FUNCTIONS

Chapter 4

Lines & Quadratics

4

SYLLABUS 1.16 2.1–2.4 2.6 2.7 2.12

In 1995 a powerful earthquake struck near Kobe, in western Japan. The Akashi Kaikyō Bridge was under construction. Its two towers already stood in the strait, but the roadway between them did not yet exist. The earthquake moved the towers apart.

The engineers did not begin the work again. They measured the new gap and changed the design to match it. The bridge opened a few years later with a main span of 1991 metres, one metre longer than the 1990 metres originally planned.

The engineers could do this because the shape of the bridge follows from a few measurements through equations. The main cables of a suspension bridge carry a level roadway of even weight. They hang in the curve called a **parabola**, the same curve as the path of a thrown ball. If one value in the equations changes, every other measurement follows from it.

This is what a **function** makes possible. A function takes an input and returns exactly one output, so a whole structure can be described by a single rule. A rule can be graphed, studied and recalculated. This chapter studies two functions that later chapters use often: lines, which change at a steady rate, and quadratics, which give the cable its curve.

A quadratic has three equivalent forms, and each shows a different feature of its curve. Where two graphs meet, both equations are true, so a picture and a calculation agree.

By the end of this chapter you will be able to:

- state the domain, range and inverse of a function, and find the equation of a line;
- solve systems of up to three linear equations, including those with no unique solution;
- move between the three forms of a quadratic, and solve quadratic equations and inequalities;
- use the discriminant to count real roots, and find their sum and product without solving.

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4.1 Functions

A function is a rule with a fixed result. Give it an input and it returns one definite output, and it returns the same output every time. Because the output is fixed, you can calculate with it and reason about it.

DEFINITION 4.1 **Function, domain and range**

A **function** f assigns exactly one output $f(x)$ to each input x. The set of inputs it accepts is the **domain**; the set of outputs that actually occur is the **range**.

The phrase "exactly one" is where the restriction lies. A rule that could return two different outputs for the same input is not a function. If it could, $f(x)$ would not name a single number, and nothing could be calculated from it. On a graph this becomes the **vertical line test**: a curve represents a function only if no vertical line crosses it more than once. The figure shows the domain and range of one graph, and the test applied to two curves.

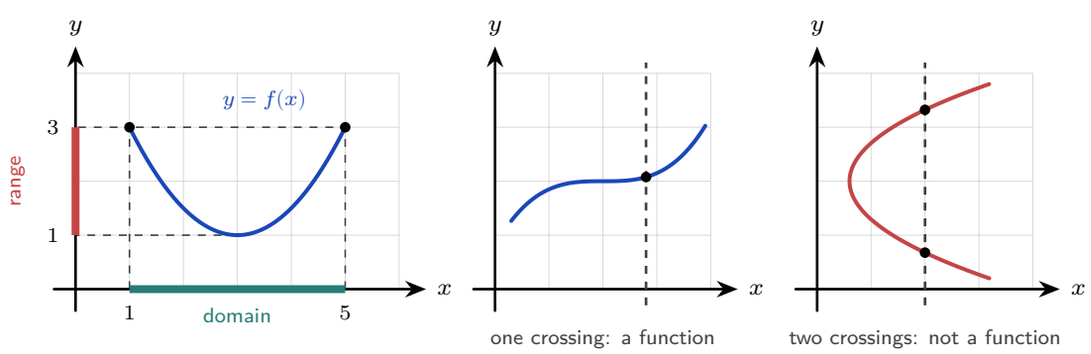


Figure 4.1 Left: the domain is the set of x-values the graph covers, here $1 \le x \le 5$, and the range is the set of y-values, here $1 \le y \le 3$. Centre and right: the vertical line test.

The letter f carries no meaning of its own. A function is named for what it measures, and the letter inside the bracket for what it depends on. A height that changes with time is written $h(t)$, a velocity $v(t)$, a cost that depends on how many items you make $C(n)$. Read $C(n)$ as "the cost of making n items", not as C multiplied by n. Examination questions name their functions this way, so read the notation as a sentence rather than as algebra.

These names show a common use of functions: describing real things. A function chosen to describe something real is called a **mathematical model**. Its letters stand for quantities you can measure, and its rule says how one of them depends on the other. With a model you must also ask a question that algebra alone cannot answer: does the answer make sense in the real situation?

Two habits follow from that: set the domain from the situation, not from the algebra alone, and check every solution against the situation. The formula $h(t) = 20t - 5t^2$ is defined for every real t. But the stone is in flight only from the moment it is thrown $(t = 0)$ until it lands $(t = 4)$, so the model holds only on $0 \le t \le 4$. And a value that solves the equation

can still be rejected, because a length or a time cannot be negative in the real situation. Every model is an approximation, so ask what it leaves out before trusting it far beyond the data it came from.

4.1.1 Reading the domain

A function need not accept every number. Three operations put limits on what you may substitute. Learn the three and you can read the domain of every function in this chapter.

Where x appears	Condition	Reason
In a denominator	$\neq 0$	you cannot divide by zero
Inside an even root	≥ 0	a negative number has no real square root
Inside a logarithm	> 0	zero and negative numbers have no logarithm

Note the difference between the last two. A square root allows zero, because $\sqrt{0} = 0$. A logarithm does not, because no power of the base gives zero. Confusing $\geq$ with $>$ here is a common slip.

Five examples show the conditions at work.

- $\sqrt{x+3}$ needs $x + 3 \geq 0$, so its domain is $x \geq -3$.
- $\sqrt{5-x}$ needs $5 - x \geq 0$, so its domain is $x \leq 5$.
- $\frac{1}{x-2}$ needs $x - 2 \neq 0$, so its domain is $x \neq 2$.
- $\ln(x-4)$ needs $x - 4 > 0$, so its domain is $x > 4$.
- $\frac{1}{\sqrt{x-1}}$ needs $x - 1 > 0$, because $x - 1$ is both under a root and in a denominator, so its domain is $x > 1$.

When a function contains more than one of these, every condition must hold at once. Write them all down, then take the values that satisfy all of them.

EXAMPLE 4.1

State the domain of $f(x) = \dfrac{\sqrt{x+4}}{x-3}$.

SOLUTION

There are two restrictions. The square root needs $x + 4 \geq 0$, so $x \geq -4$. The denominator needs $x - 3 \neq 0$, so $x \neq 3$. Both must hold, so the domain is

$$x \geq -4, \quad x \neq 3$$

EXAMPLE 4.2

State the domain of $g(x) = \ln(9 - x^2)$.

SOLUTION

A logarithm needs a positive argument (the input of the logarithm), so $9 - x^2 > 0$, that is $x^2 < 9$. This is a quadratic inequality, solved in §4.5: the parabola $y = x^2 - 9$ has x-intercepts $x = \pm 3$ and lies below the axis between them. So

$$-3 < x < 3$$

Both ends are excluded, because at $x = \pm 3$ the argument is zero and $\ln 0$ does not exist.

CAUTION Solve the condition, do not guess it. For $\sqrt{5 - x}$ the domain is $x \leq 5$, not $x \geq 5$. Write the inequality down and rearrange it every time, however obvious it looks.

4.1.2 Reading the range

The range is the set of outputs that actually occur. There is no single rule, but most range arguments use one of four facts, each saying that some expression can never be negative.

- A square is never negative: $x^2 \geq 0$.
- A modulus (the size of a number, ignoring its sign) is never negative: $|x| \geq 0$.
- The square root symbol means the non-negative root: $\sqrt{x} \geq 0$.
- A positive base raised to any power is positive: $a^x > 0$.

To use them, start from the part you know. Ask what that part can be, then follow the operations one at a time.

Function	Reasoning	Range
$y = x^2 + 1$	$x^2 \geq 0$, then add 1	$y \geq 1$
$y = \|x\| - 3$	$\|x\| \geq 0$, then subtract 3	$y \geq -3$
$y = \sqrt{x} + 2$	$\sqrt{x} \geq 0$, then add 2	$y \geq 2$
$y = 3 - \|x + 1\|$	$\|x + 1\| \geq 0$, so subtracting it from 3 gives at most 3	$y \leq 3$
$y = -(x - 2)^2 + 5$	$(x - 2)^2 \geq 0$, and multiplying by -1 reverses the inequality	$y \leq 5$

CAUTION Multiplying an inequality by a negative number reverses the inequality sign, as in the last row of the table.

A quadratic written as a square plus a constant is read in the same way, and a restricted domain can limit the range at both ends. §4.4 shows how to write any quadratic in this form.

EXAMPLE 4.3

State the range of $f(x) = (x-1)^2 + 2$ for $0 \leq x \leq 4$.

SOLUTION

Use the first fact. Since $(x-1)^2 \geq 0$, adding 2 gives $f(x) \geq 2$. The value 2 occurs at $x = 1$, which is in the domain, so 2 is the smallest output.

The square $(x-1)^2$ is largest at the input furthest from 1. On $0 \leq x \leq 4$ that input is $x = 4$, where $f(4) = 3^2 + 2 = 11$. Every value between 2 and 11 occurs as x increases from 1 to 4, so the range is $\boxed{2 \leq y \leq 11}$

Without the restriction on x the range would be $y \geq 2$.

EXAMPLE 4.4

State the domain and range of $g(x) = \sqrt{x-2}$.

SOLUTION

Domain. The square root needs $x - 2 \geq 0$, so $\boxed{x \geq 2}$

Range. As x runs from 2 upward, $x - 2$ runs from 0 upward, and so does its square root. So $\boxed{g(x) \geq 0}$

TIP On Paper 2, check a range on your GDC. Graph the function and look at the lowest and highest points the curve reaches. The algebra is still expected in the working, but the graph shows a sign error quickly.

4.1.3 Reading a graph

Most of what you need from a function can be read straight from its graph. Five features matter, and each has its own name.

Feature	What it is	How to find it
x-intercepts	the points where the curve meets the x-axis	set $y = 0$ and solve
y-intercept	the point where the curve meets the y-axis	set $x = 0$
Turning points	the local maximum and minimum points (higher or lower than every nearby point, though not always than the whole curve), where the curve changes direction	GDC, or the vertex form of a quadratic (§4.4); later, the derivative (Ch. 13; turning points in §14.1)
Asymptote	a line that the curve gets closer and closer to as x or y becomes very large (defined in §5.4)	look for where the function grows without bound, or the value it approaches for large x
Intersection	a point that lies on both of two graphs	solve the two equations together

The words *roots*, *zeros* and *x-intercepts* are often used interchangeably, and you will meet all three for the same numbers. Still, it helps to know what each one refers to strictly:

- **roots** belong to an equation: they are its solutions;
- **zeros** belong to a function: the values of x for which $f(x) = 0$;
- **x-intercepts** belong to a graph: points such as $(3, 0)$.

