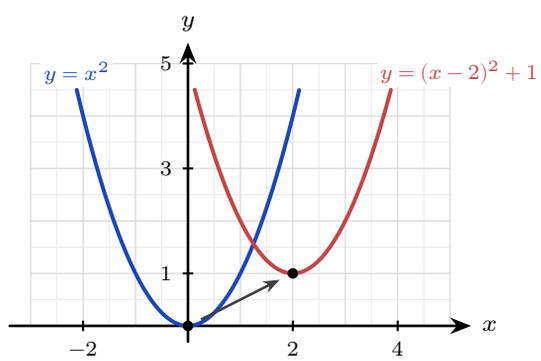


Figure 5.3 The graph of $y = (x - 2)^2 + 1$ is the graph of $y = x^2$ translated 2 units right and 1 unit up. The vertex moves from $(0, 0)$ to $(2, 1)$, and the shape is unchanged.

5.2.2 Reflections

A reflection flips the graph across an axis. Every point keeps its distance from that axis and moves to the other side of it.

For $y = -f(x)$ the sign of every output changes, so the point (p, q) moves to $(p, -q)$. That is a reflection in the x-axis. A maximum becomes a minimum, and any point already on the x-axis stays where it is.

For $y = f(-x)$ the sign of every input changes, so the point (p, q) moves to $(-p, q)$. That is a reflection in the y-axis. The y-intercept does not move, because changing the sign of 0 leaves it as 0.

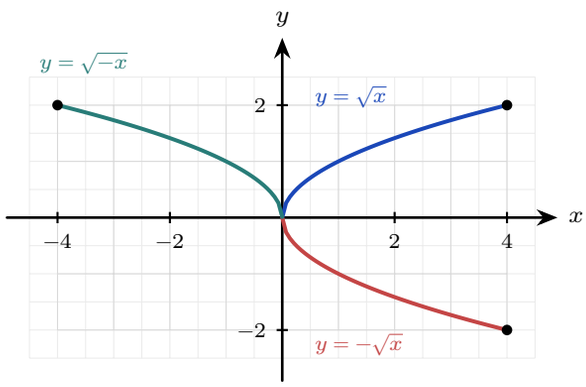


Figure 5.4 Reflections of $y = \sqrt{x}$. In the x-axis, $(4, 2)$ moves to $(4, -2)$ on $y = -\sqrt{x}$; in the y-axis, it moves to $(-4, 2)$ on $y = \sqrt{-x}$.

EXAMPLE 5.4

Describe how the graph of $y = -(x - 1)^2$ is obtained from $y = x^2$.

SOLUTION

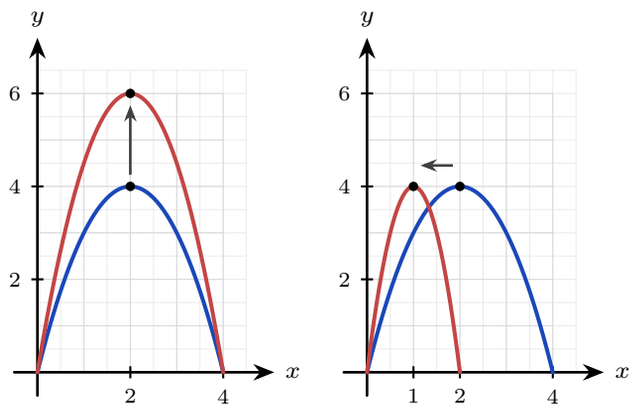
There are two changes. The -1 is inside the bracket, so it is horizontal: translate 1 unit right. The minus sign is outside the bracket, so it is vertical: reflect in the x-axis.
One change is horizontal and the other is vertical, so they do not affect each other and may be done in either order.
The vertex moves from $(0, 0)$ to $(1, 0)$, and the parabola now opens downward.

5.2.3 Stretches

A stretch changes the size of the graph in one direction only. One line stays fixed while every other point moves away from it or towards it.

For $y = a\,f(x)$ every output is multiplied by a, so (p, q) moves to (p, aq). This is a vertical stretch of scale factor a. Points on the x-axis do not move, because $a \times 0 = 0$. If a is negative, it is a stretch of scale factor $|a|$ together with a reflection in the x-axis.

For $y = f(bx)$ every input is multiplied by b, so, as shown above, every x-coordinate is divided by b and (p, q) moves to $(\frac{p}{b}, q)$. This is a horizontal stretch of scale factor $\frac{1}{b}$. Points on the y-axis do not move. If b is negative, it is a stretch of scale factor $\frac{1}{|b|}$ together with a reflection in the y-axis.



(a) $y = 1.5\,f(x)$ (b) $y = f(2x)$

Figure 5.5 $y = f(x)$ in cobalt, the stretch in red. (a) Each y-coordinate $\times 1.5$: $(2, 4)$ moves to $(2, 6)$ and the zeros stay. (b) Each x-coordinate halved: $(2, 4)$ moves to $(1, 4)$ and the zero at 4 moves to 2.

CAUTION A scale factor smaller than 1 is still called a stretch. The graph of $y = \frac{1}{2}f(x)$ is a vertical stretch of scale factor $\frac{1}{2}$, even though it makes the graph shorter. Write "stretch, scale factor $\frac{1}{2}$", not "squash".

EXAMPLE 5.5

The graph of $y = f(x)$ has a maximum point at $(6, 4)$. State the coordinates of that maximum point on the graph of (a) $y = 3f(x)$, (b) $y = f(3x)$.

SOLUTION

(a) The 3 is outside the bracket, so it is a vertical stretch of scale factor 3. It multiplies the y-coordinate and leaves the x-coordinate unchanged: $\boxed{(6, 12)}$

(b) The 3 is inside the bracket, so it is a horizontal stretch of scale factor $\frac{1}{3}$. It divides the x-coordinate and leaves the y-coordinate unchanged: $\boxed{(2, 4)}$

Each stretch changes one coordinate and leaves the other unchanged. Checking which coordinate moved is a quick way to confirm you have used the right one.

5.2.4 Composite Transformations

Many graphs in this chapter come from a familiar shape by several moves, applied one after another. This is a **composite transformation**. To describe one, split the changes into two groups, those inside the bracket and those outside it. The two groups may be done in either order, but inside each group apply the stretch before the translation.

Start by writing the target equation in one standard form, $y = a\,f(b(x - h)) + k$. Each of the four parameters controls one move:

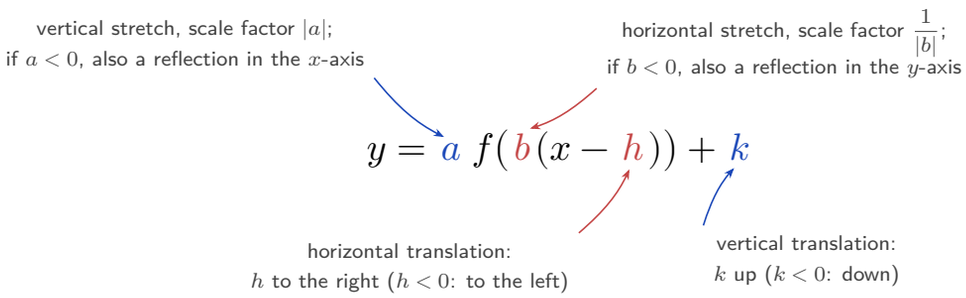


Figure 5.6 The standard form and the role of each parameter.

- Outside the bracket, a and k act on the output y, so the graph moves the way the numbers say. Inside, b and h act on the input x, **so the graph moves the opposite way**: $y = f(x + 2)$ gives the value $f(0)$ at $x = -2$, so every point moves 2 to the left, and $y = f(2x)$ gives the value $f(4)$ at $x = 2$, so every point moves to half its distance from the y-axis.

For a quadratic this standard form is the vertex form. Writing $y = a(x - h)^2 + k$ names the vertex (h, k) and the two moves that put it there, so completing the square and describing a transformation are two ways of doing the same work.

The same four moves apply to the graphs of $\sin x$ and $\cos x$, and they are part of the course there too. §8.3.1 writes the form as $y = a\sin(b(x + c)) + d$, so its c is $-h$ and its d is k.

KEY 5.4 **Describing a composite transformation**

To describe a composite transformation:

1. Split the changes into two groups: those **inside** the bracket, which are horizontal, and those **outside** it, which are vertical.
2. Do the two groups in either order. A horizontal move and a vertical move do not affect each other.
3. Inside each group the order matters. With the equation in the form $y = a\, f(b(x - h)) + k$, apply the **stretch before the translation**.

Step 3 follows from what a stretch does. A stretch multiplies every value it is applied to, including a translation that has already been applied. Do the stretch first and there is no translation for it to multiply.

Whatever sequence you write down, check it by tracking a single point. Choose a point you know on the original graph, such as a vertex, an endpoint or an intercept. Apply your moves to its coordinates. Then check that the result satisfies the target equation.

EXAMPLE 5.6

Describe a sequence of transformations that maps $y = \sqrt{x}$ onto $y = 3\sqrt{x - 2} + 1$.

SOLUTION

Take the change inside the bracket first. The -2 acts on the input, so it translates the graph 2 units *right* and gives $\sqrt{x-2}$.

Now take the vertical changes, stretch before translation. The factor 3 stretches the graph vertically by scale factor 3 and gives $3\sqrt{x-2}$. The $+1$ then translates it 1 unit *up*.

So the sequence is: translate 2 right, stretch vertically by scale factor 3, translate 1 up. Check it by tracking the endpoint. The graph of $y=\sqrt{x}$ begins at $(0,0)$. The translation carries it to $(2,0)$. The stretch leaves it there, because $3\times 0=0$. The final translation carries it to $(2,1)$. That is exactly where $y=3\sqrt{x-2}+1$ begins.

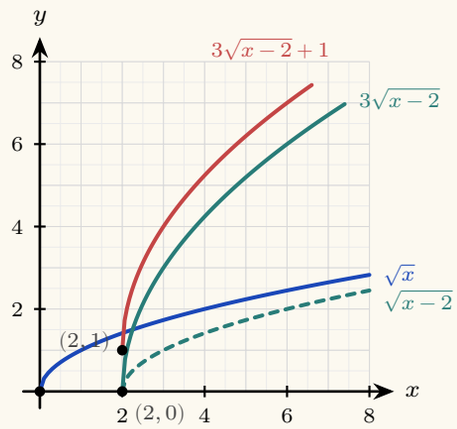


Figure 5.7 The three moves in order: translate 2 right (dashed), stretch vertically by 3, translate 1 up. The endpoint goes $(0,0)\to(2,0)\to(2,0)\to(2,1)$.

Two moves **commute** (give the same result in either order) when swapping them leaves the final graph unchanged. A horizontal move and a vertical move always commute, because they change different coordinates. A stretch and a translation in the *same* direction do not commute. The only exceptions are a scale factor of 1 or a translation of 0, where one of the two moves does nothing. The next example shows the two different answers they produce.

EXAMPLE 5.7

Starting from $y=x^2$, apply a vertical stretch by scale factor 2 and a translation 1 unit up, in both orders. Are the results the same?

SOLUTION

Stretch first, then translate. The stretch doubles every value, giving $2x^2$; the translation then adds 1:

$$x^2 \longrightarrow 2x^2 \longrightarrow 2x^2+1.$$

Translate first, then stretch. The translation adds 1, giving x^2+1; the stretch then doubles everything, *including* the 1:

$$x^2 \longrightarrow x^2+1 \longrightarrow 2(x^2+1)=2x^2+2.$$

The two curves are different. So state the moves in the order that builds the target curve.

TIP In examinations, an answer to "describe the sequence of transformations" needs named moves with their values: translation by $\binom{-3}{0}$, vertical stretch scale factor 2, translation by $\binom{0}{-1}$. Vague words such as "moved" or "squashed" are unlikely to earn the marks.

5.2.5 The Graph of $f(ax+b)$ HL

The form $f(ax+b)$ combines the horizontal translation and the horizontal stretch of this section, in the order fixed in §5.2.4. Everything in $f(ax+b)$ happens inside the bracket, so every change is horizontal. The letters a and b here are new: they are not the a and b used earlier in this section. Factorise the bracket first, so that the stretch and the translation can be read separately. For example, in $f(2x-6)=f(2(x-3))$ the translation is 3, not 6.

KEY 5.5 **Sketching $f(ax+b)$**

$f(ax+b)=f\left(a\left(x+\frac{b}{a}\right)\right)$: for $a\neq 0$, a horizontal stretch of scale factor $\frac{1}{|a|}$, with a reflection in the y-axis when $a<0$, then a horizontal translation of $-\frac{b}{a}$.

Tracking a few key points, as below, confirms the order.

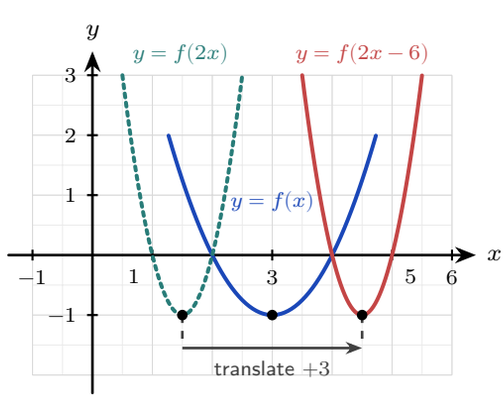


Figure 5.8 $f(x)=x^2-6x+8$ (cobalt) and $y=f(2x-6)=f(2(x-3))$ (red). The stretch $f(2x)$ (dotted) moves the vertex from $x=3$ to $x=1.5$; the translation of $-\frac{b}{a}=3$ then carries it to $x=4.5$.

EXAMPLE 5.8

Sketch $y=f(2x-6)$ for $f(x)=x^2-6x+8$, stating the images of the x-intercepts and of the vertex.

SOLUTION

Factorise inside the bracket before reading anything off:

$$f(2x-6) = f(2(x-3)).$$

Here $a = 2$ and $b = -6$, so the moves are a horizontal stretch of scale factor $\frac{1}{2}$ followed by a translation of $-\frac{b}{a} = 3$ to the right. Both are horizontal, so no y-coordinate changes.

Now track the key points of f: the x-intercepts $(2, 0)$ and $(4, 0)$ and the vertex $(3, -1)$. The stretch halves each x-coordinate and the translation then adds 3:

$$2 \to 1 \to 4, \qquad 4 \to 2 \to 5, \qquad 3 \to \tfrac{3}{2} \to \tfrac{9}{2}.$$

So the x-intercepts move to $(4, 0)$ and $(5, 0)$, and the vertex moves to $(\frac{9}{2}, -1)$: still a minimum, still with minimum value -1, and still halfway between the x-intercepts. The curve is a parabola through those three points, half as wide as f, with y-intercept $f(-6) = 36 + 36 + 8 = 80$.

Expanding confirms every number:
$f(2x-6) = ((2x-6)-2)((2x-6)-4) = (2x-8)(2x-10) = 4(x-4)(x-5)$, whose zeros are 4 and 5 and whose value at $x = \frac{9}{2}$ is $4(\frac{1}{2})(-\frac{1}{2}) = -1$.

YOUR TURN 5.2 Transformations of Graphs

Single transformations

11. Describe the single transformation that takes $y = x^2$ to each graph.

(a) $y = x^2 + 3$
(b) $y = (x+4)^2$
(c) $y = -x^2$
(d) $y = (x-1)^2$

12. Describe the single transformation that takes $y = f(x)$ to each graph.

(a) $y = f(-x)$
(b) $y = -f(x)$
(c) $y = 3f(x)$
(d) $y = f(2x)$

The image of a point

13. The point $(2, 5)$ lies on $y = f(x)$. Find the coordinates of its image on each graph.

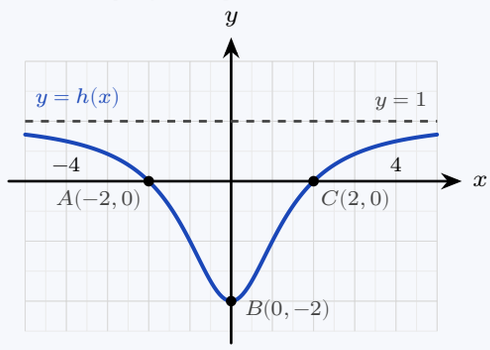
(a) $y = f(x-3) + 1$
(b) $y = 3f(x)$
(c) $y = f(2x)$
(d) $y = -f(x) + 2$

14. The point $(-2, 3)$ lies on $y = f(x)$. Find the coordinates of its image on each graph.

(a) $y = f(2x)$ (b) $y = f(x) - 4$

(c) $y = f(-x)$ (d) $y = \frac{1}{2}f(x)$

(e) The graph of $y = h(x)$ is shown below. It has x-intercepts A and C, a minimum point B, and the horizontal asymptote $y = 1$.



On separate axes, sketch each graph. On each sketch, give the coordinates of the images of A, B and C, and the equation of the asymptote.

(i) $y = h(x - 2) + 1$ (ii) $y = -h(x)$ (iii) $y = h(2x)$

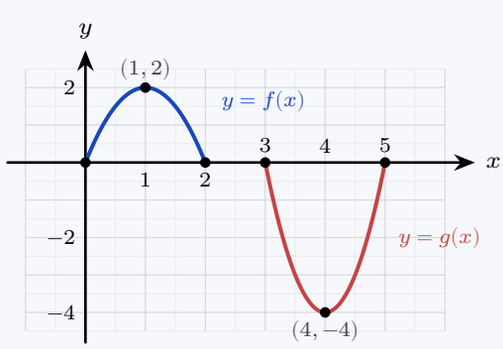
Combined transformations

15. Describe the sequence of transformations that maps $y = \sqrt{x}$ onto each graph.

(a) $y = \sqrt{x + 3}$ (b) $y = 2\sqrt{x}$

(c) $y = 2\sqrt{x + 3} - 1$ (d) $y = \sqrt{3x + 6}$ HL *(Hint: factorise inside the root first)*

(e) The graphs of $y = f(x)$, $0 \leq x \leq 2$, and $y = g(x)$, $3 \leq x \leq 5$, are shown, where $g(x) = af(x - b)$ and a and b are constants. Find the value of a and the value of b.