So the zeros of f are the roots of $f(x) = 0$, and the x-coordinates of the x-intercepts of $y = f(x)$. Using the word that matches the object makes your working clearer. When a question asks for an x-intercept, give the point, such as $(3, 0)$. In the same way the y-intercept of $y = x^2 - 5x + 4$ is the point $(0, 4)$, and "the y-intercept is 4" is shorthand for that point.

A GDC can find four of these from the graph: the zeros, the turning points, the points where two curves meet, and the y-intercept, by evaluating the function at $x = 0$. How you reach each one depends on your model. An asymptote comes from the equation, not from the screen. On Paper 2 the GDC is the intended method for these features: graph the function, find the feature, and write down what you asked the calculator to find. Many equations cannot be solved with the algebra in this course. For these, you are expected to solve them with your GDC, so make sure you can find zeros and points of intersection on your own calculator.

EXAMPLE 4.5

Use a GDC to find the y-intercept, the zeros and the turning points of $f(x) = x^3 - 2x^2 - 5x + 4$, giving coordinates to three significant figures where they are not exact.

SOLUTION

Begin with the y-intercept, which needs no calculator. Setting $x = 0$ leaves the constant term, so the curve meets the y-axis at $(0, 4)$

Graph the function over a window wide enough to show every feature; $-3 \leq x \leq 4$ works here. The curve crosses the x-axis three times, so find all three zeros on the GDC, one for each crossing:

$$(-1.86,\ 0), \qquad (0.678,\ 0), \qquad (3.18,\ 0)$$

The GDC gives both coordinates of each turning point at once:

$$\text{local maximum } (-0.786,\ 6.21), \qquad \text{local minimum } (2.12,\ -6.06)$$

You can use the turning points to check the zeros. The curve falls without limit to the left and rises without limit to the right, because the x^3 term has a positive coefficient. With the local maximum above the axis and the local minimum below it, the curve must therefore cross the axis three times, and three is what the GDC found.

TIP Know the difference between the command terms (the instruction words in a question): **sketch** means show the shape with key features labelled, not to scale; **draw** means an accurate, to-scale plot. A sketch of a parabola, the graph of a quadratic (§4.4), needs at least its turning point, the intercepts, and the correct opening direction.

4.1.4 Inverse functions

An **inverse function** reverses the rule. If f sends 3 to 7, then f^{-1} sends 7 back to 3. An inverse exists only when no two inputs give the same output. If two did, the inverse would have to send that output back to two different inputs, and a function may return only one output.

DEFINITION 4.2 **Inverse function**

The **inverse** f^{-1} of a function f undoes it: $f^{-1}(f(x)) = x$. It exists when f is **one-to-one**, meaning no two inputs share an output. On a graph, no horizontal line meets the curve more than once (the horizontal line test).

A function that is not one-to-one has no inverse as it stands, but it can often be given one by restricting its domain. On all real numbers $f(x) = x^2$ is not one-to-one, since

$f(2) = f(-2)$: the line $y = 4$ meets the graph twice. On $x \geq 0$ it passes the test, and its inverse is $\sqrt{x}$. This is developed in §5.1.5.

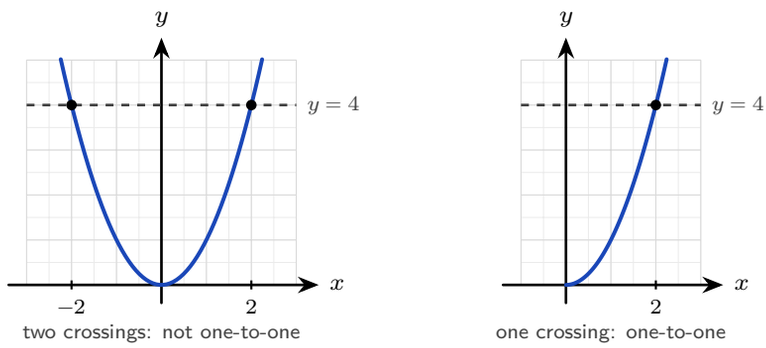


Figure 4.2 The horizontal line test. Left: on all real numbers, the line $y = 4$ meets $y = x^2$ twice, at $x = -2$ and $x = 2$, so the function is not one-to-one. Right: on $x \geq 0$ every horizontal line meets the curve at most once, so the function is one-to-one and has an inverse.

The inverse swaps input and output, so its graph is the mirror image of f in the line $y = x$, and its domain and range are those of f exchanged. The reason for the mirror image is that if (a, b) lies on the graph of f, then $f(a) = b$, so $f^{-1}(b) = a$ and (b, a) lies on the graph of f^{-1}. Exchanging the two coordinates of a point reflects it in the line $y = x$. To find a formula, write $y = f(x)$, swap x and y, then solve for y.

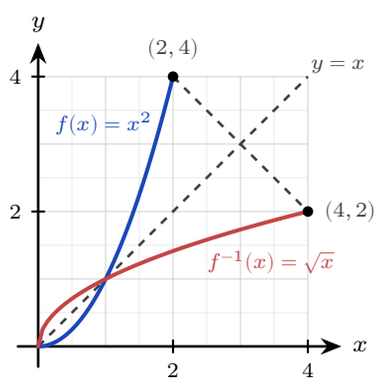


Figure 4.3 $f(x) = x^2$ for $x \geq 0$ and its inverse $f^{-1}(x) = \sqrt{x}$: the points $(2, 4)$ and $(4, 2)$ are mirror images in the line $y = x$.

EXAMPLE 4.6

Find the inverse of $f(x) = 2x + 3$.

SOLUTION

Write $y = 2x + 3$, then swap x and y:

$$x = 2y + 3.$$

Solving for y,

$$y = \frac{x-3}{2}, \qquad \text{so} \qquad \boxed{f^{-1}(x) = \frac{x-3}{2}}$$

Check: f doubles and adds three; f^{-1} subtracts three and halves. Each step of f is undone, in reverse order.

Two kinds of function need more care, because x appears in the rule twice and the steps cannot simply be undone one at a time.

- **A rational function**, such as $\frac{4x+3}{x-2}$. Swap the letters, cross-multiply to clear the fraction, expand the bracket, and collect every term in y on one side. Then y is a common factor. Divide both sides by the other factor to find y.
- **A quadratic in expanded form**, such as $x^2 - 6x + 11$. First write it in completed-square form (§4.4), so that x appears only once. Restrict the domain to one side of the vertex, where the function is one-to-one, and then undo the steps in reverse order.

The next two examples show one of each kind.

EXAMPLE 4.7

Find f^{-1} for $f(x) = \frac{4x+3}{x-2}$, and state its domain.

SOLUTION

Write $y = \frac{4x+3}{x-2}$ and swap x and y:

$$x = \frac{4y+3}{y-2}.$$

Multiply both sides by $y-2$ to clear the fraction, then gather every term containing y on one side:

$$x(y-2) = 4y+3 \implies xy - 2x = 4y+3 \implies xy - 4y = 2x+3.$$

Both terms on the left carry a factor of y, so factorise and divide:

$$y(x-4) = 2x+3 \implies \boxed{f^{-1}(x) = \frac{2x+3}{x-4}, \quad x \neq 4}$$

The domain of f^{-1} is the range of f, and this confirms it: $f(x) = 4$ would need $4x+3 = 4x-8$, which no x satisfies, so f never outputs 4.

EXAMPLE 4.8

Find the inverse of $f(x) = x^2 - 6x + 11$ for $x \geq 3$.

SOLUTION

Complete the square first. Completing the square is covered fully in §4.4; here only the result is needed:

$$f(x) = (x-3)^2 + 2.$$

The vertex is at $x = 3$, so on $x \geq 3$ the function is increasing, so it is one-to-one. Write $y = (x-3)^2 + 2$, swap x and y, and solve for y:

$$x = (y-3)^2 + 2 \implies (y-3)^2 = x - 2 \implies y = 3 \pm \sqrt{x-2}.$$

The range of f^{-1} is the domain of f, so $y \geq 3$, and only the $+$ sign is possible:

$$\boxed{f^{-1}(x) = 3 + \sqrt{x-2}, \quad x \geq 2}$$

Check with one value: $f(4) = 16 - 24 + 11 = 3$, and $f^{-1}(3) = 3 + \sqrt{1} = 4$. In expanded form the swap gives $x = y^2 - 6y + 11$, with y in two places and no single step to undo; in completed-square form y appears only once, so the steps can be undone in reverse order.

The inverse also gives another way to read an equation: solving $f(x) = 10$ is the same as evaluating $f^{-1}(10)$, because both ask for the input that f sends to 10.

EXAMPLE 4.9

For $f(x) = 2x + 3$, solve $f(x) = 10$ in two ways.

SOLUTION

Directly: $2x + 3 = 10$ gives $2x = 7$, so $\boxed{x = \frac{7}{2}}$

Using the inverse found above, $f^{-1}(x) = \dfrac{x-3}{2}$, so

$$f^{-1}(10) = \frac{10-3}{2} = \boxed{\frac{7}{2}}$$

The two methods must give the same answer, because both find the input that f sends to 10. When a question gives you f^{-1} already, the second method is shorter.

CAUTION The notation f^{-1} means the inverse, *not* the reciprocal. $f^{-1}(x)$ is not $\dfrac{1}{f(x)}$.

YOUR TURN 4.1 Functions

What a function is

1. For $f(x) = x^2 - 3x$, find each value.

(a) $f(2)$ (b) $f(-1)$

(c) $f(0)$ (d) the values of x with $f(x) = 0$

2. Decide whether each rule defines y as a function of x, and justify your answer.

(a) $y^2 = x$ (b) $y = x^2$

Domain

3. State the domain of each function.

(a) $\sqrt{2x - 6}$ (b) $\sqrt{7 - x}$

(c) $\dfrac{1}{x + 5}$ (d) $\dfrac{1}{x^2 - 4x}$

(e) $\ln(5 - x)$ (f) $\ln(2x + 1)$

(g) $\dfrac{1}{\sqrt{x + 2}}$ (h) $\sqrt{x^2 + 1}$

4. State the domain of each function. Each has more than one restriction.

(a) $\dfrac{\sqrt{x + 6}}{x - 2}$ (b) $\dfrac{\sqrt{x}}{x - 9}$

5. A ball is thrown upwards from ground level. Its height after t seconds is $h(t) = 15t - 5t^2$ metres, until it lands.

(a) Find the time at which the ball lands.

(b) State the domain of h in this model.

(c) The value $h(4)$ can be calculated. Explain why it has no meaning in this model.

Range

6. State the range of each function on the domain given.

(a) $x^2 - 4, \quad x \in \mathbb{R}$ (b) $2|x| + 1, \quad x \in \mathbb{R}$

(c) $5 - \sqrt{x}, \quad x \geq 0$ (d) $\dfrac{1}{x^2 + 1}, \quad x \in \mathbb{R}$

7. Consider $f(x) = \sqrt{2 - x}$.

(a) State the domain of f.

(b) State the range of f.

(c) Compare the domain of f with the domain of $g(x) = \sqrt{x - 2}$.

Reading a graph

8. GDC Consider $f(x) = x^3 - x^2 - 4x + 2$. Give coordinates to three significant figures where they are not exact.

(a) Write down the y-intercept of the graph of f.

(b) Use a GDC to find the zeros of f.

(c) Use a GDC to find the coordinates of the local maximum point and the local minimum point.

Inverse functions

9. Find the inverse of each function.

(a) $f(x) = 3x - 6$

(b) $f(x) = \dfrac{x+1}{2}$

(c) $f(x) = x^3 + 1$

(d) $f(x) = \dfrac{1}{x+2}$

(e) $f(x) = \dfrac{2x+1}{x-3}$

(f) $f(x) = \dfrac{x-4}{x+2}$

10. Consider $f(x) = \dfrac{3x+2}{x-1}$.

(a) State the domain of f.

(b) Find $f^{-1}(x)$.

(c) State the domain of f^{-1}.

(d) Verify that $f^{-1}(f(2)) = 2$.

11. Use an inverse function to solve each equation.

(a) For $f(x) = 5x - 3$, find $f^{-1}(x)$, and hence solve $f(x) = 17$ by evaluating $f^{-1}(17)$.

(b) A function g has inverse $g^{-1}(x) = \dfrac{2x+1}{x-3}$, $x \neq 3$. Solve $g(x) = 5$ without finding $g(x)$.

12. The function $f(x) = x^2$ has no inverse over all real numbers. State a restricted domain on which an inverse does exist, and give that inverse.

Concept check

13. A student writes: "$f(x) = 2x + 3$, *so* $f^{-1}(x) = \dfrac{1}{2x+3}$." Identify the error and find the correct $f^{-1}(x)$.

4.2 Linear Functions

The simplest function that actually changes is the line: its output changes at a constant rate. That rate is the **gradient**, the rise divided by the run. A negative gradient means the line falls from left to right.

FORMULA 4.1 **Gradient of a line** FORMULA BOOKLET · 2.1

The gradient of the line through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Needs $x_1 \neq x_2$.

The formula needs $x_1 \neq x_2$. A vertical line, $x = k$, has no gradient, because its run is zero.

A gradient and one point on the line are enough to fix the line completely. That fact gives three ways to write its equation. All three describe the same line.

FORMULA 4.2 **Equation of a straight line** FORMULA BOOKLET · 2.1

A straight line can be written

$$y = mx + c, \qquad y - y_1 = m(x - x_1), \qquad ax + by + d = 0.$$

Here m is the gradient and c the y-intercept.

The table shows when to use each form.

Form	Name	Use it when you know
$y = mx + c$	gradient-intercept	the gradient and the y-intercept
$y - y_1 = m(x - x_1)$	point-gradient	the gradient and any point
$ax + by + d = 0$	general	a question asks for it; usually written with whole-number coefficients

The second form is often the quickest to use. Many questions ask you to find a line, such as a tangent or a normal in calculus (§13.4) or a perpendicular bisector. For this form you need only one point on the line and its gradient. Find those two and the equation can be written down at once.

The second form is the gradient formula rearranged. If (x, y) is any other point on the line through (x_1, y_1), then $\frac{y - y_1}{x - x_1} = m$, so $y - y_1 = m(x - x_1)$. Taking the known point to be the y-intercept $(0, c)$ gives $y - c = mx$, which is the first form.

Two lines are **parallel** when they have the same gradient. They are **perpendicular** when you get one gradient from the other by turning the fraction upside down and changing the sign; this is called the **negative reciprocal**.

KEY 4.3 **Parallel and perpendicular lines**

For two lines with gradients m_1 and m_2:

$$m_1 = m_2 \ \text{(parallel)}, \qquad m_1 m_2 = -1 \ \text{(perpendicular)}.$$

A vertical line has no gradient, so neither rule applies to it. Two vertical lines are parallel, and a vertical line is perpendicular to a horizontal one.

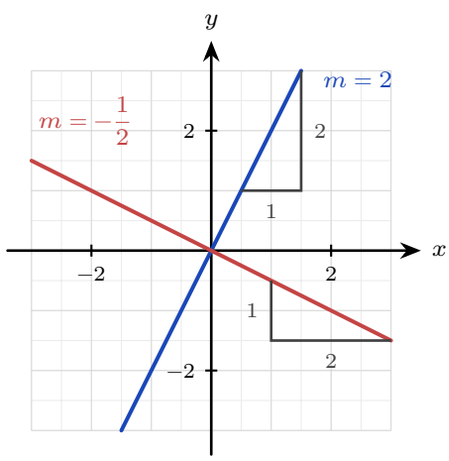


Figure 4.4 Perpendicular gradients: turn the fraction upside down and change the sign.

Turning a line through a right angle shows why. A line of gradient $m \neq 0$ rises m for every 1 across, which is the step $(1, m)$. Turn that step through a right angle anticlockwise and it becomes $(-m, 1)$: m to the left and 1 up. So the perpendicular line has gradient $\dfrac{1}{-m} = -\dfrac{1}{m}$, and the two gradients multiply to -1. In the figure, the step $(1, 2)$ along the line with $m = 2$ turns into the step $(-2, 1)$ along the line with $m = -\frac{1}{2}$.

EXAMPLE 4.10

Find the equation of the line through $(0, 4)$ perpendicular to $y = 3x - 1$.

SOLUTION

The given line has gradient 3, so the perpendicular gradient is $-\frac{1}{3}$. The point $(0, 4)$ is on the y-axis, so it is the y-intercept and $c = 4$:

$$y = -\tfrac{1}{3}x + 4$$

Two points fix more than a gradient. They also have a distance between them and a point halfway along, and both come from the coordinates directly. The distance is Pythagoras applied to the horizontal and vertical gaps; the midpoint is the average of the coordinates. To reach the midpoint from the first point, move half of the horizontal gap and half of the vertical gap. Its x-coordinate is therefore $x_1 + \frac{1}{2}(x_2 - x_1) = \frac{1}{2}(x_1 + x_2)$, and its y-coordinate is found in the same way.

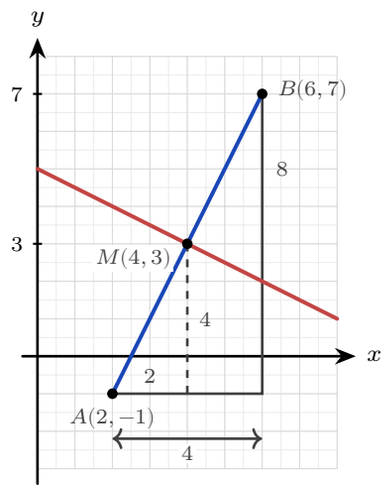


Figure 4.5 The points $A(2,-1)$ and $B(6,7)$ of the example below. The gaps 4 and 8 are the two shorter sides of a right-angled triangle, so $AB=\sqrt{4^2+8^2}$. Half of each gap, 2 across and 4 up, takes you from A to the midpoint $M(4,3)$. The red line is the perpendicular bisector of AB.

FORMULA 4.4 **Distance and midpoint** FORMULA BOOKLET

For the points (x_1,y_1) and (x_2,y_2), the distance between them and the midpoint of the segment joining them are

$$d=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}, \qquad M=\left(\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2}\right).$$

The order of the points does not matter in either formula: the differences are squared, and the sums are symmetric. Both formulas extend to three dimensions by adding a z-term to each; see Ch. 7.

The distance, the midpoint and a perpendicular gradient give the **perpendicular bisector** of a segment, the line at right angles to it through its midpoint. Every point on the perpendicular bisector of AB is the same distance from A as from B, and every point at equal distances from A and B lies on it.

EXAMPLE 4.11

The points $A(2,-1)$ and $B(6,7)$ are given. Find the length of AB, the midpoint of AB, and the equation of the perpendicular bisector of AB.

SOLUTION

The horizontal gap is 4 and the vertical gap is 8, so

$$AB=\sqrt{(2-6)^2+(-1-7)^2}=\sqrt{16+64}=\sqrt{80}=\boxed{4\sqrt{5}}$$

The midpoint is

$$M=\left(\frac{2+6}{2},\ \frac{-1+7}{2}\right)=\boxed{(4,3)}$$

The gradient of AB is $\dfrac{7-(-1)}{6-2}=2$, so the bisector has gradient $-\frac{1}{2}$. Through $M(4,3)$,

$$y-3=-\tfrac{1}{2}(x-4) \implies \boxed{y=-\tfrac{1}{2}x+5}$$

Every point of this line is the same distance from A as from B. Take $(0,5)$ on it: the distance to A is $\sqrt{4+36}=\sqrt{40}$, and to B it is $\sqrt{36+4}=\sqrt{40}$.

YOUR TURN 4.2 Linear Functions

Gradient and equation of a line

14. Find the gradient of the line through each pair of points.

(a) $(1,3)$ and $(5,11)$

(b) $(2,-1)$ and $(5,8)$

(c) $(-1,4)$ and $(3,-4)$

(d) $(0,5)$ and $(2,5)$

15. Find the equation of each line, giving your answer as $y=mx+c$.

(a) gradient 2, through $(0,-3)$

(b) through $(2,1)$ and $(4,7)$

(c) gradient $-\frac{1}{2}$, through $(4,0)$

(d) through $(2,-1)$, perpendicular to $y=-2x+3$

16. Write each equation as $y=mx+c$ and state the gradient.

(a) $2x+3y=6$

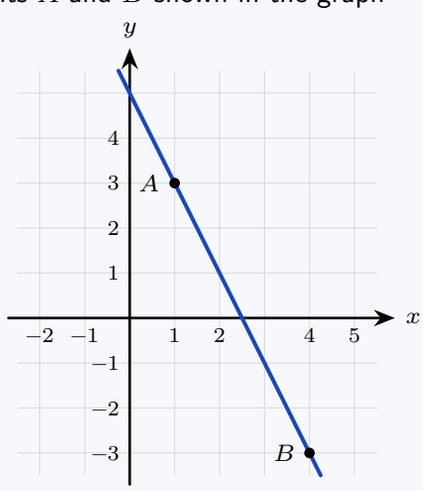
(b) $4x-2y=8$

17. Find the equation of each line in the form $ax+by+d=0$, where a, b and d are integers.

(a) gradient $\frac{2}{3}$, through $\left(\frac{1}{2},-1\right)$

(b) through $(-1,2)$ and $(3,-4)$

(c) the line through the points A and B shown in the graph



Parallel and perpendicular lines

18. Answer each question about parallel and perpendicular lines.

(a) Write down the gradient of a line perpendicular to $y = 4x + 1$.