16. Start from $y = x^2$.

(a) Translate 2 units up, then reflect in the x-axis. Write the new equation.

(b) Apply the same two moves in the opposite order. Write the new equation.

The graph of $f(ax + b)$ HL

17. Consider $y = f(4x - 8)$.

(a) Write it in the form $y = f(b(x - h))$.
(b) Describe the sequence of transformations from $y = f(x)$.

Concept check

18. A student describes two transformations of $y = f(x)$. Identify the error in each description and give the correct one.

(a) "$y = f(x + 3)$ *is a translation of* 3 *units to the right.*"
(b) "$y = f(2x)$ *is a horizontal stretch with scale factor* 2."

5.3 Symmetry and Related Graphs HL

A graph has a **symmetry** when some motion leaves it looking exactly as it did. The motion may be a reflection, a rotation or a slide. One half of the curve then determines the other half, and the symmetry gives you a way to check a sketch against the formula.

Two motions, reflection in the y-axis and a half-turn about the origin, are common enough to be named here, and each has a short algebraic test.

Type	Test	Motion that leaves the graph unchanged	Examples
Even	$f(-x) = f(x)$	reflection in the y-axis	x^2, x^4, $\lvert x\rvert$, $\cos x$
Odd	$f(-x) = -f(x)$	rotation of $180°$ about the origin	x, x^3, $\dfrac{1}{x}$, $\sin x$
Neither	no test holds	neither of the two	$x^2 + x$, $x + 1$, e^x

Many functions have neither symmetry, so "neither" is a complete and correct answer when the tests fail.

A third motion, a slide along the x-axis, belongs to **periodic** functions such as $\sin x$, whose graphs repeat at regular intervals. They are treated with trigonometry (§8.3).

5.3.1 Testing for Even and Odd

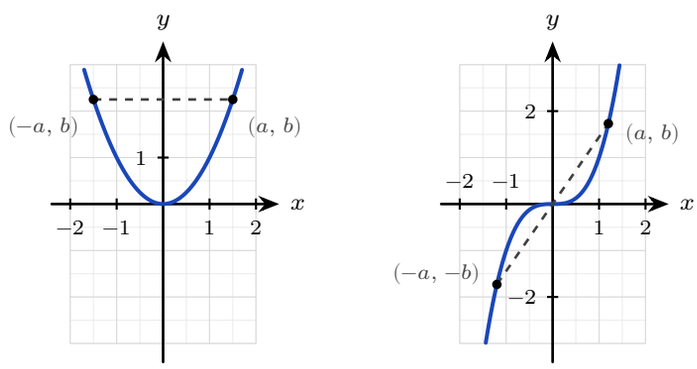
The names come from the powers. The functions $x^2, x^4, \ldots$ have even exponents and satisfy $f(-x) = f(x)$. The functions $x, x^3, \ldots$ have odd exponents and satisfy $f(-x) = -f(x)$.

DEFINITION 5.2 **Even and odd functions**

A function whose domain is symmetric about 0 is **even** if $f(-x) = f(x)$ for every x in its domain, and **odd** if $f(-x) = -f(x)$ for every x in its domain. An even graph is symmetric in the y-axis; an odd graph has rotational symmetry of order 2 (it looks the same after a half-turn) about the origin.

To test a function, substitute $-x$ for x, simplify, and compare the result with the original. There are three possible outcomes:

- the result is $f(x)$, so the function is **even**;
- the result is $-f(x)$, so the function is **odd**;
- the result is neither, so the function has **no symmetry** of this kind.



(a) $y = x^2$ is even (b) $y = x^3$ is odd

Figure 5.9 (a) Each point (a, b) has a partner $(-a, b)$, so reflection in the y-axis leaves the curve unchanged. (b) The partner is $(-a, -b)$, so a rotation of $180°$ about the origin leaves it unchanged.

EXAMPLE 5.9

Decide whether each function is odd, even or neither: (a) $f(x) = x^3 - x$, (b) $f(x) = x^4 - x^2$, (c) $f(x) = x^2 + x$.

SOLUTION

(a) Substitute $-x$ for x, simplify, then look for a factor of -1:

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The result is $-f(x)$, so by the definition the function is **odd**. Its graph therefore has rotational symmetry of order 2 about the origin.

(b) For $f(x) = x^4 - x^2$,

$$f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x),$$

so the function is **even**.

(c) For $f(x) = x^2 + x$, $f(-x) = x^2 - x$. This is not $f(x)$, and it is not $-f(x) = -x^2 - x$ either, so the function is **neither**. Failing the even test does not make a function odd: each test has to be checked on its own.

5.3.2 Self-Inverse Functions

A fourth symmetry belongs to functions that undo themselves, and it is usually met as an algebraic expression rather than as a graph. Recall that f^{-1} reverses f. If applying f twice already returns the input, then f reverses itself.

DEFINITION 5.3 **Self-inverse function**

A function is **self-inverse** if $f(f(x)) = x$ for every x in its domain. Then $f = f^{-1}$, and the graph of f is symmetric in the line $y = x$.

That symmetry has a simple reason. Reflecting any graph in the line $y = x$ produces the graph of its inverse, because the reflection swaps the two coordinates of every point. If f is its own inverse, the reflection has to return the same curve.

To test for self-inverse, substitute the whole expression for $f(x)$ in place of x, simplify, and check whether the result is x.

EXAMPLE 5.10

Show that $f(x) = \dfrac{1}{x}$ is self-inverse.

SOLUTION

Apply the function twice. Substitute the whole expression $\frac{1}{x}$ in place of x:

$$f(f(x)) = \frac{1}{\frac{1}{x}} = x.$$

Applying f twice returns the input, so $f = f^{-1}$, and the figure below confirms the symmetry in the line $y = x$.

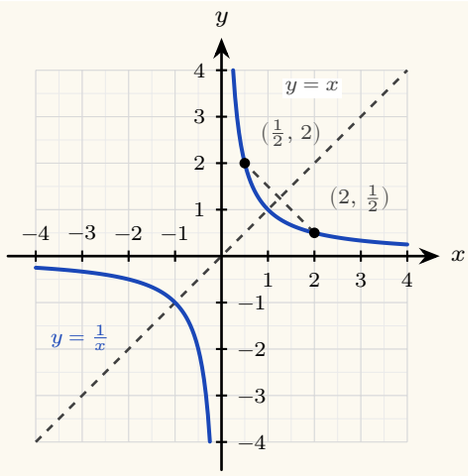


Figure 5.10 The reciprocal is symmetric in the line $y = x$. Each point on the curve has a partner with its two coordinates swapped, so reflecting in $y = x$ returns the same curve.

Other self-inverse functions take more algebra to check. The function $f(x) = 6 - x$ is self-inverse, and so is $f(x) = \dfrac{2x+1}{x-2}$. Substitute the whole expression, then multiply the top and the bottom by $x - 2$:

$$f(f(x)) = \frac{2\left(\dfrac{2x+1}{x-2}\right)+1}{\dfrac{2x+1}{x-2}-2} = \frac{2(2x+1)+(x-2)}{(2x+1)-2(x-2)} = \frac{5x}{5} = x.$$

5.3.3 The Modulus Graph $|f(x)|$

The next three graphs, $y = |f(x)|$, $y = f(|x|)$ and $y = [f(x)]^2$, are each built from $y = f(x)$ by one operation, and each has one visual rule. None of them needs a table of values.

The modulus of a number is its size without its sign, so $|f(x)|$ is never negative. The rule below follows from this.

KEY 5.6 **Sketching $|f(x)|$**

To sketch $y = |f(x)|$: keep every part of $y = f(x)$ that lies on or above the x-axis, and reflect every part below the axis upward in the x-axis. The x-intercepts do not move, because $|0| = 0$.

A zero where f *crosses* the axis with a non-zero gradient (not flat) becomes a corner; a zero where f only touches the axis, as in $y = x^2$, stays smooth.

The reason for the corner: on one side of the zero the curve is kept, and on the other it is reflected. If f crosses with a non-zero gradient, the reflection turns an increasing piece into a decreasing one, so the two pieces meet at a sharp point. Where f only touches the axis, both sides are kept, or both are reflected, so the curve stays smooth.

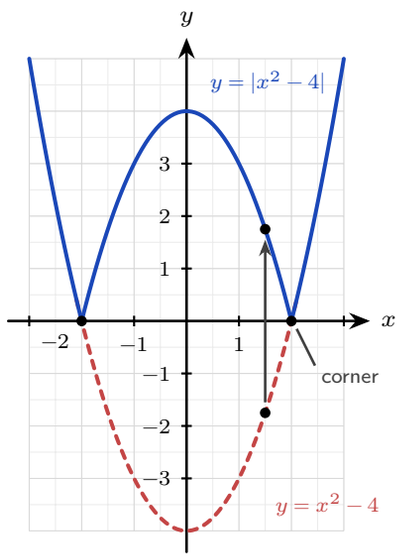


Figure 5.11 $y = |x^2 - 4|$ (cobalt) against $y = x^2 - 4$ (dashed). The part below the x-axis is reflected upward, so $(1.5, -1.75)$ goes to $(1.5, 1.75)$. The zeros stay at $x = \pm 2$, and the curve has corners there.

EXAMPLE 5.11

Sketch $y = |f(x)|$ for $f(x) = x^2 - 6x + 8$, stating the coordinates of the intercepts, the corners and the turning point.

SOLUTION

Find the key points of f first. Factorising gives $f(x) = (x - 2)(x - 4)$, so the zeros are 2 and 4, and the vertex lies halfway between them at $x = 3$, where $f(3) = 9 - 18 + 8 = -1$. So f has x-intercepts $(2, 0)$ and $(4, 0)$ and a minimum at $(3, -1)$.

Only the arc between the x-intercepts lies below the x-axis, so only that arc is reflected. Everything outside $2 \leq x \leq 4$ is already above the axis and stays unchanged.

Now move the key points:

Point of f	Point of $\lvert f(x) \rvert$	What happens
$(2, 0)$, $(4, 0)$	$(2, 0)$, $(4, 0)$	stay, as $\lvert 0 \rvert = 0$; corners, as f crosses there
$(3, -1)$	$(3, 1)$	reflected; now a local maximum
$(0, 8)$	$(0, 8)$	already above the axis, so unchanged

So the curve falls from the upper left to the corner $(2, 0)$, rises to the peak $(3, 1)$, falls to the corner $(4, 0)$, then rises again to the upper right, and no part of it lies below the x-axis.

5.3.4 The Graph of $f(|x|)$

The previous rule applied the modulus to the *output*. Applying it to the *input* instead gives a different graph and a different rule, and a question may test whether you can tell them apart.

Since $|x|$ and $|-x|$ are the same number, f receives the same input at x and at $-x$, so the two outputs are equal. The graph is therefore its own mirror image in the y-axis.

KEY 5.7 **Sketching $f(|x|)$**

To sketch $y = f(|x|)$: keep the part of $y = f(x)$ with $x \geq 0$, and reflect it in the y-axis to make the part with $x < 0$. The original graph to the left of the y-axis is discarded. The result is always an even function.

The two halves meet on the y-axis. Unless the graph of f is flat there, they meet at a corner, which is a local maximum or minimum of $f(|x|)$.

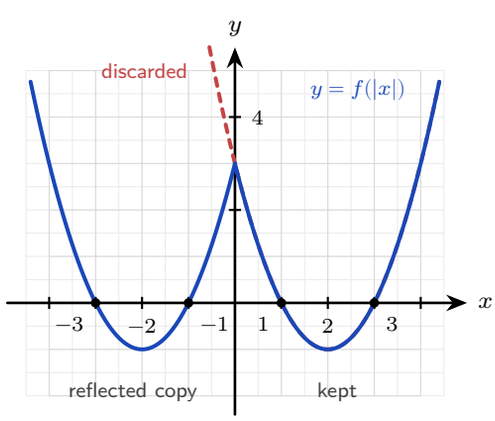


Figure 5.12 $y = f(|x|)$ for $f(x) = x^2 - 4x + 3$. The half with $x \geq 0$ is copied across the y-axis, so the zeros 1 and 3 gain partners at -1 and -3. The short dashed piece of f is discarded.

EXAMPLE 5.12

Sketch $y = f(|x|)$ for $f(x) = x^2 - 6x + 8$, stating the coordinates of the intercepts, the corner and the turning points.

SOLUTION

Keep the half of $y = f(x)$ with $x \geq 0$ and discard the rest. That half carries the y-intercept $(0, 8)$, the x-intercepts $(2, 0)$ and $(4, 0)$, and the minimum $(3, -1)$. Reflecting it in the y-axis copies each of those points across:

Point with $x \geq 0$	Partner with $x \leq 0$
$(0, 8)$	itself, since it lies on the axis
$(2, 0)$	$(-2, 0)$
$(3, -1)$	$(-3, -1)$
$(4, 0)$	$(-4, 0)$

So the graph meets the x-axis at $(-4, 0)$, $(-2, 0)$, $(2, 0)$ and $(4, 0)$, and it has two minima, at $(-3, -1)$ and $(3, -1)$, both of value -1. The y-intercept is $(0, 8)$.
The y-axis is where the two halves join, and they do not join smoothly. Just to the right of it the kept half is falling, from $f(0) = 8$ towards $f(1) = 3$; just to the left the mirror image is rising towards $(0, 8)$. The two pieces therefore meet at a **corner** at $(0, 8)$, which is a local maximum of value 8. The result is a W shape, symmetric in the y-axis.
The formula agrees: $f(|x|) = |x|^2 - 6|x| + 8 = x^2 - 6|x| + 8$, which is unchanged when x is replaced by $-x$, so it is even, and the values of f for $x < 0$ are not used.

5.3.5 The Graph of $[f(x)]^2$

Squaring the output also removes the sign, so $[f(x)]^2$ and $|f(x)|$ share a rule: neither is ever negative. They differ in size, because squaring changes the size and a modulus does not.

KEY 5.8 **Sketching $[f(x)]^2$**

To sketch $y = [f(x)]^2$: no output is negative, so the whole curve lies on or above the x-axis. Each zero of f becomes a point where the new curve **touches** the axis instead of crossing it. Where $|f(x)| > 1$ the squared value is larger, so the curve lies further from the axis than $|f(x)|$. Where $|f(x)| < 1$ it is smaller, so the curve lies closer to the axis.

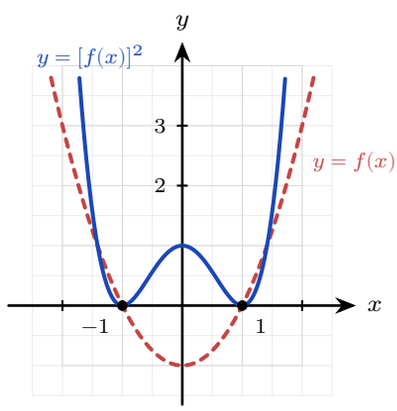


Figure 5.13 $y = [f(x)]^2$ for $f(x) = x^2 - 1$: the zeros $x = \pm 1$ become points where it touches the axis. For $|x| < \sqrt{2}$ it lies no further from the axis than f; for $|x| > \sqrt{2}$ it rises above f.

EXAMPLE 5.13

Sketch $y = [f(x)]^2$ for $f(x) = x^2 - 6x + 8$, stating where the curve meets the x-axis and the coordinates of its turning points.

SOLUTION

The key points of f are again the x-intercepts $(2, 0)$ and $(4, 0)$ and the minimum $(3, -1)$. Squaring acts on the output only, so each of them keeps its x-coordinate. Square the second coordinate of each point:

Point of f	Point of $[f(x)]^2$	What happens
$(2, 0)$, $(4, 0)$	$(2, 0)$, $(4, 0)$	$0^2 = 0$; the curve touches the axis, so minima
$(3, -1)$	$(3, 1)$	$(-1)^2 = 1$; a local maximum
$(5, 3)$	$(5, 9)$	$\lvert f\rvert > 1$; now further from the axis

The curve cannot cross the axis at $x = 2$ or $x = 4$, because no output is negative, so it touches the axis there and turns back.
Between the two zeros the squared value rises from 0 to 1 and falls back to 0, which makes $(3, 1)$ a local **maximum**.
Further out, $|f(x)|$ passes 1 at $x = 3 \pm \sqrt{2}$ and keeps growing. Beyond those points the squared curve increases much faster than $|f(x)|$.
So the curve decreases to touch the axis at $(2, 0)$, where it is flat, increases to $(3, 1)$, decreases to touch again at $(4, 0)$, then increases steeply. Inside the arch $|f(x)| \leq 1$, so squaring keeps the curve closer to the axis than $|f(x)|$.

5.3.6 Modulus Equations and Inequalities

The modulus removes the sign of a number. To solve a modulus equation, consider both possible signs. How you do this depends on what is on the other side.

There are five forms, and they must be separated before any algebra starts. $|A| = |B|$ needs no check. $|A| = k$ has no solution when $k < 0$. $|A| = B$ needs a check at the end, to reject any root that makes B negative. $|A| \lessgtr k$ is read from the V-shaped graph. The last, $|f(x)| \lessgtr g(x)$, has x on both sides and may have no algebraic method at all. The table lists the five forms in the order they are treated below.

Form	Method	Check
$\|A\| = k$	$A = k$ or $A = -k$	no solution if $k < 0$
$\|A\| = \|B\|$	$A = B$ or $A = -B$	nothing; both roots are valid
$\|A\| = B$	two cases, then substitute back	reject any root making B negative
$\|A\| \lessgtr k$	$-k < A < k$, or two intervals	for linear A, expect one interval for $<$ and two for $>$
$\|f(x)\| \lessgtr g(x)$	find where the sides are equal, then test each interval	often needs a GDC

A modulus equal to a number

If $|A| = k$ and k is positive, then A has size k, so it is k or $-k$. Both cases must be written down.

KEY 5.9 **Modulus equation $|A| = k$**

For $k > 0$, $|A| = k$ means $A = k$ or $A = -k$.

For $k = 0$ it means $A = 0$. For $k < 0$ there is no solution, because a modulus is never negative.

EXAMPLE 5.14

Solve $|2x - 1| = 5$.

SOLUTION

The right side is positive, so there are two cases:

$$2x - 1 = 5 \implies x = 3, \qquad 2x - 1 = -5 \implies x = -2.$$

So $x = 3$ or $x = -2$

The graph agrees. The V-shaped curve $y = |2x - 1|$ meets the line $y = 5$ once on each arm.

A modulus equal to a modulus

Two sizes are equal when the numbers are equal or opposite.

KEY 5.10 **Modulus equation** $|A| = |B|$

$|A| = |B|$ means $A = B$ or $A = -B$.

Squaring both sides gives the same two answers, because $|A|^2 = A^2$.

EXAMPLE 5.15

Solve $|2x - 1| = |x + 4|$.

SOLUTION

Take the two cases, $A = B$ and $A = -B$:

$$2x - 1 = x + 4 \implies x = 5, \qquad 2x - 1 = -(x + 4) \implies 3x = -3 \implies x = -1.$$

Both work: $|9| = |9|$ and $|-3| = |3|$. So $x = 5$ or $x = -1$

A modulus equal to an expression in x

This form looks like the first, but a modulus is never negative, so the other side cannot be negative either.

CAUTION For $|A| = B$, solve $A = B$ and $A = -B$ as usual, then **substitute each answer back** and discard any that makes B negative. These are **extraneous solutions** (values produced by the working that do not satisfy the original equation) (§1.3). Here an extraneous solution can come from $A = B$ or from $A = -B$. For example, $|2x| = x - 3$ gives $x = -3$ from $A = B$ and $x = 1$ from $A = -B$; both make $x - 3$ negative, so the equation has no solution.

EXAMPLE 5.16

Solve $|x - 1| = 2x - 4$.

SOLUTION

Two cases, as before:

$$x - 1 = 2x - 4 \implies x = 3, \qquad x - 1 = -(2x - 4) \implies 3x = 5 \implies x = \tfrac{5}{3}.$$

Now test each against the right side. At $x = 3$: $2(3) - 4 = 2$, which is positive, and $|3 - 1| = 2$. That works.
At $x = \frac{5}{3}$: $2\left(\frac{5}{3}\right) - 4 = -\frac{2}{3}$, which is negative. A modulus cannot equal a negative number, so this value is rejected.
The only solution is $x = 3$

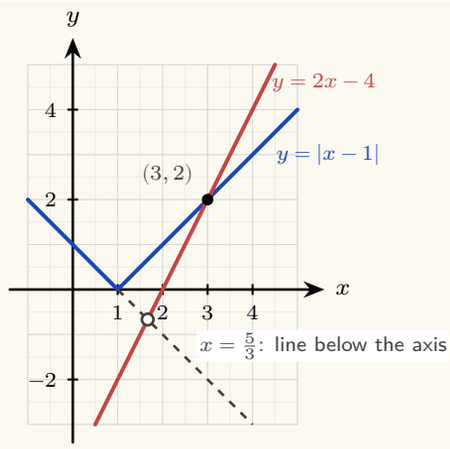


Figure 5.14 The V and the line meet only at $(3, 2)$. The second case meets the dashed line $y = 1 - x$ at $x = \frac{5}{3}$, where $2x - 4 < 0$ and the V is not, so that value is rejected.

Inequalities

Read these from the V rather than memorising them. When A is linear in x, the graph of $y = |A|$ is a V with its lowest point on the x-axis, and a horizontal line $y = k$ with $k > 0$ meets it twice. Below the line is the piece between the crossings; above it are the two pieces outside.

KEY 5.11 **Modulus inequalities**

For $k > 0$:

$$|A| < k \text{ means } -k < A < k, \qquad |A| > k \text{ means } A < -k \text{ or } A > k.$$

When A is linear in x: one interval for $<$, two for $>$. Use $\leq$ and $\geq$ to include the endpoints.

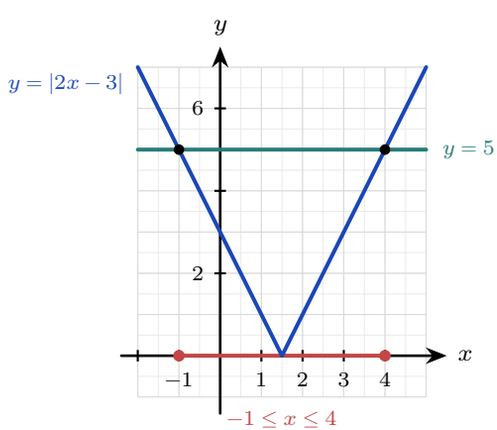


Figure 5.15 The V meets $y = 5$ at $x = -1$ and $x = 4$. Between them the V is below the line, so $|2x - 3| \leq 5$ on the red interval; outside them $|2x - 3| > 5$.

EXAMPLE 5.17

Solve (a) $|2x-3| \leq 5$ and (b) $|3x+1| > 7$.

SOLUTION

(a) One interval, so write the double inequality and work on all three parts at once:

$$-5 \leq 2x-3 \leq 5 \implies -2 \leq 2x \leq 8 \implies \boxed{-1 \leq x \leq 4}$$

(b) Two intervals, so write two separate inequalities:

$$3x+1 > 7 \implies x > 2, \qquad 3x+1 < -7 \implies x < -\tfrac{8}{3}.$$

So $\boxed{x < -\frac{8}{3} \text{ or } x > 2}$

EXAMPLE 5.18

Solve $|x^2-5| < 4$.

SOLUTION

The same two rules work when the expression inside is not linear. Write the double inequality:

$$-4 < x^2-5 < 4 \implies 1 < x^2 < 9.$$

So $x^2 > 1$ and $x^2 < 9$, which gives $\boxed{-3 < x < -1 \text{ or } 1 < x < 3}$
Here $<$ gives two intervals, not one: the one-interval pattern holds only when A is linear.

When both sides contain x

An inequality such as $|f(x)| > g(x)$ has no rule of its own, because the number of intervals depends on the two curves. The method is to find where the two sides are equal. Those points divide the x-axis into intervals, and on each interval the answer cannot change, so one test value is enough for each interval. A vertical asymptote, or any value of x excluded from the domain, is also a boundary between intervals. §5.4.5 uses the same idea in a sign table.

KEY 5.12 **Modulus against a function**

To solve $|f(x)| > g(x)$ or $|f(x)| < g(x)$:

1. Sketch $y = |f(x)|$ and $y = g(x)$ on the same axes.
2. Find every point where they meet, by solving $|f(x)| = g(x)$ or from a calculator.
3. Read the intervals where the curve you want is the higher one.

EXAMPLE 5.19

Solve $|2x| > x + 6$.

SOLUTION

Solve $|2x| = x + 6$ first:

$$2x = x + 6 \implies x = 6, \qquad 2x = -(x + 6) \implies x = -2.$$

Both make the right side positive (12 and 4), so both are crossing points. They divide the x-axis into three intervals. Test one value in each:

$$x = -3 : \ 6 > 3 \text{ is true}, \qquad x = 0 : \ 0 > 6 \text{ is false}, \qquad x = 7 : \ 14 > 13 \text{ is true}.$$

So the solution is $x < -2$ or $x > 6$

On a sketch, the V lies above the line outside the two crossing points.

TIP Many modulus inequalities cannot be solved by the algebra of this course. The syllabus example $|3x \arccos x| > 1$ is one (the function arccos is defined in §8.5). Graph both sides on your GDC, find the crossing points, and read the intervals from the picture. State the intersection coordinates you used, to three significant figures.

YOUR TURN 5.3 Symmetry and Related Graphs HL

Odd and even functions

19. Determine whether each function is odd, even, or neither.

(a) $f(x) = x^4 - 3x^2$

(b) $f(x) = x^3 + x$

(c) $f(x) = x^2 + x$

(d) $f(x) = x^5$

(e) $f(x) = 5$

(f) $f(x) = \dfrac{1}{x}$

Self-inverse functions

20. A function f is self-inverse when $f(f(x)) = x$ for every x in its domain.

(a) Show that $f(x) = \dfrac{1}{x}$ is self-inverse.

(b) Show that $f(x) = 6 - x$ is self-inverse.

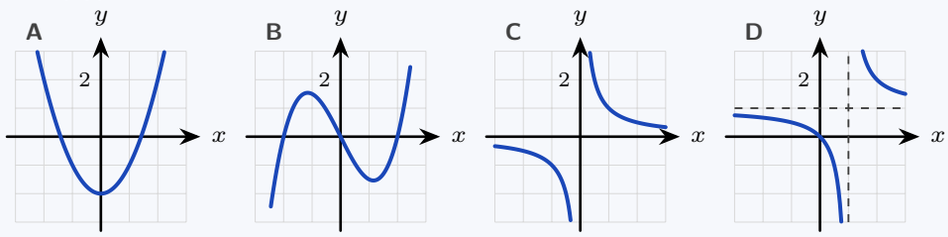
(c) Show that $f(x) = \dfrac{2x + 1}{x - 2}$ is self-inverse.

(d) Explain why $f(x) = 2x$ is not self-inverse.

(e) Explain why the graph of a self-inverse function is symmetric in the line $y = x$.

(f) Find the value of a for which $f(x) = \dfrac{4x + 1}{2x + a}$ is self-inverse.

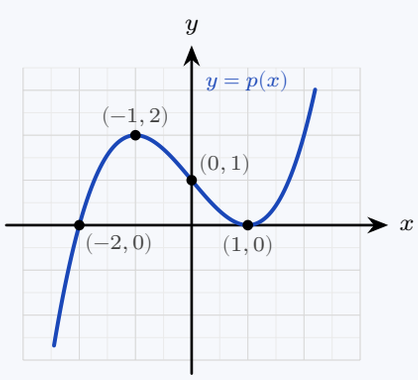
(g) The four graphs below are drawn with equal scales on the two axes. For each graph, state whether the function is odd, even or neither, and whether it is self-inverse.



The graphs of $|f(x)|$, $f(|x|)$ and $[f(x)]^2$

21. The graph of $y = f(x)$ crosses the x-axis at $x = 2$, has no other x-intercept, and has a minimum point at $(0, -4)$.

(a) State where $y = |f(x)|$ meets the x-axis.
(b) State where $y = [f(x)]^2$ meets the x-axis.
(c) State the point that $(0, -4)$ becomes on $y = |f(x)|$.
(d) State the zeros of $f(|x|)$ and the coordinates of its minimum point.
(e) The graph of $y = p(x)$ is shown, with its intercepts and its two turning points marked.



On separate axes, sketch each graph, giving the coordinates of every intercept and turning point.

(i) $y = |p(x)|$ (ii) $y = p(|x|)$ (iii) $y = [p(x)]^2$

22. Let $f(x) = x^2 - 2x - 3$. On separate axes, sketch each graph, stating the coordinates of every intercept and turning point.

(a) $y = |f(x)|$
(b) $y = f(|x|)$
(c) $y = [f(x)]^2$

Modulus equations and inequalities

23. Solve each equation.

(a) $|x - 3| = 2$ (b) $|2x + 1| = 7$
(c) $|3 - x| = 1$ (d) $|x| = 5$

24. Solve each equation. One part has no solution.

(a) $|x-2| = |2x+1|$ (b) $|3x| = |x+4|$

(c) $|x-1| = 3x-5$ (d) $|2x+1| = x-1$

25. Solve each inequality.

(a) $|x-4| < 3$ (b) $|2x+5| \geq 9$

(c) $|3-x| \leq 1$ (d) $|x^2-9| < 7$

26. GDC Both sides now contain x.

(a) Sketch $y = |x-2|$ and $y = x$ on the same axes, and hence solve $|x-2| < x$.

(b) Solve $|2x| > x+3$ algebraically.

(c) Use a GDC to solve $|x^3 - 3x| > 1$ for $0 \leq x \leq 3$, giving answers to three significant figures.

27. Solve $|x^2-4| < 3$.

Concept check

28. A student writes: "$f(x) = x^3+1$, *so* $f(-x) = -x^3+1 \neq f(x)$. *The function is not even, so it is odd.*" Identify the error and classify f correctly.

29. A student solves $|x-3| = 2x$ and gives the answer "$x = -3$ *or* $x = 1$". Identify the error and give the correct solution.

5.4 Reciprocal and Rational Functions

A line, a parabola and a cubic have a value at every input. Apart from a horizontal line, each one increases or decreases without limit as $|x|$ grows. Divide by a variable and both of those stop being true. There are now inputs where the function has no value, and there are lines, called asymptotes, that the curve approaches. Finding those lines from the formula is what makes a rational curve quick to sketch. The same work also tells you where the function is positive and where it is negative.

A **rational function** is one polynomial (a sum of whole-number powers of x with constant coefficients) divided by another, such as $\dfrac{2x+1}{x-1}$. The simplest function of this kind is the **reciprocal** $y = \dfrac{1}{x}$, and its graph has both new features: a value of x where it is undefined, and lines that the curve approaches.

It is undefined at $x = 0$. As x approaches 0, the output grows without bound, towards $+\infty$ on one side and $-\infty$ on the other. For large $|x|$ the opposite happens: the output becomes very small, and the curve approaches zero. So the reciprocal has two asymptotes, the vertical line $x = 0$ and the horizontal line $y = 0$.

It is defined for every input except $x = 0$, so its domain is all real numbers with $x \neq 0$. It produces every output except 0, so its range is all real numbers with $y \neq 0$. It is also self-inverse, so its graph is symmetric in the line $y = x$. See §5.3.2.

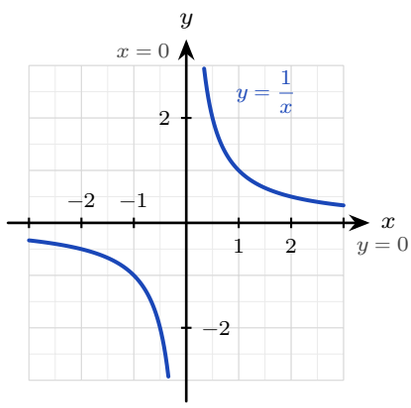


Figure 5.16 The reciprocal $y = \frac{1}{x}$. Its asymptotes are the two axes: the curve approaches $x = 0$ and $y = 0$ without meeting either. The domain is every real number except 0, and so is the range.

DEFINITION 5.4 **Asymptote**

An **asymptote** is a line that the graph of a function approaches arbitrarily closely (as close as you like). A **vertical asymptote** is a line $x = a$, where the function is undefined and $|f(x)|$ grows without bound as x approaches a. A **horizontal asymptote** is a line $y = c$, where c is the value that $f(x)$ approaches as $x \to \infty$ or $x \to -\infty$.

5.4.1 Linear over Linear

Provided $c \neq 0$ and $ad - bc \neq 0$, every function of the form $\dfrac{ax + b}{cx + d}$ is a reciprocal curve that has been translated and stretched, so all of them have the same two-branch shape. Writing the numerator as a multiple of the denominator plus a remainder shows this directly: the function becomes a constant plus a multiple of $\dfrac{1}{x + \frac{d}{c}}$. Its two asymptotes cross at the point where the translated reciprocal has its centre of symmetry. If $c = 0$ the function is linear and has no asymptote. If $ad - bc = 0$ the numerator is a multiple of the denominator, so the function is a constant with a hole at $x = -\frac{d}{c}$.

KEY 5.13 **Asymptotes of $(ax+b)/(cx+d)$**

For $f(x)=\dfrac{ax+b}{cx+d}$, with $c\neq 0$ and $ad-bc\neq 0$:

$$\text{vertical asymptote at } x=-\frac{d}{c}, \qquad \text{horizontal asymptote at } y=\frac{a}{c}.$$

The vertical one is where the denominator is zero. The horizontal one is the ratio of the leading coefficients (the coefficients of the highest power of x, here a and c).

Since $ax+b=\frac{a}{c}(cx+d)+\frac{bc-ad}{c}$,

$$\frac{ax+b}{cx+d}=\frac{a}{c}+\frac{bc-ad}{c(cx+d)}.$$

The fraction on the right is undefined where $cx+d=0$, and it approaches zero for large $|x|$, so $f(x)$ approaches $\dfrac{a}{c}$.

EXAMPLE 5.20

Sketch $f(x)=\dfrac{2x+1}{x-1}$, showing its asymptotes and intercepts.

SOLUTION

Find the vertical asymptote first. The denominator is zero at $x=1$, so the line $x=1$ is the vertical asymptote. Next the horizontal asymptote. The ratio of the leading coefficients gives $y=\dfrac{2}{1}=2$. Rewriting the function confirms the shape. The numerator is $2x+1=2(x-1)+3$, so

$$\frac{2x+1}{x-1}=2+\frac{3}{x-1},$$

a reciprocal curve centred on the crossing of the two asymptotes, $(1,2)$.

Now find the intercepts. Substitute $x=0$ for the y-intercept: $f(0)=\dfrac{1}{-1}=-1$. Set the numerator to zero for the x-intercept: $2x+1=0$, so $x=-\frac{1}{2}$. Both are marked on the figure below. Two asymptotes and two intercepts determine the whole sketch.

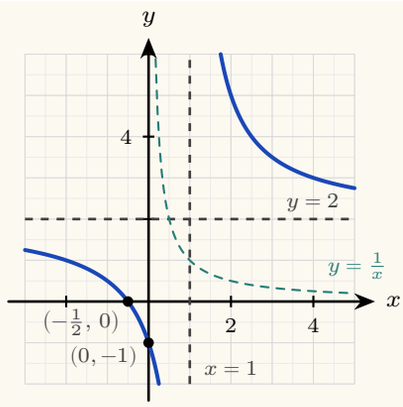


Figure 5.17 The curve $y = \frac{2x+1}{x-1}$, with its two asymptotes dashed and its two intercepts marked. The teal dashed curve is one branch of $y = \frac{1}{x}$: a vertical stretch by 3 and a move of the centre to $(1, 2)$ give the cobalt curve.