(b) Write down the gradient of a line parallel to $3x + y = 7$.

(c) Are $y = 2x + 1$ and $y = 2x - 3$ parallel? Give a reason.

(d) Are $y = 3x - 1$ and $y = -\frac{1}{3}x + 2$ perpendicular? Give a reason.

Distance, midpoint and perpendicular bisector

19. For each pair of points, find the exact distance between them and the midpoint of the segment joining them.

(a) $(1, 2)$ and $(4, 6)$

(b) $(-3, 5)$ and $(1, -1)$

20. The points $A(-1, 4)$ and $B(5, 0)$ are given.

(a) Find the midpoint M of AB.

(b) Find the equation of the perpendicular bisector of AB.

(c) The point $P(k, 8)$ lies on the perpendicular bisector. Find k, and verify that $PA = PB$.

Concept check

21. A student writes: "*The line $2x + 3y = 6$ has gradient 2, so a line perpendicular to it has gradient $-\frac{1}{2}$.*" Identify the error and find the correct perpendicular gradient.

4.3 Systems of Linear Equations HL

Two lines in a plane usually cross at one point. Finding that point means finding the values of x and y that satisfy *both* equations at once, a **system** of equations. The standard tool is **elimination**: combine the equations to cancel one variable, solve for the other, then substitute back.

EXAMPLE 4.12

Solve the simultaneous equations $2x + y = 7$ and $x - y = 2$.

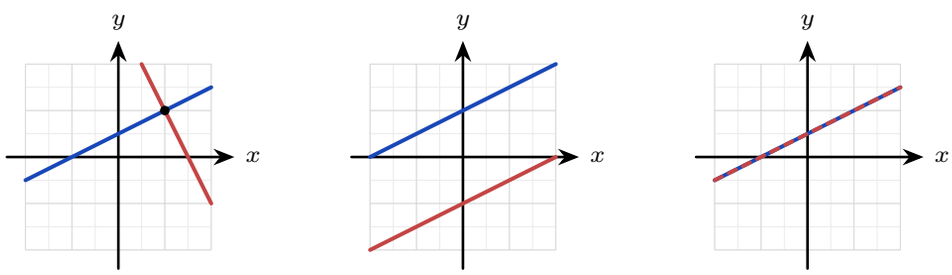
SOLUTION

Add the two equations, which cancels y:

$$(2x + y) + (x - y) = 7 + 2 \implies 3x = 9 \implies x = 3.$$

Substituting into $x - y = 2$ gives $3 - y = 2$, so $y = 1$. The solution is $\boxed{x = 3,\ y = 1}$, and the lines cross at $(3, 1)$.

Not every system has a single answer, and the picture shows why. There are three possibilities for two lines. They cross once, which gives a **unique solution**. They stay parallel and never meet, so there is **no solution**. Or they are the same line drawn twice, which gives **infinitely many solutions**.



(a) cross once (b) parallel (c) the same line

Figure 4.6 Two lines cross once, never meet, or are the same line drawn twice: a unique solution, no solution, or infinitely many solutions.

CAUTION If the two lines have the same gradient but different intercepts, such as $2x + y = 3$ and $4x + 2y = 10$, the lines are parallel and the system has *no* solution. Elimination then produces a false statement such as $0 = 4$.

4.3.1 The augmented matrix

The same idea extends to three linear equations in three unknowns, where each equation is a plane in space. Use the first equation to eliminate one unknown, say x, from the other two, then use the new second equation to eliminate y from the third. You still have three equations, but now in a **triangular form**: the last contains only z, and the one above it only y and z. Solve the last equation for z and substitute back up to find y and then x. This systematic process is called **Gaussian elimination**, or **row reduction**. In practice you carry it out on an augmented matrix, as below, because the matrix keeps every step in the same order and layout.

Use your GDC to check your work. You still need the algebra when a calculator is not allowed, and to show the working when a system has no solution or infinitely many, which questions often ask for.

Start with the equations themselves, written in columns and labelled E_1, E_2, E_3:

$$\begin{cases} E_1: & x + y + z = 6 \\ E_2: & 2x - y + z = 7 \\ E_3: & x + 2y - z = 3 \end{cases}$$

Step 1: remove x from E_2 and E_3, using E_1. Replace E_2 by $E_2 - 2E_1$ and E_3 by $E_3 - E_1$. The x-terms become $0x$:

$$\begin{cases} E_1: & x + & y + & z = & 6 \\ E_2: & 0x - & 3y - & z = & -5 \\ E_3: & 0x + & y - & 2z = & -3 \end{cases}$$

Step 2: remove y from the last equation. Swap E_2 and E_3, so that the equation whose y has coefficient 1 comes first; it makes the next step easier. Then replace the new E_3 by $E_3 + 3E_2$:

$$\begin{cases} E_1: & x + & y + & z = & 6 \\ E_2: & 0x + & y - & 2z = & -3 \\ E_3: & 0x + & 0y - & 7z = & -14 \end{cases}$$

The system is now triangular: each equation has one unknown fewer than the one above it.

Step 3: back-substitute. E_3 gives $z = 2$, then E_2 gives $y = -3 + 2(2) = 1$, then E_1 gives $x = 6 - 1 - 2 = 3$. Working upwards like this is called **back-substitution**.

In every step only the numbers changed; the letters just marked the columns. An **augmented matrix** keeps the numbers and drops the letters: write the coefficients and the constants, keeping the unknowns in the same order in every equation.

$$\begin{cases} E_1: & x + & y + & z = & 6 \\ E_2: & 2x - & y + & z = & 7 \\ E_3: & x + & 2y - & z = & 3 \end{cases} \quad \longrightarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & -1 & 3 \end{array}\right)$$

Each **row** is one equation. The first three columns hold the coefficients of x, y and z; the column after the bar holds the constants, where the equals signs were. The second row, for instance, means $2x - y + z = 7$.

Keep the columns in order. If an unknown is missing from an equation, write a zero in its column: $x - 2z = 5$ becomes $(1 \;\; 0 \;\; -2 \mid 5)$.

Elimination works on equations; in matrix form you do the same things to rows. Write R_1, R_2, R_3 for the first, second and third rows. Three **row operations** never change the

solutions. Each one can be undone (swap back, multiply by $\frac{1}{k}$, or subtract k times the other row again), so any solution of the new system is a solution of the old one, and the reverse.

KEY 4.5 **Elementary row operations**

The three elementary row operations:

Operation	Notation	Meaning
Swap two rows	$R_1 \leftrightarrow R_2$	change the order of the equations
Multiply a row by a non-zero number	$R_2 \to kR_2$, $k \neq 0$	multiply every entry in the row by k
Add a multiple of another row	$R_2 \to R_2 + kR_1$	replace row 2 by row 2 plus k times row 1

Apply each operation to the whole row, including the constant after the bar.

In matrix form the same steps become a fixed routine. In each column, the entry you use to clear the entries below it is the **pivot**. Work one column at a time. Choose a pivot, a 1 if you can (swap rows, or divide a row, to get one). Make every entry beneath the pivot zero. Then move one row down and one column right, and repeat. Every system is handled the same way, so no step depends on finding a clever combination.

The target of row reduction is **row echelon form**. In it, the first non-zero entry in each row is that row's pivot, and each pivot lies further right than the pivot above it and has only zeros beneath it, so the pivots form a 'staircase' (each step goes one row down and one column right). Any all-zero rows go at the bottom. Read back as equations, the last row then contains only z and the row above it only y and z, which is the triangular form you solve from the bottom up.

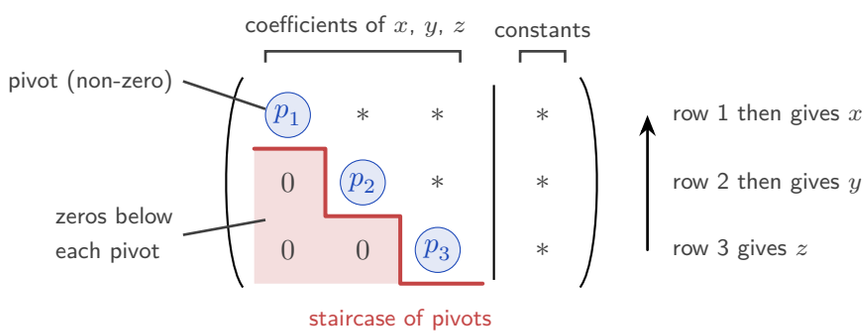


Figure 4.7 Row echelon form of a 3×3 system. Each pivot is non-zero and lies to the right of the one above it, with zeros beneath it; $*$ stands for any number. Eliminate downwards to reach this shape, then back-substitute upwards.

EXAMPLE 4.13

Use row reduction to solve the system

$$x + y + z = 6, \qquad 2x - y + z = 7, \qquad x + 2y - z = 3.$$

SOLUTION

These are the steps done on the equations above, now on the rows: each E becomes an R. Step 1 is $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$. The swap $R_2 \leftrightarrow R_3$ brings the row whose y-entry is 1 into row 2, so the second pivot is 1, as the routine prefers. Then $R_3 \to R_3 + 3R_2$ is Step 2:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & 1 & 7 \\ 1 & 2 & -1 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -1 & -5 \\ 0 & 1 & -2 & -3 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & -3 & -1 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & -7 & -14 \end{array}\right).$$

The pivots 1, 1 and -7 form the staircase of the figure above, and the rows read back as the same triangular system as before. Work upwards: $-7z = -14$ gives $z = 2$, then $y - 2z = -3$ gives $y = 1$, then $x + y + z = 6$ gives $x = 3$. Check the values in the original equations: $3 + 1 + 2 = 6$, $2(3) - 1 + 2 = 7$ and $3 + 2(1) - 2 = 3$. The solution is $(x, y, z) = (3, 1, 2)$

TIP The method in one line: eliminate downwards, then substitute upwards.

This system has one solution. Row reduction also shows when a system has no solution or infinitely many, as §4.3.2 explains.

4.3.2 When a system has no unique solution

Once a system is in row echelon form, its rows determine the outcome. There are only three possible outcomes for a 3×3 system.

Rows reduce to	What it means	The three planes
last row $(0\ 0\ a \mid b)$, $a \neq 0$	one unique solution	meet at a single point
any row $(0\ 0\ 0 \mid c)$, $c \neq 0$	no solution: the row says $0 = c$, which is false	have no common point: at least two are parallel, or each pair meets in a line and the three lines are parallel (a triangular prism)
$(0\ 0\ 0 \mid 0)$, and no row of the type above	infinitely many solutions: the row says $0 = 0$, which is true but tells us nothing	share a whole line, or a whole plane if two rows become zero

A system that has at least one solution is **consistent**; a system with no solution is **inconsistent**. In the table, the first and third rows are consistent, and the second is inconsistent.