TIP Marks for a sketch usually depend on labelled asymptotes and intercepts. Write the equation of each asymptote beside its dashed line, and write the coordinates beside each intercept.

The rewritten form also gives the range of any quotient of two linear expressions. A sketch shows which heights the curve reaches, but to state a range you need a reason, not only a picture. Writing the function as a constant plus a fraction gives that reason. In that form, the function can never equal the constant, because the fraction is never zero. Every other value is reached, which you show by setting $f(x) = k$ and solving for x. So the range of $\dfrac{ax+b}{cx+d}$, with $c \neq 0$ and $ad \neq bc$, is every real number except the horizontal asymptote, $y = \dfrac{a}{c}$. The function below is the one whose inverse was found in §5.1.4.

EXAMPLE 5.21

State the range of $f(x) = \dfrac{3x-1}{x-1}$, $x \neq 1$, justifying your answer.

SOLUTION

Write the numerator as a multiple of the denominator plus a remainder, as in the example above. Since $3x - 1 = 3(x-1) + 2$,

$$f(x) = 3 + \frac{2}{x-1}.$$

The second term is a fraction with numerator 2, so it is never 0, whatever x is. Adding something that is never 0 to 3 never gives 3, so $f(x) \neq 3$ for every x in the domain. Every other value is produced. Set $f(x) = k$ and solve back for x:

$$\frac{2}{x-1} = k - 3 \implies x - 1 = \frac{2}{k-3} \implies x = 1 + \frac{2}{k-3},$$

which is a real input, and never equal to 1, whenever $k \neq 3$. So the range is $y \in \mathbb{R},\ y \neq 3$

5.4.2 Reading the Asymptotes from the Degrees HL

The two rules above are one case of a general rule: the kind of asymptote a rational function has as $x \to \pm\infty$ depends only on how the degree (the highest power of x; see §6.1) of the numerator compares with the degree of the denominator.

Degrees	Asymptote as $x \to \pm\infty$	Example
numerator < denominator	horizontal, $y = 0$	$\frac{x+1}{x^2-4}$
numerator = denominator	horizontal, at the ratio of leading coefficients	$\frac{2x+1}{x-1}$
numerator = denominator + 1	oblique, a slanted line	$\frac{x^2-2x+3}{x-1}$

One method gives the first two rows. Divide the numerator and the denominator by the highest power of x in the denominator. Every term of the form $\frac{k}{x^m}$ then approaches zero as $|x|$ grows, and what remains gives the asymptote:

$$\frac{x+1}{x^2-4} = \frac{\frac{1}{x}+\frac{1}{x^2}}{1-\frac{4}{x^2}} \to \frac{0}{1} = 0, \qquad \frac{2x+1}{x-1} = \frac{2+\frac{1}{x}}{1-\frac{1}{x}} \to \frac{2}{1} = 2.$$

The third row is derived in §5.4.3.

In every case the vertical asymptotes are at the zeros of the denominator, once any common factor has been cancelled.

A horizontal or oblique asymptote describes the curve only for large $|x|$. Nearer the origin the curve may cross it. A curve cannot cross a vertical asymptote, because the function is undefined there.

The syllabus names one more shape: a linear numerator over a quadratic denominator, $\frac{ax+b}{cx^2+dx+e}$. The denominator has the higher degree, so the horizontal asymptote is $y = 0$, and each real zero of the denominator gives its own vertical asymptote. This form can therefore have two.

With two asymptotes and an intercept, you still need to know whether each branch lies above or below the x-axis. To decide, find the sign of f on each interval. A rational function changes sign only where it is zero or where it is undefined. To see why, factorise the numerator and the denominator. The sign of the function is then the product of the signs of the factors. A linear factor $x - r$ changes sign only at $x = r$, and a quadratic factor with no real root never changes sign. So those values divide the x-axis into intervals, and the sign is the same throughout each one. Testing one point per interval is therefore enough.

The sign beside a vertical asymptote gives the direction of that branch. Where f is positive, $f(x) \to +\infty$ as x approaches the asymptote. Where f is negative, $f(x) \to -\infty$. §5.4.5 sets the same work out as a table.

EXAMPLE 5.22

Find the key features of the graph of $f(x) = \dfrac{x+1}{x^2-4}$: its asymptotes, its intercepts and its sign on each interval.

SOLUTION

Factorise the denominator to find the vertical asymptotes: $x^2 - 4 = (x-2)(x+2)$, so there are *two*, at $x = 2$ and $x = -2$.

The numerator has lower degree than the denominator, so the horizontal asymptote is $y = 0$.

For the intercepts, set the numerator to zero to get $x = -1$, and substitute $x = 0$ to get $f(0) = -\frac{1}{4}$.

Now the signs. The three values -2, -1 and 2 divide the x-axis into four intervals. Take the sign of each factor of $\dfrac{x+1}{(x-2)(x+2)}$, one interval at a time:

$$x < -2 : \ \frac{-}{(-)(-)} < 0, \qquad -2 < x < -1 : \ \frac{-}{(-)(+)} > 0,$$

$$-1 < x < 2 : \ \frac{+}{(-)(+)} < 0, \qquad x > 2 : \ \frac{+}{(+)(+)} > 0.$$

Read the branches from those signs. The middle branch is negative to the right of $x = -1$ and positive to the left of it, and it passes through both intercepts. In particular it crosses its horizontal asymptote $y = 0$ at $x = -1$.

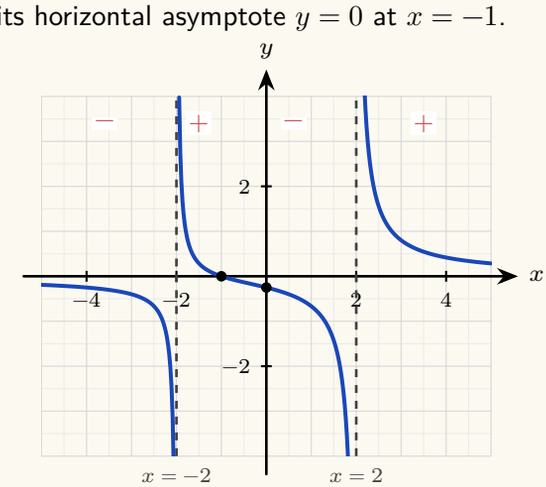


Figure 5.18 $y = \dfrac{x+1}{x^2-4}$: the sign on each interval (red) tells you whether the branch is above or below the x-axis. The middle branch crosses $y = 0$ at $x = -1$ and meets the y-axis at $(0, -\frac{1}{4})$.

EXT A zero of the denominator does not always produce an asymptote. This is why the rule above says "once any common factor has been cancelled". Sometimes the same factor also divides

the numerator and cancels completely from the denominator. Then all that is left at that value of x is a **hole** (a single missing point). Its formal name is a **removable discontinuity**. The function still has no value there, so the domain still excludes it, but the two sides of the curve meet at that point without a jump. If a copy of the factor remains in the denominator, as in $\frac{x-3}{(x-3)^2}$, the asymptote remains. To decide which case you have, substitute the zero of the denominator into the numerator. If the result is not 0, no factor cancels and there is a vertical asymptote. If it is 0, factorise and cancel, then check whether a copy of the factor is left in the denominator.

EXAMPLE 5.23

Describe the behaviour of $f(x) = \dfrac{x^2-9}{x-3}$ at $x = 3$, and state its domain. Compare it with $g(x) = \dfrac{x^2-9}{x+4}$ at $x = -4$.

SOLUTION

Factorise the numerator first, because the factors show whether anything cancels:

$$x^2 - 9 = (x+3)(x-3).$$

For f the factor $x-3$ appears in both the numerator and the denominator, so it cancels:

$$f(x) = \frac{(x+3)(x-3)}{x-3} = x+3, \qquad x \neq 3.$$

The graph of f is therefore the straight line $y = x+3$ with a single point removed. That point is at $x = 3$, where the line has height $3+3 = 6$, so the hole is at $(3, 6)$. There is no vertical asymptote at all, and the domain is $x \in \mathbb{R},\ x \neq 3$

For g the denominator is zero at $x = -4$. Substitute that into the numerator: $(-4+3)(-4-3) = 7$, which is not 0, so nothing cancels. As x approaches -4 the numerator approaches 7 while the denominator approaches 0, so $|g(x)|$ grows without bound and $x = -4$ is a vertical asymptote.

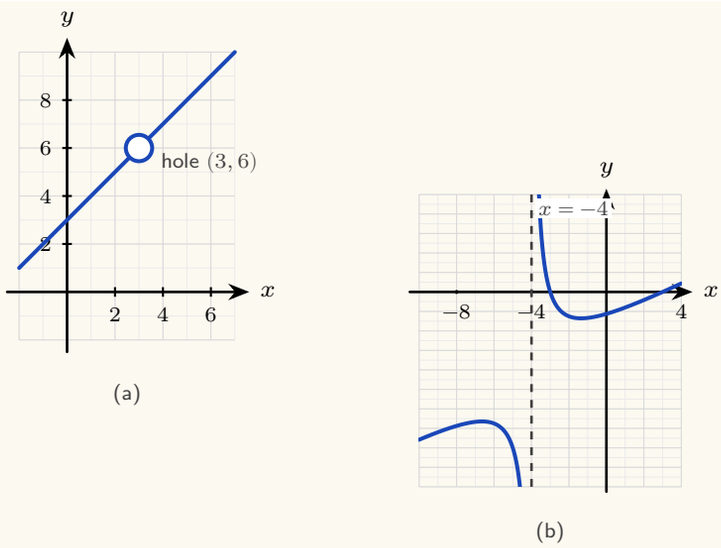


Figure 5.19 (a) $f(x) = \frac{x^2-9}{x-3}$: the common factor cancels, leaving the line $y = x + 3$ with one point missing. (b) $g(x) = \frac{x^2-9}{x+4}$: nothing cancels, so $x = -4$ is a vertical asymptote.

5.4.3 Oblique Asymptotes HL

When the numerator has a higher degree than the denominator, there is no horizontal asymptote. When the difference in degree is exactly one, the curve approaches a slanted line instead, and finding that line is a division problem.

Polynomial division splits the function into a linear part plus a fraction, the remainder divided by the denominator. That fraction approaches zero for large $|x|$, so the linear part is the **oblique** (slant) asymptote. The general method is in §6.2.2; here it is enough to write the numerator as a multiple of the denominator plus a remainder.

The simplest case is $y = x + \dfrac{1}{x}$. It is undefined at $x = 0$, so $x = 0$ is a vertical asymptote. For large $|x|$ the term $\dfrac{1}{x}$ approaches zero, so the curve approaches the oblique asymptote $y = x$.

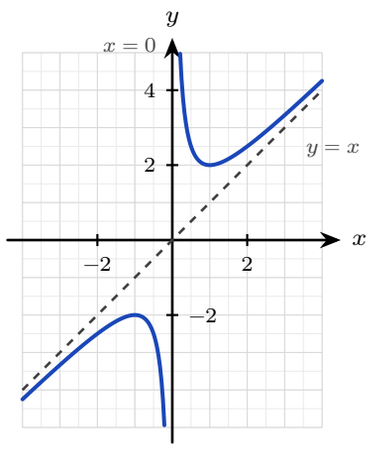


Figure 5.20 The two branches of $y = x + \frac{1}{x}$, each approaching the vertical asymptote $x = 0$ on one side and the oblique asymptote $y = x$ on the other.

In $x + \frac{1}{x}$ the split into a line plus a remainder was already written out. In general you produce it yourself.

EXAMPLE 5.24

Find all the asymptotes of $f(x) = \dfrac{x^2 - 2x + 3}{x - 1}$.

SOLUTION

Start from the denominator. It is zero at $x = 1$, so there is a vertical asymptote,

$$x = 1$$

The numerator's degree exceeds the denominator's by one, so there is an oblique asymptote. Multiply the denominator by a linear expression chosen to match the first two terms of the numerator: $(x - 1)(x - 1) = x^2 - 2x + 1$, which is 2 less than the numerator. So $x^2 - 2x + 3 = (x - 1)(x - 1) + 2$, and

$$\frac{x^2 - 2x + 3}{x - 1} = x - 1 + \frac{2}{x - 1}.$$

As $|x|$ grows the remainder term $\frac{2}{x-1}$ approaches zero, so the curve approaches the line

$$y = x - 1$$

which is the oblique asymptote.

TIP Show how the numerator was rewritten, not only the answer. The linear part *is* the asymptote, and the working can earn method marks.

5.4.4 The Reciprocal of Any Function HL

Every reciprocal so far has been the reciprocal of something simple: of x, of a linear expression, of one polynomial divided by another. The same reading works for the reciprocal of any function at all. Given a function f, its reciprocal is

$$y = \frac{1}{f(x)},$$

defined at every x where f is defined and $f(x) \neq 0$.

You do not need to calculate points to sketch it. Every feature of the reciprocal can be read from the graph of f, one feature at a time.

Where f does this	The reciprocal does this	Reason
$f(x) = 0$	vertical asymptote	you cannot divide by zero
$\lvert f(x)\rvert$ is large	the curve approaches the x-axis	a large number has a small reciprocal
$f(x) = 1$ or -1	the two graphs meet	1 and -1 are their own reciprocals
$f(x) > 0$	$\frac{1}{f(x)} > 0$	the sign is unchanged
a local maximum	a local minimum, and a minimum becomes a maximum	a large output has a small reciprocal

Where f is large the reciprocal is small, so a high point of f becomes a low point of $\frac{1}{f}$. This holds wherever f is not zero, whatever the sign of f.

CAUTION $\dfrac{1}{f(x)}$ is *not* $f^{-1}(x)$. The reciprocal divides 1 by the output; the inverse undoes the function. For $f(x) = x + 3$ the reciprocal is $\dfrac{1}{x+3}$ and the inverse is $x - 3$, which are not the same. The notation f^{-1} never means "one over f".

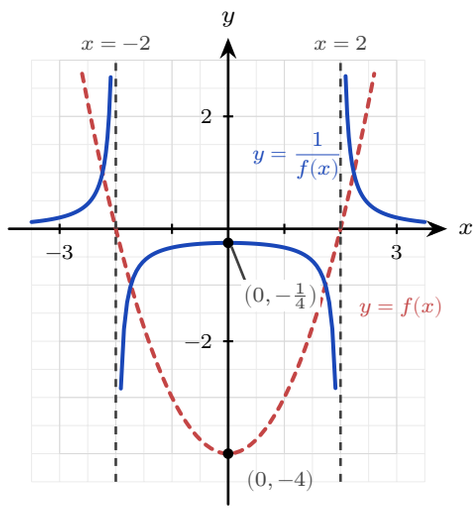


Figure 5.21 $y = 1/(x^2 - 4)$ against $y = x^2 - 4$. The zeros $x = \pm 2$ become asymptotes, the minimum $(0, -4)$ becomes the maximum $(0, -\frac{1}{4})$, and large values of f give values near zero.

EXAMPLE 5.25

The graph of $y = f(x)$ touches the x-axis at $x = 5$ and has no other x-intercept, it has a local minimum at $(2, 4)$, and $f(x) \to +\infty$ as $x \to \pm\infty$. Describe the corresponding features of $y = \dfrac{1}{f(x)}$.

SOLUTION

Take the features one at a time.

The zero at $x = 5$ makes the denominator zero, so $y = \dfrac{1}{f(x)}$ has a **vertical asymptote at** $x = 5$. Since the curve only touches the axis there, f is positive on both sides of $x = 5$, so $\dfrac{1}{f(x)} \to +\infty$ on both sides of that asymptote.

The minimum at $(2, 4)$ has $f(2) = 4$, so the reciprocal passes through $(2, \frac{1}{4})$. A minimum of f becomes a **maximum** of the reciprocal, so this is a local maximum at $(2, \frac{1}{4})$.

Between $x = 2$ and $x = 5$, f must also have a local maximum, because it increases from 4 and then comes back down to 0. That maximum becomes a local minimum of $\dfrac{1}{f(x)}$.

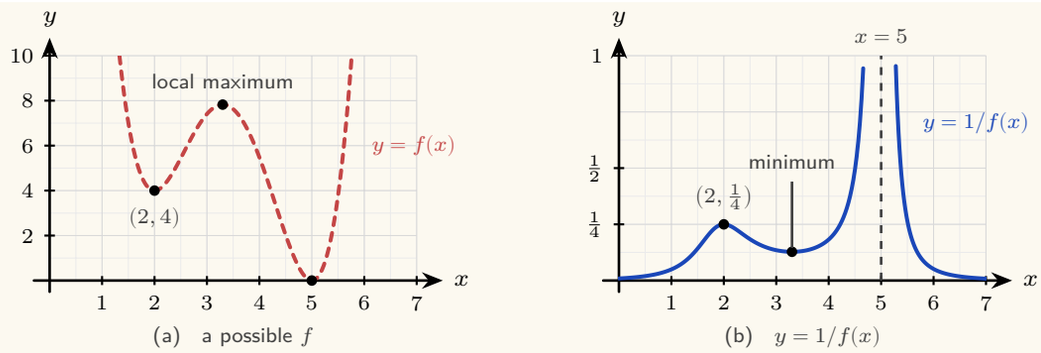


Figure 5.22 (a) One possible f with the given features: a minimum at $(2, 4)$, a local maximum between $x = 2$ and $x = 5$, and a zero at $x = 5$ where it touches the axis. (b) Its reciprocal: the asymptote $x = 5$ with both branches tending to $+\infty$, the maximum $(2, \frac{1}{4})$, and a local minimum from the maximum of f.

Since $f(x) \to \infty$ at both ends, $\dfrac{1}{f(x)} \to 0$, so $y = 0$ is a **horizontal asymptote**.

TIP To sketch $y = \dfrac{1}{f(x)}$ in an examination:

1. Draw a dashed vertical asymptote at each zero of f.
2. Mark the points where $f(x) = 1$ or $f(x) = -1$; they lie on both graphs.
3. Mark each turning point of f with its y-coordinate replaced by its reciprocal.
4. Draw the curve through those points, one branch at a time.

Four graphs in this chapter are built from $y = f(x)$ by one operation. They are easy to confuse, so here they are side by side.

Graph	Zeros of f	Where $f(x) < 0$	Turning points
$\lvert f(x)\rvert$	stay; a corner where f crosses	reflected up in the x-axis	a minimum below the axis becomes a maximum
$f(\lvert x\rvert)$	those with $x \geq 0$ stay and are mirrored	only the part with $x \geq 0$ is used	those with $x > 0$ are mirrored in the y-axis
$[f(x)]^2$	stay; the curve touches the axis	the output becomes positive	a minimum below the axis becomes a maximum
$\dfrac{1}{f(x)}$	become vertical asymptotes	the output stays negative	a maximum becomes a minimum, and the reverse

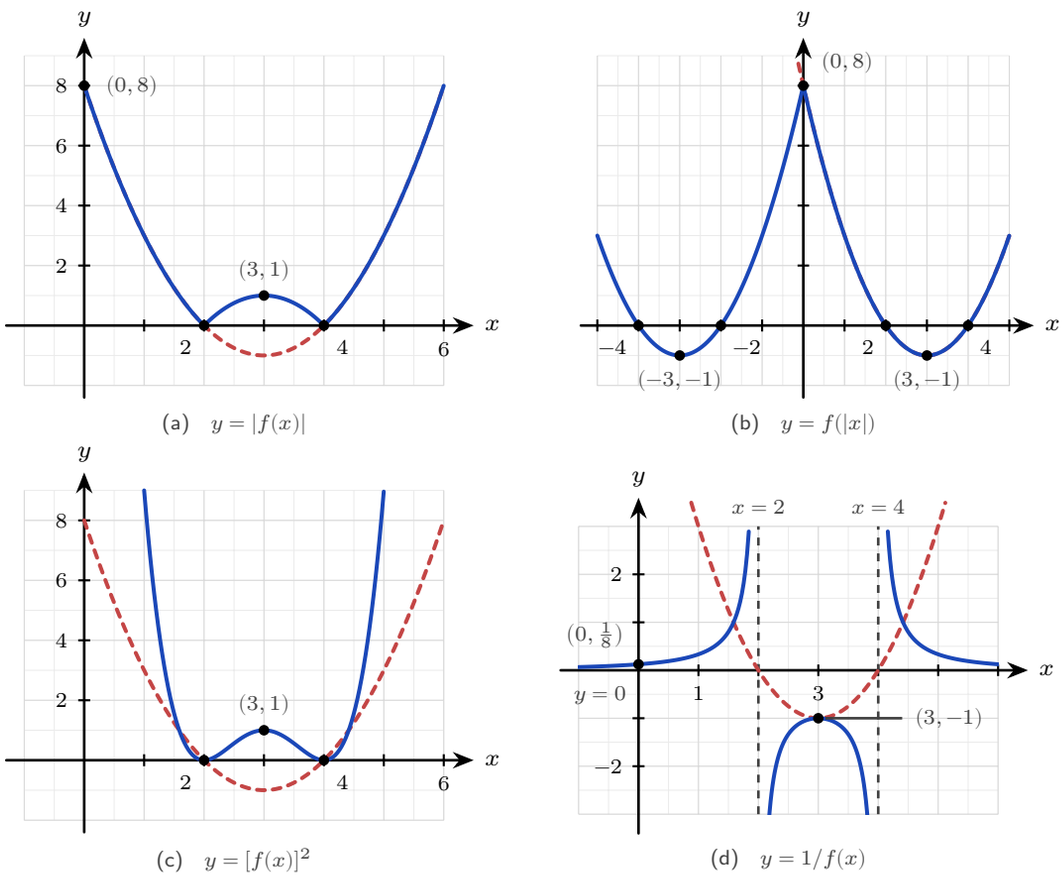


Figure 5.23 The four related graphs of $f(x) = x^2 - 6x + 8$ (dashed). Compare each panel with its row of the table.

5.4.5 Rational Inequalities HL

Solving an inequality means finding the intervals on which an expression is positive and those on which it is negative. A sketch shows the same information.

First rewrite it so that one side is zero: collect every term on one side, so that the inequality compares a single fraction with zero.

The rest is a **sign table**. A rational expression changes sign only where it is zero or where it is undefined, so those **critical values** divide the x-axis into intervals on which the sign never changes. Give the table one row for each factor and one column for each interval. Fill in the sign of every factor, then multiply down each column to get the sign of the whole expression. The last row is the answer, read interval by interval.

KEY 5.14 **Solving a rational inequality** HL

To solve a rational inequality:

1. Collect every term on one side, combine into a single fraction and factorise.
2. Mark the **critical values**, where the numerator or the denominator is zero.
3. Build a sign table on the intervals between them, and read the intervals with the sign you need.
4. Include a zero of the numerator when the inequality is $\geq$ or $\leq$; never include a zero of the denominator.

CAUTION Do not multiply both sides by an expression whose sign you do not know. Multiplying an inequality by a negative number reverses it, so multiplying by a denominator such as $x-2$ would split the question into two cases, one for each sign. Instead, collect every term on one side and combine into a single fraction.

The same table works when there is no fraction at all, so the quadratic inequalities of Ch. 4 can be solved this way too.

EXAMPLE 5.26

Solve $\dfrac{x+3}{x-2} \geq 2$.

SOLUTION

Collect everything on one side and combine into a single fraction:

$$\frac{x+3}{x-2} - 2 \geq 0 \implies \frac{(x+3)-2(x-2)}{x-2} \geq 0 \implies \frac{7-x}{x-2} \geq 0.$$

Now the critical values. The numerator is zero when $7-x=0$, that is $x=7$. The denominator is zero when $x=2$. These two values divide the x-axis into three intervals.

Interval	$x<2$	$2<x<7$	$x>7$
$7-x$	$+$	$+$	$-$
$x-2$	$-$	$+$	$+$
$\frac{7-x}{x-2}$	$-$	$+$	$-$

Only the middle interval gives the sign required. Now decide the two endpoints. Exclude $x=2$, because the fraction is undefined there. Include $x=7$, because the inequality allows equality and the fraction is zero there. So

$$2 < x \leq 7$$

A graph agrees: $y = \dfrac{x+3}{x-2}$ lies above the line $y = 2$ only between the asymptote and the crossing point. The boundary of the solution is a vertical asymptote at one end and a crossing at the other, and both appear as critical values in the sign table.

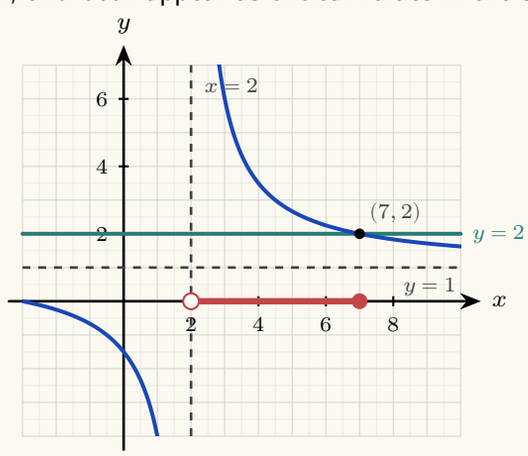


Figure 5.24 $y = \dfrac{x+3}{x-2}$ lies on or above $y = 2$ only for $2 < x \leq 7$ (red): the asymptote $x = 2$ is excluded, and the crossing point $x = 7$ is included.

TIP A GDC can solve an equation or inequality of this kind numerically once every term is on one side. Rewrite $f(x) = g(x)$ as $f(x) - g(x) = 0$, graph the single function $y = f(x) - g(x)$, and read its zeros; for an inequality, read the intervals where that curve is above or below the axis. This is an alternative to finding where two graphs meet. Both methods work when the algebra of this course cannot solve an equation, such as $e^x = 3 - x$.

YOUR TURN 5.4 Reciprocal and Rational Functions

Linear over linear

30. State the vertical and horizontal asymptotes of each function.

(a) $f(x) = \dfrac{1}{x-3}$

(b) $f(x) = \dfrac{3}{x+2}$

(c) $f(x) = \dfrac{2x}{x+1}$

(d) $f(x) = \dfrac{x+1}{x-2}$

(e) $f(x) = \dfrac{5}{2x-1}$

(f) $f(x) = \dfrac{4x+1}{2x-3}$

31. For each function, state the domain, the range, both asymptotes and both intercepts. Where an intercept does not exist, say why.

(a) $f(x) = \dfrac{1}{x-5}$

(b) $f(x) = \dfrac{1}{x} + 4$

(c) $f(x) = \dfrac{x-1}{x+2}$

(d) $f(x) = \dfrac{2x+3}{x-1}$

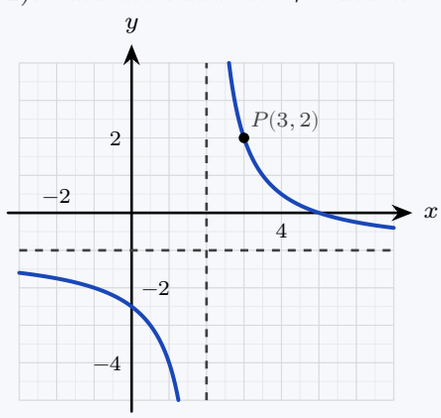
32. Consider $f(x) = \dfrac{3x-6}{x+1}$.

(a) Write $f(x)$ in the form $A + \dfrac{B}{x+1}$.

(b) Hence state the point at which the two asymptotes cross.

(c) Sketch the curve, showing both asymptotes and both intercepts.

(d) The graph of $y = \dfrac{a}{x-b} + c$ is shown, with its two asymptotes dashed. It passes through the point $P(3,2)$. Find the values of a, b and c.



Asymptotes and holes HL

33. For each function, state every vertical asymptote, and state the kind and the equation of the asymptote as $x \to \pm\infty$.

(a) $f(x) = \dfrac{1}{(x-1)(x+3)}$

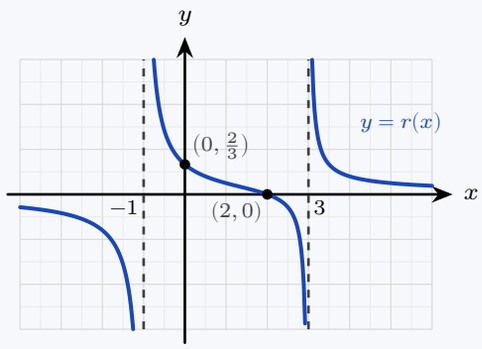
(b) $f(x) = \dfrac{x}{x^2-9}$

(c) $f(x) = \dfrac{x^2+1}{x^2-4}$

(d) $f(x) = \dfrac{x^2+1}{x-2}$

(e) $f(x) = \dfrac{x^2-9}{x+2}$

(f) The graph of $y = r(x)$ is shown, where $r(x)$ is a polynomial divided by a polynomial. The vertical asymptotes are dashed, and the two intercepts are marked.



Write down the equations of the three asymptotes. Hence find a possible expression for $r(x)$, and use the y-intercept to fix any constant factor.

34. For each function, describe the graph at every value of x where the denominator is zero, and state the domain.

(a) $f(x) = \dfrac{x^2 - 4}{x - 2}$

(b) $f(x) = \dfrac{x^2 + x - 6}{x + 3}$

(c) $f(x) = \dfrac{x^2 - 1}{x^2 + x}$

(d) $f(x) = \dfrac{2x + 4}{x + 2}$

Oblique asymptotes HL

35. Write each function as a linear expression plus a fraction (the remainder divided by the denominator), and state the oblique asymptote.

(a) $f(x) = \dfrac{x^2 + 3}{x}$

(b) $f(x) = \dfrac{x^2}{x - 1}$

(c) $f(x) = \dfrac{2x^2 - x + 3}{x - 1}$

(d) $f(x) = \dfrac{x^2 + 4x + 1}{x + 1}$

36. Consider $f(x) = x + \dfrac{2}{x}$.

(a) State the vertical asymptote.

(b) State the oblique asymptote.

(c) Explain why the curve never crosses the oblique asymptote.

(d) Sketch the two branches.

37. Consider $f(x) = x + \dfrac{1}{x}$.

(a) State the vertical asymptote.

(b) State the oblique asymptote.

(c) State whether each branch lies above or below the oblique asymptote, and give a reason.

The reciprocal of a function HL

38. For each function f, sketch $y = \dfrac{1}{f(x)}$, showing every asymptote and the coordinates of any turning point.

(a) $f(x) = x - 4$ (b) $f(x) = x^2 - 1$

(c) $f(x) = x^2 + 1$ (d) $f(x) = (x-1)(x-3)$

39. The graph of $y = f(x)$ crosses the x-axis at $x = -2$ and at $x = 2$ and has no other x-intercept. It has a local maximum at $(0, 4)$, and $f(x) \to -\infty$ as $x \to \pm\infty$.

(a) State the equations of the vertical asymptotes of $y = \dfrac{1}{f(x)}$.

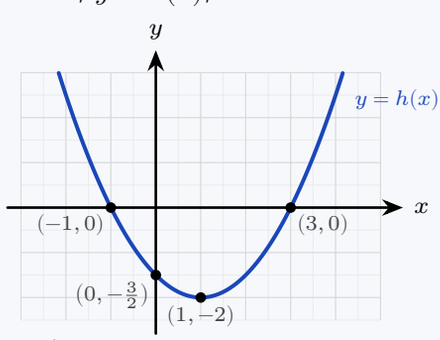
(b) Find the coordinates of the turning point of $y = \dfrac{1}{f(x)}$ at $x = 0$, and state its type.

(c) Describe the behaviour of $y = \dfrac{1}{f(x)}$ as $x \to \pm\infty$.

(d) Given that $f(x) = 4 - x^2$, find the x-coordinates of the points where $y = f(x)$ and $y = \dfrac{1}{f(x)}$ meet.

(e) Another function g touches the x-axis at $x = 3$ and has no other x-intercept. It has a local minimum at $(0, 2)$, and $g(x) \to +\infty$ as $x \to \pm\infty$. Explain why g must also have a local maximum between $x = 0$ and $x = 3$, and state what this point becomes on the graph of $y = \dfrac{1}{g(x)}$.

(f) The graph of a third function, $y = h(x)$, is shown.



Sketch the graph of $y = \dfrac{1}{h(x)}$, showing its asymptotes and the coordinates of its turning point and its y-intercept.

40. Let $f(x) = x - 2$.

(a) On one set of axes, sketch $y = f(x)$ and $y = \dfrac{1}{f(x)}$.

(b) Find the coordinates of the two points where the graphs meet. *(Hint: they meet where $f(x) = \pm 1$)*

(c) State the interval on which $\dfrac{1}{f(x)} > f(x)$ for $x > 2$.

Rational inequalities HL

41. Solve each inequality.

(a) $x^2 \leq x + 6$

(b) $x^2 > 2x + 3$

(c) $\dfrac{x-4}{x+1} \geq 0$

(d) $\dfrac{3}{x-2} > 1$

(e) $\dfrac{x+2}{x-1} \geq 3$

(f) $\dfrac{2x-1}{x+2} \leq 1$

42. Solve $x^2 \leq 3x + 10$ using a sign table.

Concept check

43. A student solves $\dfrac{x+1}{x-3} \geq 2$ by multiplying both sides by $x-3$: "$x+1 \geq 2x-6$, *so* $x \leq 7$." Identify the error and give the correct solution.

44. A student says: "*a graph never crosses its horizontal asymptote.*" Give a counter-example, and explain why the claim is false.

5.5 Exponential and Logarithmic Functions

Pour a cup of coffee at $90\,^{\circ}\text{C}$ and set it down in a room at $20\,^{\circ}\text{C}$. It cools quickly at first, then more slowly, then so slowly that you can barely measure the change. Look at the numbers rather than the coffee. In each fixed interval of time the difference between the cup and the room falls by the same *fraction*, not by the same amount. Isaac Newton measured that and found the rule. No polynomial behaves this way.

An **exponential function** $y = a^x$ does. In it, the variable is the exponent, not the base. As a result, an exponential grows at a rate proportional to its own size, so a quantity twice as large grows twice as fast. For a step of one unit, $a^{x+1} - a^x = (a-1)\,a^x$, so the change over the next unit of x is a fixed multiple of the current value. The same kind of function, translated where needed, describes cooling coffee, growing populations and decaying atoms. Ch. 1 sets up the algebra of exponents and logarithms. This section is about their graphs.

The base is always positive and never 1. For $a > 1$ the graph rises, and rises more and more steeply: this is **growth**. For $0 < a < 1$ it falls towards zero: this is **decay**. Either way the graph passes through $(0, 1)$, stays positive everywhere, and approaches the horizontal asymptote $y = 0$ on one side. The domain is all real numbers and the range is $y > 0$.

The graph passes through $(0, 1)$ because $a^0 = 1$. A power of a positive number is positive, so the graph stays above the x-axis. For $a > 1$, each step of 1 to the left divides the output by a, so the outputs approach zero and do not reach it. For $0 < a < 1$ the same happens to the right.

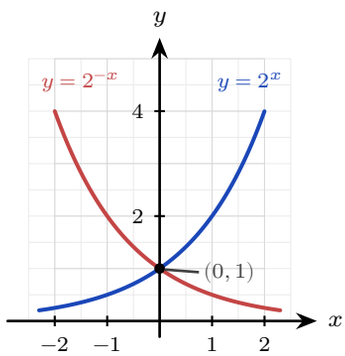


Figure 5.25 Growth and decay are reflections of each other in the y-axis. Both pass through $(0, 1)$ and approach $y = 0$, but on opposite sides.

Calculus usually uses one base: $e \approx 2.718$, the base whose curve has steepness equal to its height at every point. Because of this property, the derivative of e^x is e^x itself, with no extra factor. It is also why every other exponential is usually rewritten in terms of e. This course does not prove the property; §13.5 reads it from the graph of $y = e^x$.