First look for a row $(0\ 0\ 0 \mid c)$ with $c \neq 0$. Any such row, in any position, means there is no solution: the system is inconsistent. If there is none, count the rows that become all zeros. No zero row gives one solution. One zero row gives a line of solutions, described by one **free variable**, an unknown that can take any value. Two zero rows mean all three equations describe the same plane, and the solutions form that whole plane, described by two free variables. The geometry of three planes is treated in §9.6.3.

When one row reduces to $0 = 0$, choose as the free variable an unknown whose column contains no pivot. Set it equal to a parameter $(\text{a letter that can take any real value})$, such as $z = t$. This is often the last unknown, but not always: in $x + z = 2$, $y = 3$ the free variable is z, although the last non-zero row contains only y. Back-substitute up the echelon form to write each other unknown in terms of t. The result, (x, y, z) in terms of t with $t \in \mathbb{R}$, is the **general solution**, and it describes the line where the planes meet. In the next example, the outcome depends on the value of a constant k in the equations.

EXAMPLE 4.14

Find the value of k for which the system

$$x + y + 2z = 2, \qquad 2x - y + z = 1, \qquad x + 4y + 5z = k$$

has infinitely many solutions, and give the general solution. Show that every other value of k gives no solution.

SOLUTION

Write the augmented matrix and reduce it. Use $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$, then $R_3 \to R_3 + R_2$:

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 2 & -1 & 1 & 1 \\ 1 & 4 & 5 & k \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -3 \\ 0 & 3 & 3 & k-2 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 2 & 2 \\ 0 & -3 & -3 & -3 \\ 0 & 0 & 0 & k-5 \end{array}\right).$$

The last row says $0 = k - 5$. Read it against the table.
If $k \neq 5$, the last row is $(0\ 0\ 0 \mid c)$ with $c = k - 5 \neq 0$. It says $0 = c$, which is false, so the system is inconsistent: it has no solution, for every such k.
If $k = 5$, the last row is $(0\ 0\ 0 \mid 0)$. It is true but gives no new information, so the system is consistent and has infinitely many solutions. No pivot lies in the z-column, so let the free variable be $z = t$. Row 2 gives $-3y - 3t = -3$, so $y = 1 - t$. Row 1 gives

$$x = 2 - y - 2z = 2 - (1 - t) - 2t = 1 - t.$$

The general solution is

$$(x, y, z) = (1 - t,\ 1 - t,\ t), \quad t \in \mathbb{R}$$

This is a whole line of solutions: the line where the three planes meet. §9.6.3 writes such a solution as a vector equation of a line. Check with $t = 0$: the point $(1, 1, 0)$ satisfies all three original equations. So does $(0, 0, 1)$, from $t = 1$. One free parameter gives one line, so the geometry and the algebra give the same result.

CAUTION "No unique solution" covers two different cases, no solution and infinitely many solutions, so always separate them. Examination questions can ask for both, and for the general solution in terms of a parameter.

YOUR TURN 4.3 Systems of Linear Equations HL

Two equations in two unknowns

22. Solve each system by elimination.

(a) $x + y = 5, \quad x - y = 1$
(b) $3x - y = 5, \quad x + y = 3$
(c) $x + y = 4, \quad 2x + y = 7$
(d) $2x + 3y = 12, \quad x - y = 1$

23. In each system the first equation gives y in terms of x. Substitute it into the second equation, solve for x, then find y.

(a) $y = 2x, \quad x + y = 6$
(b) $y = x - 1, \quad 2x + y = 8$

24. State how many solutions each system has, and give a reason.

(a) $2x + y = 4, \quad 4x + 2y = 9$

(b) $y = x + 1, \quad y = x + 3$

(c) $x + y = 3, \quad 2x + 2y = 6$

(d) $x + y = 3, \quad x - y = 1$

Three equations in three unknowns

25. Solve each system of three equations.

(a) $x + y + z = 6, \quad x - y + z = 2, \quad x + y - z = 0$

(b) $x + y + z = 4, \quad 2x - y + z = 8, \quad x + 2y - z = -3$

(c) $x + y + z = 2, \quad 2x - y + 3z = -1, \quad x - 2y + z = -1$

(d) $2x + y - z = -4, \quad x + 3y + 2z = 3, \quad 3x - y + z = 9$

Row reduction and the number of solutions

26. For each system, write the augmented matrix, reduce it to row echelon form, and state the number of solutions.

(a) $x + y + z = 6, \quad x + 2y - z = 2, \quad 2x + 3y + z = 11$

(b) $x + y + z = 3, \quad 2x + 2y + 2z = 7, \quad x - y + z = 1$

(c) $x + y + z = 6, \quad x - y + z = 2, \quad 2x + 2z = 8$

(d) $x + y + z = 3, \quad x + y + 2z = 5, \quad 2x + 2y + 3z = 8$. Give the general solution.

27. Each system contains one or two unknown constants.

(a) For $x + y + z = 3, \ 2x + y + 3z = 7, \ 3x + 2y + 4z = k$, find the value of k for which the system has infinitely many solutions, and state what happens for every other value of k.

(b) For $x + y - z = 2, \ x + 2y + z = 5, \ 2x + 3y = k$, find the value of k for which the system has infinitely many solutions, and give the general solution for that value.

(c) For $x + y + z = 1, \ x + 2y + 3z = 4, \ 2x + 3y + kz = 5$, find the value of k for which the system does not have a unique solution. Show that for this value there are infinitely many solutions, and give the general solution.

(d) For $x + y + z = 2, \ x - y + 2z = 1, \ 3x + y + az = b$, find the conditions on a and b for which the system has (i) a unique solution, (ii) no solution, (iii) infinitely many solutions.

Concept check

28. A student eliminates x from $x + 2y = 4$ and $2x + 4y = 8$, obtains $0 = 0$, and writes "*no solution*".

(a) Explain why this conclusion is wrong, and state the correct number of solutions.

(b) The second equation is changed to $2x + 4y = 10$. State what elimination now gives, and what that result means.

4.4 Quadratic Functions and Equations

Quadratics describe the path of a thrown ball, the cable of a bridge and the area of a rectangle with a fixed perimeter. To use one you need its graph, which shows where it turns, and its equation, which gives where it crosses the axis.

4.4.1 The graph and its three forms

A **quadratic** is a function of the form $ax^2 + bx + c$ with $a \neq 0$, so its highest power is x^2. Its graph is the parabola, the curve of a thrown ball: a smooth U-shaped curve, opening up or down, that is symmetric about a vertical line through its turning point, which is called the **vertex**.

The same quadratic can be written in three forms, and each form shows a different feature immediately.

KEY 4.6 **Three forms of a quadratic**

A quadratic can be written in three forms:

$$y = ax^2 + bx + c, \qquad y = a(x - p)(x - q), \qquad y = a(x - h)^2 + k.$$

Expanded (or **standard**) form shows the y-intercept c; **factorised** form shows the zeros p and q; **vertex** form shows the turning point (h, k). The factorised form exists only when the quadratic has real zeros.

The sign of a sets the direction. If $a > 0$ the parabola opens upward and the turning point is a minimum. If $a < 0$ it opens downward and the turning point is a maximum. The curve is symmetric, and its mirror line passes through the vertex.

FORMULA 4.7 **Axis of symmetry** FORMULA BOOKLET · 2.6

The graph of $f(x) = ax^2 + bx + c$ has axis of symmetry

$$x = -\frac{b}{2a}.$$

When the zeros p and q are real, the axis lies halfway between them: $x = \frac{p+q}{2}$, the average of the roots, gives the same line.

To move from expanded form to vertex form, **complete the square**. Halve the coefficient of x and square the result: $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$. The bracket squared adds $\left(\frac{b}{2}\right)^2$, so you subtract it again to keep the value unchanged. When the coefficient of x^2 is not 1, take it out as a factor of the x^2 and x terms first, as in the second example below. The method is used again elsewhere in the course, for example to find an inverse (§4.1) and to integrate (Ch. 15).

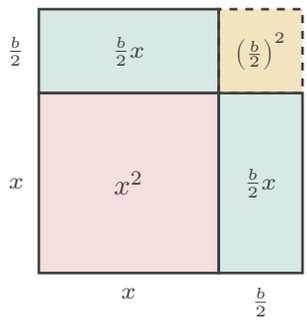


Figure 4.8 $x^2 + bx$ is a square and two strips. Adding the corner $\left(\frac{b}{2}\right)^2$ completes a square of side $x + \frac{b}{2}$, so $x^2 + bx = \left(x + \frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2$.

EXAMPLE 4.15

Write $y = x^2 - 10x + 21$ in vertex form.

SOLUTION

Take the $x^2 - 10x$ part and complete the square. Half of -10 is -5, and $(-5)^2 = 25$:

$$x^2 - 10x = (x - 5)^2 - 25.$$

So

$$y = (x - 5)^2 - 25 + 21 = \boxed{(x - 5)^2 - 4}$$

The vertex is $(5, -4)$ and the axis of symmetry is $x = 5$. Because $a = 1 > 0$, this vertex is the minimum point.

The same steps with letters give the axis of symmetry. Take out the factor a and complete the square:

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a},$$

which is vertex form with $h = -\dfrac{b}{2a}$. The inputs $x = h + t$ and $x = h - t$ give the same square, t^2, so they give the same output, and the curve is symmetric about $x = h$. When $a > 0$ the term $a(x - h)^2$ is never negative, so $y \geq k$ and the vertex is a minimum. When $a < 0$ that term is never positive, so the vertex is a maximum.

EXAMPLE 4.16

Write $y = -3x^2 + 12x - 5$ in vertex form.

SOLUTION

Take the factor -3 out of the x^2 and x terms only, then complete the square inside the bracket:

$$-3x^2 + 12x - 5 = -3\left(x^2 - 4x\right) - 5 = -3\left[(x - 2)^2 - 4\right] - 5.$$

Multiply the bracket out. The -4 is inside the bracket, so it is multiplied by -3 as well:

$$= -3(x-2)^2 + 12 - 5 = \boxed{-3(x-2)^2 + 7}$$

The vertex is $(2, 7)$, and because $a = -3 < 0$ it is the maximum point. Check by expanding: $-3(x^2 - 4x + 4) + 7 = -3x^2 + 12x - 5$.

Vertex form therefore gives the range of a quadratic by the method of §4.1. The square is never negative, so k is the smallest output when $a > 0$ (range $y \geq k$) and the largest output when $a < 0$ (range $y \leq k$).

EXAMPLE 4.17

Sketch $y = x^2 - 2x - 3$, showing the vertex, axis of symmetry, and all intercepts.