FORMULA 5.15 **Exponentials in base e** FORMULA BOOKLET · 1.7

$$a^x = e^{x \ln a}$$

Valid for $a > 0$. Here $\ln a = \log_e a$, the natural logarithm (Ch. 1).

This follows because the exponential and the logarithm undo each other: $\log_a(a^x) = x$ and $a^{\log_a x} = x$ (Ch. 1). So a^x and $\log_a x$ are inverse functions, as §5.5.3 explains. Taking base e in the second identity gives $a = e^{\ln a}$. Raise both sides to the power x and apply the power-of-a-power law of Ch. 1:

$$a^x = \left(e^{\ln a}\right)^x = e^{x \ln a}.$$

EXAMPLE 5.27

For $f(x) = 5(0.4)^x$, decide whether the function grows or decays, and state its domain, range and horizontal asymptote.

SOLUTION

Look at the base first. Here $a = 0.4$, and $0 < 0.4 < 1$, so the function **decays**. The factor 5 stretches the graph vertically. It moves the y-intercept to $(0, 5)$ but does not change the domain, the range or the asymptote. The outputs of $(0.4)^x$ are always positive. So the domain is $x \in \mathbb{R}$, the range is $y > 0$, and the horizontal asymptote is $y = 0$

5.5.1 Transforming an Exponential Graph

An exponential often appears in a model as

$$y = k\,a^x + c,$$

and each constant has a separate effect. The factor k stretches the graph vertically. If $k < 0$ the graph is also reflected in the x-axis, so it lies below its asymptote. The constant c translates it vertically. Here a is the base, so these letters are not those of §5.2.

The vertical translation is the one that changes the asymptote. The graph of a^x approaches $y = 0$, so adding c makes it approach $y = c$ instead. **The horizontal asymptote moves with the graph.**

EXAMPLE 5.28

Sketch $y = 3(2)^x - 5$, stating the asymptote and both intercepts.

SOLUTION

Start with the asymptote. The -5 translates the whole graph down by 5, so the horizontal asymptote is $y = -5$, not $y = 0$.
For the y-intercept, substitute $x = 0$:

$$y = 3(2)^0 - 5 = 3 - 5 = -2.$$

For the x-intercept, set $y = 0$ and solve:

$$3(2)^x = 5 \implies 2^x = \tfrac{5}{3} \implies x = \log_2 \tfrac{5}{3} \approx 0.737.$$

The curve therefore rises from just above $y = -5$ on the left. It crosses the axes at $(0, -2)$ and at $(0.737, 0)$, then rises steeply.

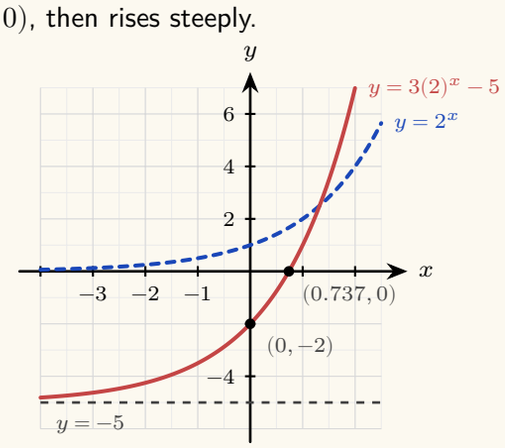


Figure 5.26 $y = 3(2)^x - 5$: the asymptote moves down with the graph, from $y = 0$ to $y = -5$.

5.5.2 Exponential Models

Many real quantities are modelled with base e and time as the variable:

$$A(t) = A_0\, e^{kt},$$

where A_0 is the amount at $t = 0$. If $k > 0$ the model shows growth, and if $k < 0$ it shows decay. A larger $|k|$ gives faster change. Ch. 1 solves these models algebraically; what is added here is reading them off a graph.

Two data points determine the whole model. If one point is at $t = 0$, its value is A_0. For the second, substitute its t and A into $A = A_0 e^{kt}$, divide by A_0 and take logarithms:

$$k = \frac{1}{t} \ln\left(\frac{A}{A_0}\right).$$

If neither point is at $t = 0$, write $A = A_0 e^{kt}$ for each point and divide one equation by the other. A_0 cancels, leaving an equation in k alone.

CAUTION A model fitted to two points assumes the percentage change per unit of time stays the same. If the real rate is itself changing, the prediction is less reliable. Report such a figure as what would happen if nothing changed, not as a forecast.

EXAMPLE 5.29

Japan's population was about 128.1 million in 2010 and about 126.1 million in 2020. Model it as $P(t) = P_0 e^{kt}$, with t in years after 2010, and use the model to estimate the population in 2030.

SOLUTION

Take $P_0 = 128.1$, the value at $t = 0$, then use the 2020 figure to find k. Substitute $t = 10$ and $P = 126.1$:

$$126.1 = 128.1\, e^{10k} \implies k = \frac{1}{10} \ln\left(\frac{126.1}{128.1}\right) \approx \boxed{-0.00157}$$

The sign is negative, which confirms decay: the population falls by about 0.16% each year. For 2030, substitute $t = 20$:

$$P(20) = 128.1\, e^{-0.00157 \times 20} \approx \boxed{124.1 \text{ million}}$$

Japan's rate of decline is itself increasing, so official projections for 2030 are lower than this figure.

5.5.3 The Logarithm as an Inverse

Since $y = a^x$ is one-to-one, it has an inverse, and that inverse is the logarithm $y = \log_a x$. It is the answer to the reversed question: not "what is a to the x?" but "to what power must I raise a to get x?" Everything about its graph follows from being the mirror image of the exponential in the line $y = x$.

Reflecting swaps the axes' roles. The exponential's horizontal asymptote $y = 0$ becomes the logarithm's *vertical* asymptote $x = 0$. The exponential's domain, all real numbers, becomes the logarithm's range. Its range, the positive numbers, becomes the logarithm's domain. So $\log_a x$ has domain $x > 0$ and range all real numbers; for $a > 1$ it passes through $(1, 0)$ and rises without bound, more and more slowly.

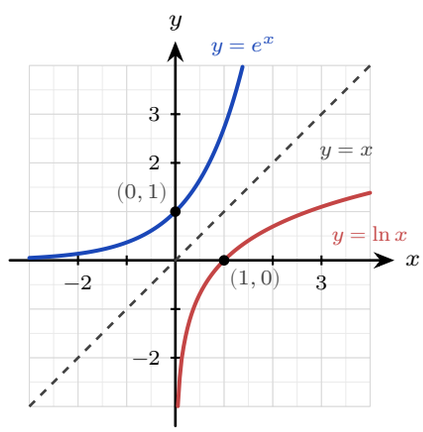


Figure 5.27 The logarithm is the exponential reflected in $y = x$, so $(0, 1)$ on $y = e^x$ becomes $(1, 0)$ on $y = \ln x$, and the horizontal asymptote becomes vertical.

Finding the domain

A logarithm is defined only for positive inputs. So every logarithmic function has a condition on its domain that the formula does not show. To find the domain, set whatever sits inside the logarithm greater than zero, and solve. For $\log(5 - 2x)$, add $2x$ to both sides rather than dividing by -2, so the inequality cannot be reversed by mistake: $5 - 2x > 0$ gives $5 > 2x$, so $x < \frac{5}{2}$.

Transforming a logarithmic graph

The same translations apply as for any function. A horizontal translation moves the *vertical* asymptote; a vertical translation leaves it where it is.

In $y = \ln(x - h)$ the change is inside the bracket, so it moves the graph h units right. The asymptote moves with it, from $x = 0$ to $x = h$.

The x-intercept moves with it too. A logarithm is zero exactly when its input is 1, so $\ln(x - h) = 0$ when $x - h = 1$, that is at $x = h + 1$.

EXAMPLE 5.30

Sketch $y = \ln(x - 5)$, stating its asymptote, its domain and its x-intercept.

SOLUTION

The input must be positive, so $x - 5 > 0$ and the domain is $x > 5$.
That same condition gives the asymptote. As x approaches 5 from above, $x - 5$ approaches 0 and $\ln(x - 5)$ decreases without bound, so $x = 5$ is a vertical asymptote.
For the x-intercept, set $y = 0$. A logarithm is zero when its input is 1:

$$x - 5 = 1 \implies x = 6.$$

So the curve rises from the asymptote at $x = 5$, crosses the axis at $(6, 0)$, and continues upward more and more slowly.

Using the inverse relationship

The exponential and the logarithm undo each other. So a logarithm solves for a variable in an exponent, and an exponential solves for a variable inside a logarithm. This is the extra step needed in the method of §5.1.4: at step 3, where you solve for y, apply a logarithm if y sits in an exponent and an exponential if y sits inside a logarithm.

EXAMPLE 5.31

Find the inverse of $f(x) = 2e^{3x} - 5$, and state its domain.

SOLUTION

Write $y = f(x)$ and exchange the letters:

$$x = 2e^{3y} - 5 \implies \frac{x + 5}{2} = e^{3y}.$$

The variable is now in an exponent, so apply the natural logarithm to both sides:

$$3y = \ln\left(\frac{x + 5}{2}\right) \implies \boxed{f^{-1}(x) = \tfrac{1}{3}\ln\left(\frac{x + 5}{2}\right)}$$

Now the domain. The domain of f^{-1} is the range of f. Since $e^{3x} > 0$ for every x, we have $2e^{3x} > 0$ and so $f(x) > -5$. The domain of f^{-1} is therefore $\boxed{x > -5}$, which agrees with requiring $\frac{x+5}{2} > 0$ inside the logarithm.

TIP State the domain whenever a question asks for an inverse involving a logarithm. It often earns a separate mark, and a quick check is that **the domain of f^{-1} is the range of f.**

5.5.4 Equations Solved Numerically

Some equations cannot be solved by the algebra of this course. In $\ln x = x - 2$ the unknown appears inside a logarithm and outside it, and taking a logarithm or an exponential of both sides does not bring x into one place. An equation of this kind has no **analytic** (exact, by algebra) method within this course, so it is solved from a graph or with technology. The same is true of an equation such as $e^x = \sin x$ (Ch. 8).

The written answer has three parts: the equation, a sketch, and the roots.

EXAMPLE 5.32

Solve $\ln x = x - 2$.

SOLUTION

Write the equation down, then graph $y = \ln x$ and $y = x - 2$ on a GDC for $0 < x \leq 5$ and copy the sketch. The logarithm curve rises steeply from its asymptote $x = 0$ and then flattens, while the line rises at a steady rate. So the line is above the curve near $x = 0$. At $x = 1$ it is below the curve, since $\ln 1 = 0 > -1$. For large x it is above the curve again. The two graphs therefore cross twice. The GDC gives the x-coordinates of the two crossing points, and these are the roots:

$$x \approx 0.159 \quad \text{or} \quad x \approx 3.15$$

Check one of them: $\ln 3.15 \approx 1.147$ and $3.15 - 2 = 1.15$. Give such roots to three significant figures unless the question says otherwise, and mark the crossing points on the sketch.

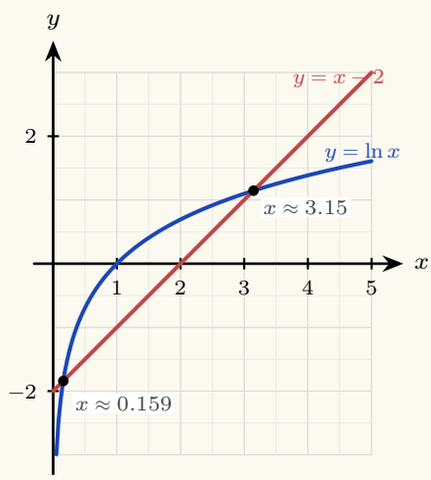


Figure 5.28 The sketch that goes with the answer: $y = \ln x$ and $y = x - 2$ cross twice, at $x \approx 0.159$ and $x \approx 3.15$.

The sketch is part of the answer. It shows how many roots there are, so that none is missed, and it is the evidence for a root that cannot be found by algebra.

Copying a graph from the screen to paper needs care, because the screen usually shows no axis labels or scale.

1. Draw and label both axes, with a scale that shows every key feature in the domain asked for.
2. Mark each intercept and each turning point with its coordinates, to three significant figures where they are not exact.
3. Draw each asymptote as a dashed line, with its equation beside it.
4. Join the marked points with a smooth curve of the correct shape, and stop the curve at the ends of the domain.

A GDC also graphs a sum or a difference of two functions. Enter f and g, then enter a third function as $f(x)+g(x)$ or $f(x)-g(x)$; the calculator adds or subtracts the two heights at every x.

EXAMPLE 5.33

Sketch $y=\ln x-(x-2)$ for $0<x\leq 5$, the difference of the two functions in the example above.

SOLUTION

Enter it as a third function and read the features from the screen. The GDC gives the turning point $(1,1)$, and $\ln 1-(1-2)=1$ confirms it. It also gives the x-intercepts $(0.159,0)$ and $(3.15,0)$. These are the two roots found above, because the difference is zero exactly where the two graphs cross. There is no y-intercept: the curve falls without bound as x approaches 0, so $x=0$ is a vertical asymptote. At the right-hand end, $y=\ln 5-3\approx -1.39$. On paper, mark the points $(0.159,0)$, $(1,1)$, $(3.15,0)$ and $(5,-1.39)$, and label the asymptote $x=0$. Then draw one smooth curve. It rises steeply from beside the asymptote to the maximum, then falls through the second intercept to the end point.

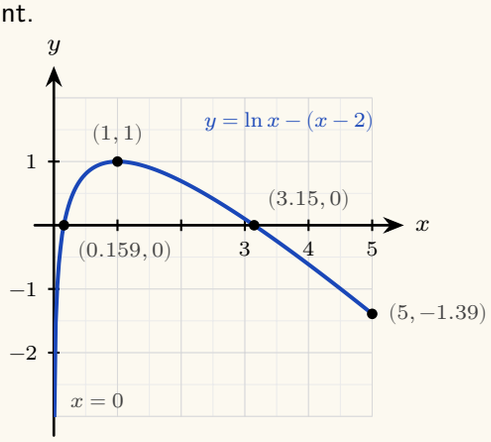


Figure 5.29 The difference $\ln x-(x-2)$ is zero exactly where the two graphs in the previous example cross.

YOUR TURN 5.5 Exponential and Logarithmic Functions

Exponential and logarithmic graphs

45. Evaluate each expression.

(a) 2^{-2}
(b) e^0
(c) $\log_2 8$
(d) $\log_5 125$
(e) $\ln 1$
(f) $\ln e$

46. State whether each function grows or decays, and give a reason.

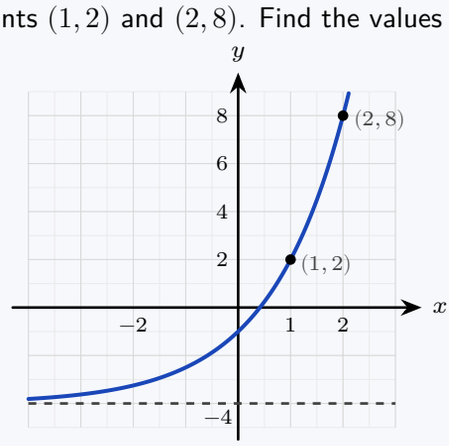
(a) $y = (0.5)^x$
(b) $y = 3^x$
(c) $y = 5e^{-0.2x}$
(d) $y = 2^{-x}$

47. For each graph, state its asymptote and the coordinates of every point where it crosses an axis.

(a) $y = 5 \cdot 2^x$
(b) $y = 2^x + 5$
(c) $y = 4(3)^x - 1$
(d) $y = \ln x$
(e) $y = \ln(x - 2)$
(f) $y = \log(x + 5)$

48. Sketch each graph, stating its asymptote and the exact coordinates of both intercepts.

(a) $y = 2(3)^x - 6$
(b) $y = 8 - 2^{x+1}$
(c) $y = 5e^{-x} - 1$
(d) The graph of $y = a \times b^x + c$, where $b > 0$, is shown, with its asymptote dashed. It passes through the points $(1, 2)$ and $(2, 8)$. Find the values of a, b and c.



Base e and exponential equations

49. Write each in the required form.

(a) 2^x in terms of base e
(b) 5^x in terms of base e

50. Solve each equation, giving exact answers.

(a) $2^x = 8$
(b) $\log_3 x = 2$
(c) $5 \cdot 3^x = 45$
(d) $\ln x = 3$
(e) $3(2)^x = 4$
(f) $\log_2(x-1) = 4$

Exponential models

51. GDC Two exponential models.

(a) A population is modelled by $P(t) = 800e^{0.03t}$, with t in years. State the initial population $P(0)$, and say whether the population grows or decays.
(b) Find the population after 10 years.
(c) Find how long it takes the population to reach 1200.
(d) A radioactive sample decays according to $A(t) = 50e^{-0.2t}$, with t in days and A in grams. Find $A(6)$, correct to three significant figures.
(e) Find the half-life of the sample.

52. GDC A colony of bacteria numbers 500 at $t = 0$ and 1800 after 4 hours. It is modelled by $N(t) = N_0 e^{kt}$, with t in hours.

(a) Find the exact value of k, and give it correct to three significant figures.
(b) Estimate the size of the colony after 6 hours.
(c) Find how long it takes the colony to reach $10\,000$.
(d) A second colony follows the same kind of model. It numbers 600 after 2 hours and 1500 after 5 hours. Find its value of k and its initial size N_0.

53. GDC A machine is worth $24\,000$ dollars when new and $15\,000$ dollars three years later. Its value after t years is modelled by $V(t) = V_0 e^{kt}$.

(a) Find k, correct to three significant figures, and explain what its sign tells you.
(b) Find the value of the machine after 5 years.
(c) Find the time it takes for the value to halve.

54. GDC After a sauna is switched off, its temperature in °C after t minutes is $T(t) = 20 + 70e^{-0.12t}$.

(a) State the temperature at the moment it is switched off.
(b) Find the temperature after 5 minutes.
(c) Find how long it takes to cool to $40\,^\circ$C.
(d) State the temperature the sauna approaches, and explain what it represents.