SOLUTION

Factorise for the zeros: $x^2 - 2x - 3 = (x-3)(x+1)$, so the x-intercepts are $(-1, 0)$ and $(3, 0)$. The y-intercept is $(0, -3)$. Completing the square, $y = (x-1)^2 - 4$, so the vertex is $(1, -4)$ and the axis of symmetry is $x = 1$, which is exactly halfway between the two zeros.

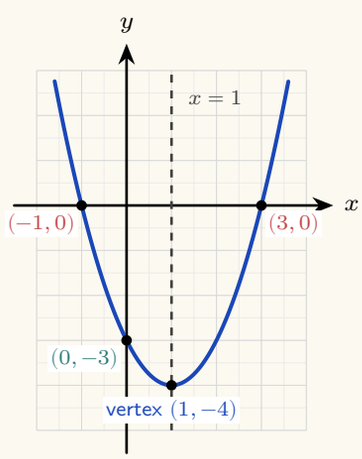


Figure 4.9 $y = x^2 - 2x - 3$, with its vertex, axis of symmetry and intercepts. The colours match the card of the three forms: the zeros in red, the y-intercept in teal and the vertex in blue.

The three forms also work in reverse. Given points on a parabola, choose the form that already contains what you know, then use the remaining point to fix a. When you know the two zeros and one further point, use the factorised form.

EXAMPLE 4.18

A parabola passes through $(0, -8)$, $(-1, 0)$ and $(4, 0)$. Find its equation.

SOLUTION

Two of the points lie on the x-axis, so they are the x-intercepts, and the zeros are -1 and 4. The factorised form gives

$$y = a(x+1)(x-4),$$

with a still unknown: infinitely many parabolas share a pair of zeros, and they differ only in how steeply they open. Substitute the third point, $(0,-8)$, to find a:

$$-8 = a(0+1)(0-4) = -4a \implies a = 2.$$

Expanding,

$$y = 2(x+1)(x-4) = \boxed{2x^2 - 6x - 8}$$

Check the three points: at $x = 0$, $y = -8$; at $x = -1$ and $x = 4$, $y = 0$. Had the curve passed through $(0,-4)$ instead, the same substitution would give $a = 1$, halving every y-coordinate while leaving the zeros where they are.

Setting a quadratic equal to zero is the same as asking where its parabola crosses the x-axis. There are three ways to find those crossings, and all three give the same answers. Choosing the right method saves time in an examination.

Method	Use it when	Be careful with
Factorisation	the quadratic factorises with whole numbers	many quadratics do not factorise, and searching for factors takes time
Completing the square	you also want the vertex, or an exact surd answer	the arithmetic is harder when $a \neq 1$
The formula	always. It works for every quadratic	slower than factorising when factors are obvious

4.4.2 Solving by factorisation

If the quadratic factorises, you can read the roots immediately. The reason is the zero product rule: a product is zero only when one of its factors is zero.

EXAMPLE 4.19

Solve $x^2 - 8x + 15 = 0$.

SOLUTION

Find two numbers that multiply to 15 and add to -8. They are -3 and -5:

$$(x-3)(x-5)=0.$$

A product is zero only when a factor is zero, so $x-3=0$ or $x-5=0$, giving $x=3$ or $x=5$

When the leading coefficient a is not 1, first take out any common factor: $2x^2-6x-8=2(x^2-3x-4)$. If $a\neq 1$ remains, split the middle term:

1. find two numbers that multiply to ac and add to b;
2. use them to split bx into two terms;
3. factorise the first pair and the second pair, and take out the common bracket.

EXAMPLE 4.20

Solve $3x^2+10x-8=0$.

SOLUTION

Here $ac=3\times(-8)=-24$. The numbers 12 and -2 multiply to -24 and add to 10, so split $10x$ as $12x-2x$:

$$3x^2+12x-2x-8=3x(x+4)-2(x+4)=(3x-2)(x+4)=0.$$

Expanding the brackets is a quick check: $(3x-2)(x+4)=3x^2+10x-8$. So $x=\frac{2}{3}$ or $x=-4$

CAUTION The zero product rule needs a zero on the right. From $(x-2)(x-3)=2$ you may *not* conclude $x-2=2$. Expand, collect everything on one side, and factorise again.

4.4.3 Solving by completing the square

Completing the square works whether or not the quadratic factorises, and it gives exact answers in surd form. Rewrite the quadratic as a perfect square plus a constant, then take square roots of both sides.

EXAMPLE 4.21

Solve $x^2-4x-1=0$, giving exact answers.

SOLUTION

Half of -4 is -2, and $(-2)^2 = 4$:

$$(x-2)^2 - 4 - 1 = 0 \implies (x-2)^2 = 5 \implies x - 2 = \pm\sqrt{5}.$$

This quadratic does not factorise with whole numbers, which is why the answer contains a surd: $x = 2 \pm \sqrt{5}$

TIP When $a \neq 1$, divide the whole equation by a first. For $2x^2 - 8x + 5 = 0$, divide by 2 to get $x^2 - 4x + 2.5 = 0$, then complete the square as usual. This works for an equation, because the right side is 0. To rewrite an expression, take a out as a factor instead, as in the vertex-form examples of §4.4.1.

4.4.4 Solving by formula

The formula works for every quadratic, and it is in the formula booklet.

FORMULA 4.8 **The quadratic formula** FORMULA BOOKLET · 2.7

The solutions of $ax^2 + bx + c = 0$, with $a \neq 0$, are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

- You do not need to memorise the formula, because the booklet prints it. But see where it comes from: it is **what completing the square gives** when the coefficients are letters instead of numbers, so you can derive it yourself from $ax^2 + bx + c = 0$.

Divide through by a, then complete the square on $x^2 + \frac{b}{a}x$:

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \implies \left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a} = 0.$$

Move the two constants to the right and write them over the common denominator $4a^2$:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Take the square root of both sides, which is where the $\pm$ enters, and then subtract $\frac{b}{2a}$:

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a} \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The $\pm$ matches the sketch of $y = x^2 - 2x - 3$ in §4.4.1: the parabola is symmetric, with one x-intercept on each side of its axis. Adding the two roots cancels the $\pm$ terms, so their

average is

$$\frac{1}{2}\left(\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}\right)=-\frac{b}{2a},$$

the axis of symmetry. This confirms the note in the axis of symmetry card of §4.4.1: for $y=x^2-2x-3$ the roots -1 and 3 average to 1.

EXAMPLE 4.22

Solve $2x^2-5x-12=0$ using the quadratic formula.

SOLUTION

Identify the coefficients: $a=2$, $b=-5$, $c=-12$. The formula gives

$$x=\frac{5\pm\sqrt{25+96}}{4}=\frac{5\pm 11}{4},$$

so $\boxed{x=4 \text{ or } x=-\tfrac{3}{2}}$

4.4.5 Quadratics in disguise

Some equations with powers higher than x^2 can be solved with nothing more than the methods above. The equation $x^4-5x^2+4=0$ has highest power x^4, yet it contains only x^4 and x^2, and the first is the square of the second: $x^4=(x^2)^2$. Give x^2 a name of its own and the equation becomes a quadratic.

EXAMPLE 4.23

Solve $x^4-5x^2+4=0$.

SOLUTION

Put $u=x^2$:

$$u^2-5u+4=0 \implies (u-1)(u-4)=0,$$

so $u=1$ or $u=4$. The question asked for x, not u, so convert each value back:

$$x^2=1 \implies x=\pm 1, \qquad x^2=4 \implies x=\pm 2.$$

An equation with highest power x^4 can have up to four roots, and this one has all four: $\boxed{x=\pm 1 \text{ or } x=\pm 2}$

An equation like this is called a **'quadratic in disguise'**: it is not a quadratic in x, but it becomes one after a substitution. It contains one expression and its square, and the unknown appears nowhere else. Look for a term that is the square of another.

Equation	Substitution	Values of u allowed
$x^4 - 5x^2 + 4 = 0$	$u = x^2$	$u \geq 0$
$x^6 - 7x^3 - 8 = 0$	$u = x^3$	any real u
$x - 5\sqrt{x} + 6 = 0$	$u = \sqrt{x}$	$u \geq 0$
$(\ln x)^2 - 3\ln x + 2 = 0$	$u = \ln x$	any real u
$2^{2x} - 10 \cdot 2^x + 16 = 0$	$u = 2^x$ (§1.5.3)	$u > 0$

The last column is a check that is easy to forget. Solve for u, reject any value that u cannot take, and only then convert back to x.

EXAMPLE 4.24

Solve $x^6 - 7x^3 - 8 = 0$.

SOLUTION

Here $x^6 = (x^3)^2$, so put $u = x^3$:

$$u^2 - 7u - 8 = 0 \implies (u - 8)(u + 1) = 0,$$

so $u = 8$ or $u = -1$. A cube can be negative, so neither value is rejected:

$$x^3 = 8 \implies x = 2, \qquad x^3 = -1 \implies x = -1.$$

Each cube has exactly one real cube root, so the equation has two real solutions, not six: $x = 2$ or $x = -1$

EXAMPLE 4.25

Solve $(\log_2 x)^2 - \log_2 x - 6 = 0$.

SOLUTION

The unknown appears only inside $\log_2 x$, once squared and once not, so put $u = \log_2 x$:

$$u^2 - u - 6 = 0 \implies (u - 3)(u + 2) = 0,$$

so $u = 3$ or $u = -2$. A logarithm can take any real value, so both are kept. Convert back with the definition of a logarithm:

$$\log_2 x = 3 \implies x = 2^3 = 8, \qquad \log_2 x = -2 \implies x = 2^{-2} = \tfrac{1}{4}.$$

Both are positive, so both lie in the domain of $\log_2 x$, and $x = 8$ or $x = \frac{1}{4}$

Note that $(\log_2 x)^2$ is the *square* of the logarithm, not $\log_2(x^2)$, which equals $2\log_2 x$ by the power law.

YOUR TURN 4.4 Quadratic Functions and Equations

Completing the square

29. Write each quadratic in vertex form and state its vertex.

(a) $y = x^2 + 4x + 1$ (b) $y = x^2 - 8x + 10$

(c) $y = x^2 + 3x + 1$ (d) $y = x^2 + 2x + 5$

30. Write each quadratic in the form $a(x-h)^2 + k$. *(Hint: take out the factor a from the x^2 and x terms first)*

(a) $2x^2 + 12x + 5$ (b) $3x^2 - 6x + 1$

(c) $-x^2 + 6x - 4$ (d) $-2x^2 - 8x + 3$

Vertex, axis and range

31. State the vertex and the axis of symmetry of each quadratic.

(a) $y = (x-2)^2 + 5$ (b) $y = x^2 - 8x + 3$

(c) $y = -(x+1)^2 + 2$ (d) $y = 2(x-3)^2 - 1$

32. Find the range of each quadratic by completing the square. Each is defined for all real x.

(a) $x^2 - 6x + 11$ (b) $x^2 + 5x$

(c) $2x^2 - 4x + 7$ (d) $1 + 4x - 2x^2$

33. State whether each quadratic has a maximum or a minimum, give a reason, and find that maximum or minimum value.

(a) $y = -x^2 + 2x$ (b) $y = 3x^2 - x + 1$

(c) $y = x^2 - 6x + 5$ (d) $y = 5 + 4x - x^2$

Zeros, intercepts and sketches

34. Consider $y = (x-1)(x+5)$.

(a) Write down the zeros of the function. (b) Find its y-intercept.