Domains and inverses

55. State the domain of each function.

(a) $y = \ln(3x + 1)$
(b) $y = \log(10 - 2x)$
(c) $y = \ln(2x - 6)$
(d) $y = \ln(x + 1) + \ln(3 - x)$
(e) $y = \ln(x^2 - 4x)$
(f) $y = \ln(x^2 - 1)$

56. Find the inverse of each function, and state its domain.

(a) $f(x) = e^x + 2$

(b) $f(x) = \ln(x - 1)$

(c) $f(x) = 3e^{2x} + 1$

Equations that need a graph

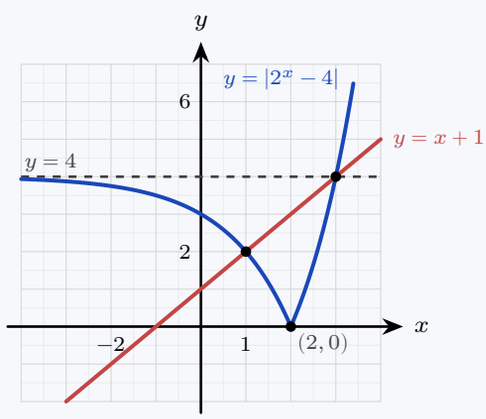
57. GDC Consider the equation $e^x = 3 - x$.

(a) Explain why it has exactly one real solution.

(b) Find that solution, correct to three significant figures.

(c) State why a graph is the appropriate method for this equation.

(d) The graphs of $y = |2^x - 4|$ and $y = x + 1$ are shown. They meet at $(1, 2)$ and $(3, 4)$, and the line $y = 4$ is an asymptote of the first graph. Use the graphs to solve $|2^x - 4| > x + 1$.



(e) Use the graph of $y = |2^x - 4|$ in part (d) to write down the number of solutions of the equation $|2^x - 4| = k$ when (i) $k = 0$, (ii) $k = 2$, (iii) $k = 5$. Then state the values of k for which the equation has exactly two solutions.

58. GDC Use your GDC to answer each part. In each case write down the equation or function you graphed, and give answers to three significant figures where they are not exact.

(a) For $y = x^3 - 3x + 1$, find the coordinates of the two turning points, and solve $x^3 - 3x + 1 = 0$.

(b) Solve $x \ln x = 1$.

(c) Find the coordinates of the points where the graphs of $y = 2^x$ and $y = x + 3$ meet.

Concept check

59. A student states that the graph of $y = 3 \times 2^x - 5$ has the asymptote $y = 0$ and the y-intercept $(0, -5)$. Identify both errors, and give the correct asymptote and y-intercept.

Chapter 5 · Functions & Graphs

Reflections

TOK Slide the graph of $y = x^2$ two units right and you get $y = (x-2)^2$. Is this the same function or a different one? It has the same shape, the same name ("a parabola"), yet a different rule, and different values at every input except $x = 1$. When we call two things "the same" in mathematics, we are choosing what to ignore. In a transformation you can see that choice: the shape is kept and the position is ignored.

IM At twenty-three, the German mathematician Felix Klein proposed a new way to organise geometry. In his view, a geometry *is* a set of transformations together with the properties they leave unchanged. Euclidean geometry studies the properties that stay unchanged under translations, rotations and reflections; other geometries allow other moves. The transformations of this chapter are a small example of the same idea. The question "what stays the same when I move this?" has guided much of geometry since Klein. *Try it:* which features of $y = x^2$ stay unchanged under a translation? Which stay unchanged under a vertical stretch?

RW Hikers in the Alps once found a body frozen in the ice. The man is now called Ötzi. He died about 5300 years ago, and we know this from exponential decay. The amount of carbon-14 in a sample halves every 5730 years, so after t years the fraction left is $\left(\frac{1}{2}\right)^{t/5730}$. For $t = 5300$ this simple model gives about 0.53: samples from him hold just over half the carbon-14 of living tissue, and measuring that fraction dates him. The same exponential, with a growth factor in place of $\frac{1}{2}$, describes compound interest. When growth has an upper limit, a different model is used: the logistic curve, which describes, for example, how a rumour spreads. *Try it:* what fraction of its carbon-14 is left in a bone 10 000 years old?

EXT A self-inverse function (§5.3.2) is also called an **involution**. Examples are $f(x) = \dfrac{1}{x}$, $f(x) = 5 - x$ and every $f(x) = \dfrac{ax + b}{cx - a}$ with $c \neq 0$ and $a^2 + bc \neq 0$. The quarter circle $f(x) = \sqrt{1 - x^2}$, $0 \leq x \leq 1$, is another. Can you explain from its graph why it must be its own inverse, and find an involution of your own that is neither a line nor a hyperbola?

TOK Name something in your life that behaves like an asymptote. Is the comparison mathematics, or poetry?

End-of-Chapter Problems
Chapter 5: Functions & Graphs

1. Let $f(x) = 2x - 3$ and $g(x) = x^2$.

(a) Find $(f \circ g)(x)$. [2]
(b) Find $(g \circ f)(x)$. [2]
(c) Solve $(f \circ g)(x) = 5$. [2]

2. Let $f(x) = (x+1)^2 - 4$ for $x \geq -1$.

(a) Explain why f has an inverse on this domain. [1]
(b) Find $f^{-1}(x)$, stating its domain. [4]

3. Let $f(x) = \dfrac{x}{1+x}$, $x \neq -1$.

(a) Find $(f \circ f)(x)$, simplifying your answer fully. [3]
(b) Hence find $(f \circ f \circ f)(x)$. [2]
(c) Find $f^{-1}(x)$. [2]
(d) State the domain of $f \circ f$. [2]

4. Let $g(x) = 3 + \sqrt{x-2}$, $x \geq 2$.

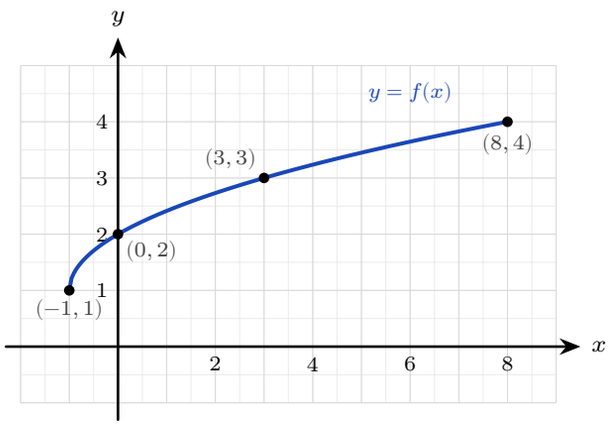
(a) State the range of g. [1]
(b) Find $g^{-1}(x)$, and state its domain. [3]
(c) Sketch the graphs of $y = g(x)$ and $y = g^{-1}(x)$ on the same axes, together with the line $y = x$. [2]
(d) Hence solve $g(x) = g^{-1}(x)$, giving your answer in exact form. [3]

5. GDC Oil leaking from a ship spreads out as a circle. The radius of the circle t hours after the leak starts is $r(t) = 2 + 1.5\sqrt{t}$ km, and the area of a circle of radius r is $A(r) = \pi r^2$.

(a) Find an expression for $(A \circ r)(t)$, and state what it represents. [2]
(b) Find the area of the oil spill after 4 hours. [1]
(c) Find the time at which the area of the oil spill reaches 100 km^2. [2]
(d) Find $r^{-1}(t)$, and state what it represents. [2]

Give your answers to three significant figures.

6. The graph of $y = f(x)$, for $-1 \leq x \leq 8$, is shown.



(a) Write down the value of $f(3)$. [1]

(b) Let $g(x) = 2x + 1$. Find $(f \circ g)(1)$. [2]

(c) Write down the value of $f^{-1}(2)$. [1]

(d) Write down the range of f. [1]

(e) Sketch the graph of $y = f^{-1}(x)$. Give the coordinates of its end points and of the images of $(0, 2)$ and $(3, 3)$. [3]

(f) Given that $f(x) = \sqrt{x + 1} + 1$, solve the equation $f(x) = g(x)$. [4]

7. The graph of $y = f(x)$ has a maximum point at $(1, 4)$. State the coordinates of the maximum of:

(a) $y = f(x) + 2$; [1]

(b) $y = f(x - 3)$; [1]

(c) $y = 2f(x)$; [1]

(d) $y = -f(x)$ (state the resulting minimum). [1]

8. For each quadratic, write $f(x)$ in vertex form and describe the transformations that map $y = x^2$ onto $y = f(x)$.

(a) $f(x) = x^2 - 4x + 5$; state also the range of f. [3]

(b) $f(x) = x^2 - 6x$; state also the vertex. [2]

9. The graph of $y = f(x)$ passes through the points $A(0, 1)$, $B(2, 0)$ and $C(3, 4)$.

(a) Find the images of A, B and C on the graph of $y = f(x + 1) - 2$. [2]

(b) Describe the transformations that map $y = f(x)$ onto $y = f(2x - 2)$, and find the images of A, B and C on the graph of $y = f(2x - 2)$. HL [3]

(c) The graph of $y = f(x)$ is reflected in the x-axis and then translated 3 units up. Write down the equation of the new graph in terms of f, and find the image of C. [2]

10. Determine whether each function is odd, even, or neither.

(a) $f(x) = x^3 - 2x$ [1] (b) $f(x) = |x|$ [1]

(c) $f(x) = x^2 + x^3$ [1] (d) $f(x) = \dfrac{x}{x^2+1}$ [1]

(e) $f(x) = \dfrac{1}{x^2+1}$ [1] (f) $f(x) = x + 1$ [1]

11. Solve each equation.

(a) $|x-2| = |x+4|$ [3]

(b) $|x| = 2 - x$, and state the coordinates of the point where $y = |x|$ meets $y = 2 - x$. [2]

(c) $|x-1| = 3x + 5$, showing clearly why one candidate is rejected. [3]

12. Reasoning from the definitions of odd and even.

(a) The function f is even with $f(3) = 7$, and g is odd with $g(2) = -5$. Write down the values of $f(-3)$ and $g(-2)$. [2]

(b) Prove that if f and g are both odd functions, then $f \circ g$ is also odd. [3]

13.

(a) Solve $|x-2| = |3x+1|$. [4]

(b) Hence, or otherwise, solve $|x-2| \geq |3x+1|$. [3]

14. The function $f(x) = ax + b$ satisfies $f(f(x)) = x$ for all real x. Find all possible values of a and b. [4]

15. Show that $f(x) = \dfrac{2x+3}{x-2}$ is self-inverse, that is, $(f \circ f)(x) = x$. [3]

16. Let $f(x) = x^2 - 4x$.

(a) Sketch $y = |f(x)|$ for $-1 \leq x \leq 5$, showing the coordinates of both corners and of the turning point. [3]

(b) Hence write down the values of k for which the equation $|f(x)| = k$ has exactly four solutions. [2]

(c) Solve $|x^2 - 4x| = 3$. [3]

(d) On separate axes, sketch $y = [f(x)]^2$, stating where the curve meets the x-axis and the coordinates of its turning points. [3]

17. For each function, state every asymptote, and say what happens at the value of x where the denominator is zero.

(a) $f(x) = \dfrac{x^2-1}{x}$; state also the x-intercepts. [3]

(b) $f(x) = \dfrac{x^2+x-2}{x-1}$; simplify the expression first. [3]

18. Let $f(x) = \dfrac{3x+1}{x-1}$, $x \neq 1$.

(a) Show that $f(x) = 3 + \dfrac{4}{x-1}$. [2]

(b) Hence describe a sequence of transformations that maps the graph of $y = \dfrac{1}{x}$ onto the graph of $y = f(x)$. [3]

(c) Write down the equations of the asymptotes of the graph of f. [2]

(d) Sketch the graph of $y = f(x)$, showing both asymptotes and the coordinates of both intercepts. [3]

19. GDC A workshop makes x chairs a month. The average cost of making one chair, in dollars, is modelled by

$$C(x) = \frac{0.02x^2 + 15x + 2000}{x}, \qquad x > 0.$$

(a) Write down the equation of the oblique asymptote of the graph of C, and state what the vertical asymptote $x = 0$ means in context. [2]

(b) Find the number of chairs the workshop should make each month to minimise the average cost, and state the minimum average cost. [3]

(c) Find the numbers of chairs for which the average cost is less than \$30. [2]

20. HL Let $f(x) = x^2 - 4$.

(a) Sketch $y = f(x)$, showing the intercepts. [2]

(b) Write down the equations of the vertical asymptotes of $y = \dfrac{1}{f(x)}$, and explain how you found them. [3]

(c) Find the coordinates of the maximum point of $y = \dfrac{1}{f(x)}$ on the interval $-2 < x < 2$, and explain why it is a maximum even though the value is negative. [3]

(d) Describe the behaviour of $y = \dfrac{1}{f(x)}$ as $x \to \pm\infty$, and state the horizontal asymptote. [2]

(e) Sketch $y = \dfrac{1}{f(x)}$, showing all asymptotes and the point found in part (c). [4]

21. HL Consider $f(x) = \dfrac{x^2 - 4x + 5}{x - 2}$.

(a) Show that $f(x) = (x-2) + \dfrac{1}{x-2}$. [2]

(b) Write down the equations of the vertical and oblique asymptotes. [3]

(c) Find the y-intercept, and show that the curve has no x-intercept. [3]

(d) Show that the curve never meets its oblique asymptote. [2]

(e) Sketch the curve, showing both asymptotes and the y-intercept. [3]

22. Solve each equation.

(a) $e^{2x} - 3e^x + 2 = 0$ [3]

(b) $4^x - 2^{x+1} = 8$ [4]

23. GDC After a single dose, the amount of a medicine in a patient's bloodstream is modelled by $A(t) = 250(0.82)^t$ mg, where t is the time in hours after the dose.

(a) State the size of the dose, and the percentage of the medicine that leaves the bloodstream each hour. [2]

(b) Find the amount of medicine in the bloodstream after 5 hours. [1]

(c) The medicine is effective while at least 40 mg remains in the bloodstream. Find the length of time for which the dose is effective. [2]

(d) A second patient is given a different medicine, and the amount in their bloodstream is modelled by $B(t) = 400e^{-0.25t}$ mg. Find the time at which the two patients have the same amount of medicine in their bloodstream. [3]

Give your answers to three significant figures.

24. GDC Let $f(x) = \ln(x + 3)$ and $g(x) = x^2 - 2x - 1$.

(a) State the domain of f and the equation of the vertical asymptote of its graph. [2]

(b) Sketch the graphs of f and g on the same axes for $-3 < x \leq 4$. [2]

(c) Find the x-coordinates of the points where the two graphs intersect. [2]

(d) Hence solve the inequality $\ln(x + 3) > x^2 - 2x - 1$. [2]

Give your answers to three significant figures.

25. Let $f(x) = \log_3(2x - 1)$ and $g(x) = \frac{1}{2}(3^x + 1)$.

(a) State the domain and the range of f. [2]

(b) Show that g is the inverse of f, by showing that $(f \circ g)(x) = x$ for every real x. [3]

(c) State the equation of the line in which the graphs of f and g are reflections of each other, and write down the coordinates of the point where the graph of g crosses the y-axis. [2]

26. Solve each inequality.

(a) $x^2 + 2 \leq 3x$ [3]

(b) $|2x - 1| < 5$ [2]

(c) $\dfrac{x - 5}{x + 1} \leq 2$ [4]

27. Let $f(x) = x^2 - 4x + 3$.

(a) Write $f(x)$ in vertex form, and state the coordinates of the vertex. [3]

(b) Find the zeros of f and the y-intercept. [3]

(c) Sketch $y = f(x)$, showing the vertex and both intercepts. [2]

(d) On separate axes, sketch $y = |f(x)|$ and $y = f(|x|)$, showing the key points of each. [4]

(e) The graph of $y = f(x)$ is translated 2 units to the right and 3 units up to give $y = g(x)$. Find $g(x)$ in the form $ax^2 + bx + c$. [3]

(f) Describe the transformations that map $y = f(x)$ onto $y = f(2x - 1)$. Hence write down the roots of $f(2x - 1) = 0$ and the coordinates of the vertex of $y = f(2x - 1)$. HL [3]

28. Let $f(x) = \dfrac{3x - 1}{x + 2}$ for $x \neq -2$, and let $g(x) = x + 2$.

(a) State the range of f, justifying your answer. [2]

(b) Find $f^{-1}(x)$, and state its domain. [3]

(c) Find $(f \circ g)(x)$, giving your answer as a single fraction. [3]

(d) State the domain of $f \circ g$. [2]

(e) Solve $(f \circ g)(x) = 2$. [3]

29. Let $f(x) = \ln(2x - 4)$.

(a) State the domain of f. [2]

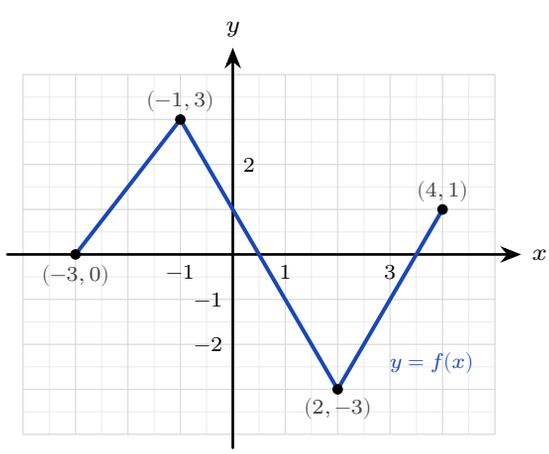
(b) Find the exact x-intercept of the graph of f. [2]

(c) State the equation of the vertical asymptote, and describe the behaviour of $f(x)$ near it. [3]

(d) Find $f^{-1}(x)$, and state its domain and range. [3]

(e) Describe a sequence of transformations that maps $y = \ln x$ onto $y = \ln(2x - 4)$. HL [3]

30. The graph of $y = f(x)$, for $-3 \leq x \leq 4$, is made of three line segments, as shown.



(a) Write down the value of $f(3)$ and the value of $(f \circ f)(0)$. [2]

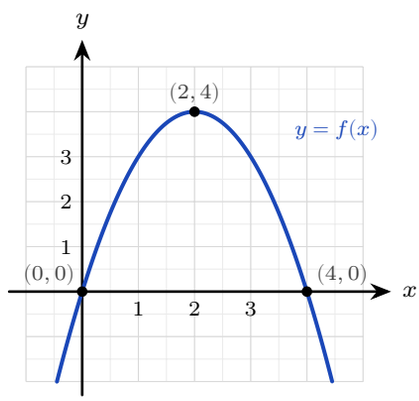
(b) Write down the range of f. [1]

(c) Sketch the graph of $y = |f(x)|$. Give the coordinates of its end points, of the points where it meets the x-axis, and of the points where its gradient changes. [3]

(d) The graph of $y = f(x)$ is reflected in the y-axis and then translated 1 unit up, to give the graph of $y = g(x)$. Write down $g(x)$ in terms of f, and find the images of the four labelled points. [3]

(e) Solve the equation $|f(x)| = 2$. [5]

31. The graph of the quadratic function $y = f(x)$, $x \in \mathbb{R}$, is shown. It meets the x-axis at $(0, 0)$ and $(4, 0)$, and its maximum point is $(2, 4)$.