(c) Find its axis of symmetry. (d) Find its vertex.

35. Sketch each parabola, showing the vertex, the axis of symmetry and all intercepts.

(a) $y = -x^2 + 2x + 8$ (b) $y = 2x^2 - 4x - 6$

(c) $y = x^2 - 4x - 5$ (d) $y = -x^2 - 6x - 5$

Finding the equation of a parabola

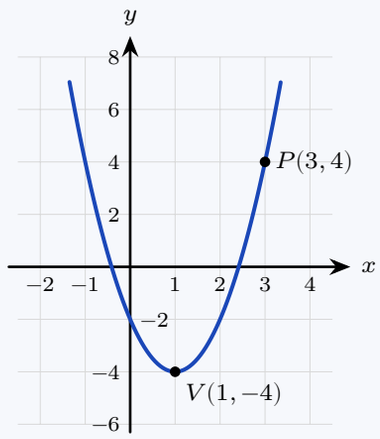
36. Find the equation of each parabola in the form $y = ax^2 + bx + c$.

(a) It has x-intercepts $(-2, 0)$ and $(3, 0)$, and passes through $(0, 12)$.

(b) It passes through $(1, 0)$, $(5, 0)$ and $(3, -8)$.

(c) Its vertex is $(2, -3)$, and it passes through $(0, 5)$.

(d) The parabola shown, with vertex $V(1, -4)$ and passing through $P(3, 4)$.



Concept check: graphs

37. A student states that the vertex of $y = (x + 3)^2 - 5$ is $(3, -5)$. Identify the error and state the correct vertex.

38. A student completes the square as follows.

$$3x^2 - 12x + 5 = 3(x - 2)^2 - 4 + 5 = 3(x - 2)^2 + 1$$

Identify the error, correct the working, and state the vertex.

Factorisation

39. Solve each equation by factorisation.

(a) $x^2 - 5x + 6 = 0$

(b) $x^2 - x - 12 = 0$

(c) $x^2 + x - 20 = 0$

(d) $2x^2 - 5x - 3 = 0$

Rearranging to zero first

40. Rearrange each equation so that one side is 0, then solve it by factorisation.

(a) $x^2 = 7x$

(b) $x(x + 4) = 21$

(c) $(x - 2)(x - 3) = 2$

(d) $3x^2 = 2(x + 4)$

Solving by completing the square

41. Solve each equation by completing the square, giving exact answers.

(a) $x^2 + 8x + 7 = 0$

(b) $x^2 - 2x - 5 = 0$

(c) $x^2 + 6x + 2 = 0$ (d) $2x^2 + 12x + 7 = 0$

The quadratic formula

42. Solve each equation with the quadratic formula, giving exact answers.

(a) $x^2 + 2x - 1 = 0$ (b) $x^2 + 6x + 4 = 0$
(c) $2x^2 - 3x - 2 = 0$ (d) $3x^2 + 2x - 1 = 0$

Quadratics in disguise

43. Each equation is a quadratic in an exponential. Solve for real x, giving exact answers.

(a) $2^{2x} - 6 \cdot 2^x + 8 = 0$ (b) $9^x - 4 \cdot 3^x + 3 = 0$
(c) $e^{2x} - 5e^x + 6 = 0$ (d) $e^{2x} - e^x - 6 = 0$
(e) $4^x - 2^{x+2} - 32 = 0$ (f) $e^x + 12e^{-x} = 7$

44. Solve each equation for real x, giving exact answers.

(a) $x^4 - 13x^2 + 36 = 0$ (b) $x^4 + 3x^2 - 4 = 0$
(c) $x^6 + 7x^3 - 8 = 0$ (d) $x - 7\sqrt{x} + 10 = 0$
(e) $(\ln x)^2 - 5\ln x + 6 = 0$ (f) $2(\log_3 x)^2 - \log_3 x - 1 = 0$

Concept check: equations

45. A student solves $x^2 = 5x$ by dividing both sides by x, and writes "$x = 5$". Identify the error and give the complete solution.

46. A student says that $(\ln x)^2$ and $\ln(x^2)$ are the same, because both equal $2\ln x$. Explain the error, and use $x = e^3$ to show that the two expressions differ.

4.5 Quadratic Inequalities

A quadratic **inequality** is a different question. It does not ask where the parabola meets the axis. It asks where the parabola lies above the axis, or below it. The method has three steps.

KEY 4.9 **Solving a quadratic inequality**

To solve a quadratic inequality:

1. Collect every term on one side, so the other side is 0.
2. Find the roots of the matching equation. These are the only places where the sign can change.
3. Sketch the parabola and read the intervals where it lies on the side you want.

With $<$ or $>$ the roots themselves are excluded; with $\leq$ or $\geq$ they are included, because the quadratic is zero there.

The sketch is the main step of the method, so here is one in full. The roots cut the axis into three stretches, and on each stretch the curve stays on one side of the axis.

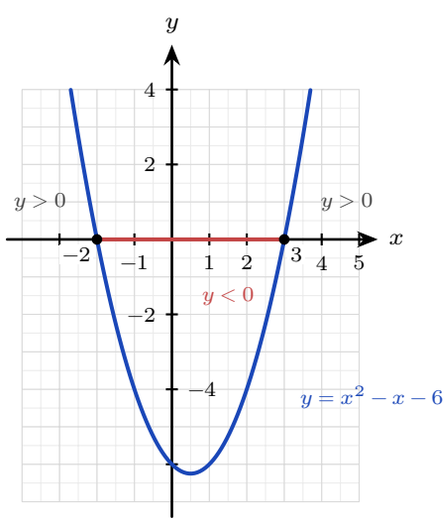


Figure 4.10 $y = x^2 - x - 6$ lies below the axis between its zeros -2 and 3 (red) and above it outside them, so $x^2 - x - 6 < 0$ exactly when $-2 < x < 3$.

Read the picture, not a rule. An upward parabola lies below the axis between its x-intercepts and above it outside them; a downward one is the other way round. The table shows the four cases, for a quadratic with two real roots.

Parabola	< 0 where	> 0 where
$a > 0$ (opens up)	between the roots	outside the roots
$a < 0$ (opens down)	outside the roots	between the roots

If the matching equation has no real roots, the parabola never meets the axis, so it lies entirely on one side of it. The inequality then holds for every real x or for none, and the sign of a decides which. Completing the square shows this directly (§4.6 gives a quicker test).

The same figure also shows that $x^2 - x - 6 > 0$ exactly when $x < -2$ or $x > 3$. A **sign table** gives the same answer; it needs no sketch and is used again for rational inequalities in

§5.4.5. The quadratic must be in factorised form first. Each factor changes sign at its own root. Multiply the signs in each column to get the sign of the product.

	$x < -2$	$-2 < x < 3$	$x > 3$
$x+2$	$-$	$+$	$+$
$x-3$	$-$	$-$	$+$
$(x+2)(x-3)$	$+$	$-$	$+$

A repeated root is the one case in which the sign does not change. Since $(x-2)^2$ is never negative, $(x-2)^2 > 0$ holds for every x except $x = 2$, while $(x-2)^2 \leq 0$ holds only at $x = 2$.

EXAMPLE 4.26

Solve $3x^2 \leq 5x + 2$.

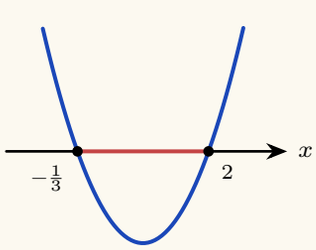
SOLUTION

First collect everything on one side:

$$3x^2 - 5x - 2 \leq 0.$$

Factorise: $(3x+1)(x-2) = 0$ gives roots $x = -\frac{1}{3}$ and $x = 2$. A quick sketch gives the answer. The parabola opens upward, so it lies *below* the axis between the roots, and the sign $\leq$ includes the roots themselves:

$$-\tfrac{1}{3} \leq x \leq 2$$



≤ 0 here

CAUTION Do not divide an inequality by a term containing x unless its sign is known. Dividing by a negative quantity reverses the inequality. Collect on one side and factorise instead.

EXAMPLE 4.27

Solve $x^2 + 3x + 4 > 0$.

SOLUTION

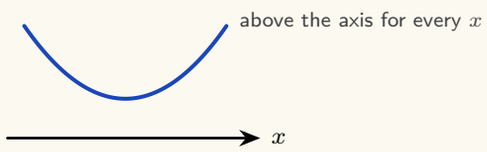
Look first for the roots of the matching equation $x^2 + 3x + 4 = 0$. It does not factorise, so use the formula:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(1)(4)}}{2} = \frac{-3 \pm \sqrt{-7}}{2}.$$

A negative number has no real square root, so there are no real roots: the parabola never meets the x-axis. Completing the square shows which side it stays on:

$$x^2 + 3x + 4 = \left(x + \tfrac{3}{2}\right)^2 + \tfrac{7}{4}.$$

A square is never negative, so the expression is at least $\frac{7}{4}$ and the parabola lies entirely above the axis. The inequality holds for *every* real x, so the solution is $x \in \mathbb{R}$
Had the sign been $<$, there would be no solution at all.



TIP Make a quick sketch for every quadratic inequality: an axis, the parabola the right way up, and its roots, if it has any. It takes seconds, and it shows at once which side of the axis you need. This is a step where marks are often lost.

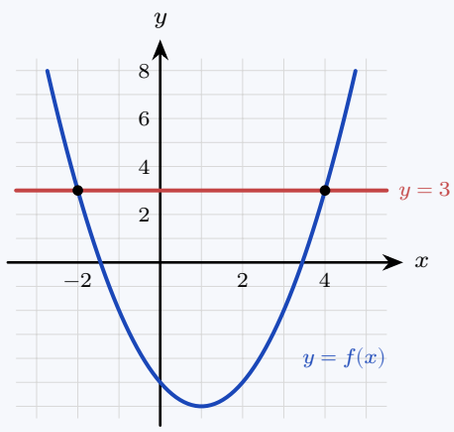
YOUR TURN 4.5 Quadratic Inequalities

Solving a quadratic inequality

47. Solve each inequality.

(a) $x^2 - 9 < 0$
(b) $x^2 + 2x - 15 < 0$
(c) $x^2 - 4x + 3 \geq 0$
(d) $-x^2 + 4x > 0$
(e) $x^2 - x - 12 > 0$, using a sign table

(f) The graph of a quadratic function $y = f(x)$ and the line $y = 3$ are shown. Use the graph to solve $f(x) < 3$.