(a) Write down the value of $f(3)$. [1]

(b) Write down the range of f. [1]

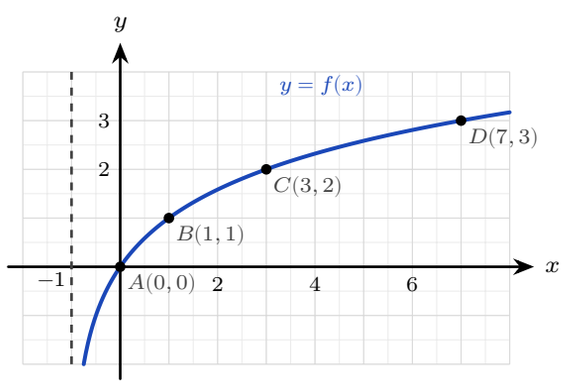
(c) Explain why f does not have an inverse. The domain of f is then restricted to $x \geq 2$, so that an inverse exists. Write down the value of $f^{-1}(3)$. [2]

(d) For the domain $x \in \mathbb{R}$, sketch the graph of $y = \dfrac{1}{f(x)}$. Show its asymptotes and the coordinates of its turning point. [4]

(e) Find an expression for $f(x)$. [2]

(f) Hence solve the inequality $\dfrac{1}{f(x)} > 1$. [5]

32. The graph of $y = f(x)$, where $f(x) = \log_2(x + b)$ and b is a constant, is shown. The line $x = -1$ is its vertical asymptote, and the graph passes through the points A, B, C and D.



(a) Write down the value of $f(3)$ and the value of $f^{-1}(3)$. [2]

(b) Solve $(f \circ f)(x) = 1$. [2]

(c) Find the value of b. [2]

(d) Find $f^{-1}(x)$, and state the domain and the range of f^{-1}. [3]

(e) Sketch the graph of $y = f(x-2)+1$. Show its asymptote and the images of A, B, C and D. [3]

(f) Solve the inequality $f(x) > f^{-1}(x)$. [3]

33. Prove that every function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an even function and an odd function. [4]

34. Consider the curve $y = \dfrac{x^2 + ax + b}{x - 1}$, where a and b are constants.

(a) For some values of b the curve has oblique asymptote $y = x + 3$. Find a. [3]

(b) Using your value of a, find the value of b for which the zero of the denominator gives a hole at $x = 1$ instead of a vertical asymptote. Describe the resulting graph, and explain why it has no oblique asymptote. [3]

35. Show that $f(x) = \dfrac{ax + b}{cx + d}$ is self-inverse (that is, $f(f(x)) = x$) if and only if $a + d = 0$, where $c \neq 0$ and $ad - bc \neq 0$. [6]

36. GDC The number of people who have heard a rumour t days after it starts is modelled by

$$N(t) = \frac{5000}{1 + 49e^{-0.8t}}.$$

(a) Find the number of people who had heard the rumour at $t = 0$. [2]

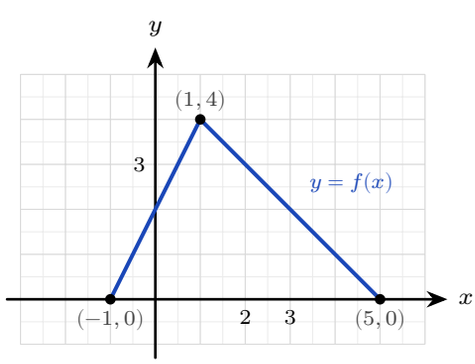
(b) Find $N(5)$, giving your answer to the nearest whole number. [2]

(c) Find the time at which 2500 people have heard the rumour, giving an exact answer and a value to three significant figures. [4]

(d) State the value N approaches as t becomes large, and interpret it in context. [3]

(e) Show that $\dfrac{5000 - N}{N} = 49e^{-0.8t}$, and hence explain why plotting $\ln\left(\dfrac{5000 - N}{N}\right)$ against t gives a straight line. State its gradient and its vertical intercept. [5]

37. The graph of $y = f(x)$, for $-1 \leq x \leq 5$, is made of two line segments, as shown.

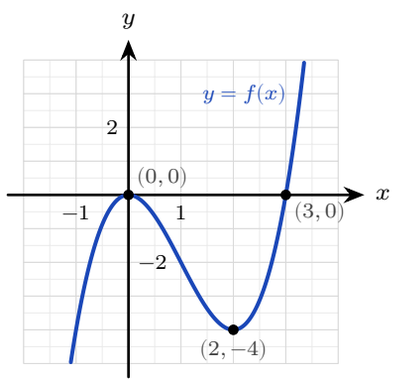


(a) On a new set of axes, sketch the graph of $y = f(2x + 3)$. Give the coordinates of the images of the three labelled points. [3]

(b) Write down the domain of $y = f(2x + 3)$. [1]

(c) Solve the equation $f(2x + 3) = 3$. [3]

38. The graph of $y = f(x)$, where $f(x) = x^3 - 3x^2$, is shown. It has a local maximum at $(0, 0)$ and a local minimum at $(2, -4)$, and it crosses the x-axis at $(3, 0)$.

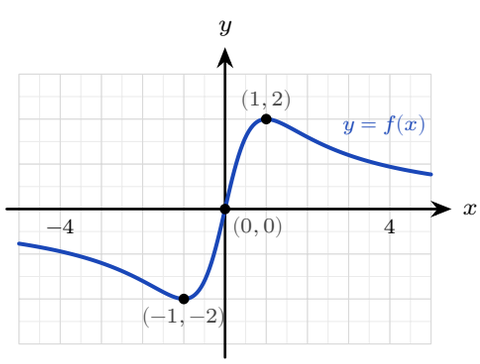


(a) On a new set of axes, sketch $y = |f(x)|$. Give the coordinates of its turning points and of the points where it meets the x-axis. [3]

(b) On another set of axes, sketch $y = f(|x|)$. Give the same information. [3]

(c) Find all the solutions of $f(|x|) = -2$, giving exact answers. [4]

39. The graph of $y = f(x)$, where $f(x) = \dfrac{4x}{x^2 + 1}$, is shown. It passes through the origin, its maximum point is $(1, 2)$ and its minimum point is $(-1, -2)$. The x-axis is a horizontal asymptote.



(a) Write down the equation of the vertical asymptote of the graph of $y = \dfrac{1}{f(x)}$. [1]

(b) On a new set of axes, sketch $y = \dfrac{1}{f(x)}$. Give the coordinates of its turning points, and describe what happens to y as $x \to \pm\infty$. [4]

(c) Show that $\dfrac{1}{f(x)} = \dfrac{x}{4} + \dfrac{1}{4x}$, and hence write down the equation of a second asymptote of the graph of $y = \dfrac{1}{f(x)}$. [2]

40. Let $f(x) = \dfrac{2x^2 - 3x}{x^2 + 1}$.

(a) Write down the equation of the horizontal asymptote of the graph of f. [1]

(b) Find the coordinates of the point where the graph crosses its horizontal asymptote. [2]

(c) Explain why the graph has no vertical asymptote. [1]

(d) Find the coordinates of the points where the graph meets the x-axis. [2]

41.

(a) Solve $|x^2 - 2x - 4| < 4$. [4]

(b) Solve $|x^2 - 2x - 4| = 2x - 4$, showing clearly why each rejected value is rejected. [5]

42. Let $f(x) = \ln(6 - 2x)$.

(a) State the domain of f. [2]

(b) Find $f^{-1}(x)$, and state its domain and range. [3]

(c) Find the exact values of $f^{-1}(1)$ and $\dfrac{1}{f(1)}$, and hence explain why $\dfrac{1}{f(x)}$ and $f^{-1}(x)$ are different functions. [2]

(d) Find the equation of the vertical asymptote of the graph of $y = \dfrac{1}{f(x)}$. [2]

Challenge Questions

C1. When does the order of two transformations matter? Section 5.2 showed that a vertical stretch and a vertical translation give different curves in different orders. This investigation finds the exact condition for any two vertical moves, and a way to read that condition from the graph.

A *vertical move* with parameters $a \neq 0$ and k replaces $y = f(x)$ by $y = af(x) + k$: a vertical stretch of scale factor a followed by a translation of k units up. Throughout, start from $y = f(x)$.

(a) Apply a vertical stretch of scale factor a and then a translation of k units up, and write down the result. Do the same in the opposite order. Deduce the exact condition on a and k under which the two agree for every function f. [3]

(b) Two vertical moves have parameters (a_1, k_1) and (a_2, k_2). Show that the first followed by the second gives $y = a_1a_2f(x) + a_2k_1 + k_2$, and find the result of the opposite order. Deduce that the two orders give the same curve for every f if and only if

$$k_1(a_2 - 1) = k_2(a_1 - 1).$$

[5]

(c) Check the condition in part (b) for two translations ($a_1 = a_2 = 1$), for two stretches ($k_1 = k_2 = 0$), and for a stretch with a translation, comparing the last with part (a). Then show that the moves with parameters $(2, 1)$ and $(3, 2)$ may be applied in either order, although neither is a pure stretch or a pure translation. [4]

(d) Show that when $a \neq 1$ the move with parameters (a, k) leaves every point on the line $y = \dfrac{k}{1-a}$ at the same height. Show that, when $a_1 \neq 1$ and $a_2 \neq 1$, the condition in part (b) says exactly that the two moves leave the same horizontal line fixed, and check this for the pair in part (c). [4]

(e) Explain why any vertical move and any horizontal move give the same curve in either order. *(Hint: consider which variable each move acts on)* [2]

(f) A horizontal move replaces x by $bx + c$, where $b \neq 0$. Investigate when two horizontal moves may be applied in either order, and whether a fixed-line rule like the one in part (d) still holds. Suggest one way to extend the investigation further. [3]

C2. GDC **Newton's law of cooling.** A body at temperature T in a room at temperature R cools so that the difference $T - R$ decays exponentially:

$$T(t) = R + (T_0 - R)\,e^{-kt}, \qquad k > 0,$$

where T_0 is the temperature at $t = 0$ and t is measured in minutes.

(a) Show that $T(t) - R$ falls by the same factor over any interval of fixed length, and that this factor does not depend on T_0. [4]

(b) Deduce that the difference halves every $\dfrac{\ln 2}{k}$ minutes. [3]

(c) Three readings T_1, T_2 and T_3 are taken at times t, $t + h$ and $t + 2h$. Use part (a) to show that $(T_2 - R)^2 = (T_1 - R)(T_3 - R)$, and hence that

$$R = \frac{T_1 T_3 - T_2^2}{T_1 + T_3 - 2T_2},$$

provided the denominator is not zero. [5]

(d) A cup of coffee in a room of unknown temperature reads 95, 65 and $47\,^\circ$C at $t = 0$, 10 and 20 minutes. Find R, and find k in exact form. [4]

(e) Find, to the nearest minute, how long the coffee takes to reach $40\,^\circ$C. [2]

(f) Explain why the model predicts that the coffee never reaches room temperature, and state which feature of the graph of T against t shows this. [2]

(g) Suppose the third reading had been recorded as $48\,^\circ$C instead of $47\,^\circ$C. Find the new estimate of R, and comment on how sensitive the method in part (c) is to a small error in one reading. Suggest how more than three readings could be used to estimate R more reliably. [3]

C3. Self-inverse rational functions. Throughout, let

$$f(x) = \frac{ax + b}{cx + d}, \qquad c \neq 0, \quad ad - bc \neq 0.$$

The function is called *self-inverse* when $(f \circ f)(x) = x$ for every x at which the composition is defined. The same restriction applies to every repeated composition below. This investigation finds exactly which of these functions are self-inverse, and then which return to x after three or four applications.

(a) Show that $f(x) = \dfrac{x + 1}{x - 1}$ is self-inverse. [2]

(b) Show that if $ad - bc = 0$ then f is constant, and explain why the condition $ad - bc \neq 0$ is therefore assumed. [3]

(c) Show that

$$(f \circ f)(x) = \frac{(a^2 + bc)\,x + b(a + d)}{c(a + d)\,x + (bc + d^2)}.$$

[4]

(d) Hence prove that f is self-inverse if and only if $a + d = 0$. *(Hint: two expressions in x are equal for all x only when their coefficients match)* [4]

(e) Let $g(x) = \dfrac{1}{1 - x}$. Show that g is not self-inverse, but that $(g \circ g \circ g)(x) = x$ for every $x \neq 0, 1$. [3]

(f) Explain why $(f \circ f \circ f)(x) = x$ for every x in the domain is equivalent to $(f \circ f)(x) = f^{-1}(x)$, where $f^{-1}(x) = \dfrac{dx - b}{-cx + a}$. By comparing this with part (c), prove that, when $a + d \neq 0$, the function f returns to x after three applications if and only if

$$(a + d)^2 = ad - bc.$$

Verify the condition for g. *(Hint: two such fractions, each with a nonzero value of "$ad - bc$", are equal for every x exactly when the four coefficients of one are a constant multiple of the four coefficients of the other)* [5]

(g) Show that $h(x) = \dfrac{x - 1}{x + 1}$ returns to x after four applications but not after two. Compare $(a + d)^2$ with $ad - bc$ for the self-inverse functions, for g and for h, and suggest a condition for f to return to x after n applications. [3]

C4. Counting the solutions of $|f(x)| = k$. Let $f(x) = x^3 - 3x$ and let $k \geq 0$ be a constant. The graph of $y = f(x)$ has a local maximum at $(-1, 2)$ and a local minimum at $(1, -2)$.

(a) Sketch $y = |f(x)|$, showing the x-intercepts in exact form and the coordinates of the local maxima. [3]

(b) Find the number of real solutions of $|f(x)| = k$ for each of $k = 0$, $k = 1$, $k = 2$ and $k = 3$. *(Hint: for $k = 2$, each of $x^3 - 3x - 2$ and $x^3 - 3x + 2$ has a repeated linear factor)* [4]

(c) Explain, using your sketch, why the number of solutions changes at $k = 0$ and $k = 2$ and nowhere else. [3]

(d) State the number of real solutions of $|f(x)| = k$ for every $k \geq 0$, giving your answer as a list of cases. [2]

(e) Now let $g(x) = x^3 - 3x + 1$, whose graph has a local maximum at $(-1, 3)$ and a local minimum at $(1, -1)$. Give the corresponding list of cases for $|g(x)| = k$, and explain why it has more cases than the list in part (d). [4]

(f) State a rule that predicts, from the graph of $y = |g(x)|$, the values of k at which the number of solutions changes. Suggest how you would test your rule on a quartic. [2]

C5. The range of a quadratic over a linear function. Section 5.4 finds the range of $\dfrac{ax+b}{cx+d}$ by writing it as a constant plus a fraction. When the numerator is a quadratic, that method no longer gives the range directly. This investigation uses the discriminant instead. Throughout,

$$f(x) = \frac{x^2 + bx + c}{x + d}, \qquad x \neq -d,$$

where b, c and d are real constants, and $N(x) = x^2 + bx + c$ is the numerator.

(a) Let $f(x) = \dfrac{x^2 + 3}{x - 1}$. Show that $f(x) = y$ leads to $x^2 - yx + (y + 3) = 0$. By requiring this equation to have a real root, show that the range of f is $y \leq -2$ or $y \geq 6$, and check that the excluded input $x = 1$ removes no value of y. [5]

(b) For the function in part (a), find the value of x at which $f(x) = -2$ and the value at which $f(x) = 6$, and explain why these points are turning points of the graph. Write $f(x)$ as $x + 1 + \dfrac{4}{x-1}$ and state the equations of the asymptotes. [4]

(c) Find the range of $\dfrac{x^2 - 4x + 3}{x - 2}$. [3]

(d) For $\dfrac{x^2 + x - 2}{x - 1}$, show that the discriminant method gives $(y - 3)^2 \geq 0$, which holds for every y. Explain why the range is nevertheless not every real number, and state the range. [4]

(e) Return to the general f. Show that $f(x) = y$ has a real solution when

$$\Delta(y) = y^2 + (4d - 2b)\,y + b^2 - 4c \geq 0,$$

and that $\Delta(y)$, as a quadratic in y, has discriminant $16N(-d)$. Deduce that when $N(-d) < 0$ the range is every real number, and that when $N(-d) > 0$ the range misses an interval. Interpret the condition $N(-d) < 0$ in terms of where the vertical asymptote lies relative to the zeros of N. [5]

(f) Show that when $N(-d) > 0$ the missing interval has width $4\sqrt{N(-d)}$, and check this against part (a). Suggest how the investigation could be extended, for example to $\dfrac{x^2 + bx + c}{x^2 + px + q}$. [3]