(g) On one set of axes, sketch the graphs of $y = x^2 - 4$ and $y = x + 2$, showing the intercepts and the points where they meet. Hence solve $x^2 - 4 < x + 2$.

48. Solve each inequality. Two of them are special cases, so justify your answers.

(a) $2x^2 \leq x + 6$ (b) $x^2 \geq 16$

(c) $x^2 + x + 1 > 0$ (d) $x^2 + 4 < 0$

49. A rectangle has width x and length $x + 2$, both in centimetres.

(a) Show that its area is $x^2 + 2x$ square centimetres.

(b) Find the values of x for which the area is less than 15 square centimetres.

Concept check

50. A student solves $x^2 > 9$ by writing "$x > \pm 3$, *so* $x > 3$ *or* $x > -3$." Identify the error and give the correct solution.

4.6 The Discriminant

The quadratic formula gives more information than the roots. Before you calculate a single root, the quantity under the square root sign already tells you how many real roots there are. That quantity, $b^2 - 4ac$, is called the discriminant.

FORMULA 4.10 **The discriminant** FORMULA BOOKLET · 2.7

The **discriminant** of $ax^2 + bx + c$ is

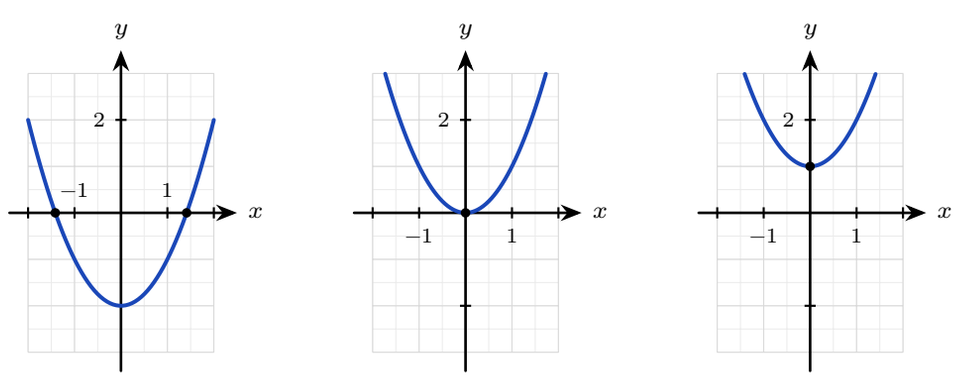
$$\Delta = b^2 - 4ac.$$

Its sign tells you how many real roots the equation has, without solving it.

The sign of Δ determines how many times the parabola meets the x-axis. The reason is the term $\pm\sqrt{\Delta}$ in the formula. When $\Delta > 0$ it gives two different values, so there are two roots. When $\Delta = 0$ it is ± 0, so the two roots are equal. When $\Delta < 0$ it has no real value, because a negative number has no real square root.

Discriminant	Real roots	Parabola and x-axis
$\Delta > 0$	two distinct	crosses twice
$\Delta = 0$	one repeated (two equal roots)	touches once
$\Delta < 0$	none	does not meet it

The three cases are three positions of the same parabola relative to the x-axis.



(a) $\Delta > 0$: crosses twice (b) $\Delta = 0$: touches once (c) $\Delta < 0$: never meets

Figure 4.11 $y = x^2 - 2$, $y = x^2$ and $y = x^2 + 1$ have two x-intercepts, one and none, so the matching equations have two distinct roots, one repeated root and no real roots.

A downward parabola behaves the same way, with the picture turned over: the sign of a decides which way it opens, and Δ still counts the crossings.

EXAMPLE 4.28

For the equation $3kx^2 + 2x + k = 0$, find the values of k that give two distinct real roots, two equal real roots, and no real roots.

SOLUTION

Here the unknown constant appears in two places: $a = 3k$ and $c = k$, with $b = 2$. So

$$\Delta = b^2 - 4ac = 4 - 4(3k)(k) = 4 - 12k^2.$$

Now consider the three cases separately.

$$\Delta > 0 : \; 4 - 12k^2 > 0 \implies k^2 < \tfrac{1}{3} \implies -\tfrac{1}{\sqrt{3}} < k < \tfrac{1}{\sqrt{3}}.$$

$$\Delta = 0 : \; k^2 = \tfrac{1}{3} \implies \boxed{k = \pm\tfrac{1}{\sqrt{3}}}$$

$$\Delta < 0 : \; k^2 > \tfrac{1}{3} \implies \boxed{k < -\tfrac{1}{\sqrt{3}} \text{ or } k > \tfrac{1}{\sqrt{3}}}$$

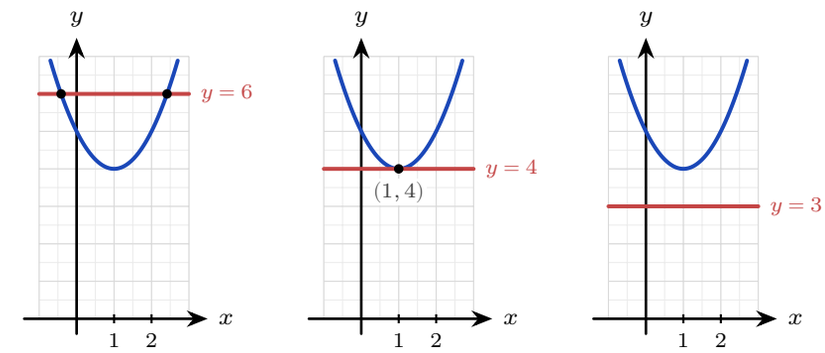
The value $k = 0$ must be checked separately: the equation then becomes $2x = 0$. That is linear, not quadratic, and it has the single root $x = 0$. So the answer for two distinct real roots is $\boxed{-\tfrac{1}{\sqrt{3}} < k < \tfrac{1}{\sqrt{3}},\ k \neq 0}$

CAUTION Before using the discriminant, check that the equation really is quadratic. If the coefficient of x^2 contains the unknown constant, the value that makes it zero must be excluded and treated separately.

TIP When the equation is a quadratic, a question that says "two distinct roots", "equal roots", "a repeated root", "touches the x-axis", or "no real solutions" usually means you need the discriminant. If you recognise the wording, you know which method to use.

4.6.1 A line meeting a parabola

The discriminant can also count the points where a line meets a parabola. A point common to two graphs satisfies both equations, so setting the two expressions for y equal gives a quadratic whose **real roots are the x-coordinates of the intersections**. The discriminant of that quadratic tells you how many there are. When that equation has one repeated root, the line meets the parabola at exactly one point. The line is then a tangent to the parabola (§13.2 gives the general definition of a tangent).



(a) $\Delta > 0$: two points (b) $\Delta = 0$: tangent (c) $\Delta < 0$: no point

Figure 4.12 The parabola $y = x^2 - 2x + 5$ and three horizontal lines. (a) $y = 6$ meets it at two points. (b) $y = 4$ touches it at the vertex $(1, 4)$, so it is a tangent. (c) $y = 3$ does not meet it. In each case the discriminant of the equation for the x-coordinates gives the number of points.

EXAMPLE 4.29

The line $y = mx + c$ meets the parabola $y = x^2 - 2x + 5$. (a) Find the discriminant of the equation for the x-coordinates of the intersections, and say what its sign tells you. (b) Find the values of m for which $y = mx + 4$ is a tangent to the parabola.

SOLUTION

(a) At any intersection both equations hold, so

$$x^2 - 2x + 5 = mx + c \implies x^2 - (m+2)x + (5-c) = 0.$$

Its coefficients are 1, $-(m+2)$ and $5-c$, so

$$\Delta = (m+2)^2 - 4(1)(5-c) = \boxed{(m+2)^2 + 4c - 20}$$

The constant term is $5 - c$, not c; the intercept c changes sign when it moves to the left-hand side. Each real root is one intersection point, so $\Delta > 0$ gives two intersections, $\Delta = 0$ gives one and the line is a tangent, and $\Delta < 0$ means the line misses the parabola entirely.

(b) For $y = mx + 4$ take $c = 4$ and apply the tangency condition $\Delta = 0$:

$$\Delta = (m+2)^2 - 4 = 0 \implies m + 2 = \pm 2 \implies \boxed{m = 0 \text{ or } m = -4}$$

Check $m = 0$: the quadratic is $x^2 - 2x + 1 = (x-1)^2$, a repeated root at $x = 1$, so the horizontal line $y = 4$ touches at the vertex $(1, 4)$. For $m = -4$ the quadratic is $x^2 + 2x + 1 = (x+1)^2$, touching at $(-1, 8)$.

A tangent gives a repeated root, so factorising the quadratic at each value of m both confirms the answer and gives the point of contact.

4.6.2 The sum and product of the roots HL

The zeros of a quadratic are linked to its coefficients before you solve anything. If $ax^2 + bx + c$ has zeros α and β (the p and q of §4.4, renamed), with $a \neq 0$ so that the expression really is quadratic, then it factorises as

$$a(x - \alpha)(x - \beta) = ax^2 - a(\alpha + \beta)x + a\,\alpha\beta.$$

Comparing the coefficient of x gives $b = -a(\alpha + \beta)$, and comparing the constant term gives $c = a\,\alpha\beta$. Since $a \neq 0$, you can divide each by a. The sum of the roots is then $-\dfrac{b}{a}$, and their product is $\dfrac{c}{a}$. The same comparison works for a polynomial of any degree, with one bracket for each root. The general result is in Ch. 6.

The quadratic formula gives the same results directly. Write the roots as

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding them cancels the square roots, and multiplying them is a difference of two squares:

$$\alpha + \beta = \frac{-2b}{2a} = -\frac{b}{a}, \qquad \alpha\beta = \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

KEY 4.11 **Sum and product of the roots** HL

If $ax^2 + bx + c = 0$ has roots α and β, then

$$\alpha + \beta = -\frac{b}{a}, \qquad \alpha\beta = \frac{c}{a}.$$

The booklet does not print this form. It prints the rule for any degree (booklet 2.12, Ch. 6), and this is its case $n = 2$.

These let you find expressions in the roots without solving the equation. Any expression that stays the same when α and β are swapped can be rewritten in terms of $\alpha + \beta$ and $\alpha\beta$; for example, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$.

EXAMPLE 4.30

The roots of $2x^2 - 6x + 3 = 0$ are α and β. Find $\alpha^2 + \beta^2$ without solving the equation.

SOLUTION

Read the sum and product from the coefficients. Here $\alpha + \beta = -\dfrac{-6}{2} = 3$ and $\alpha\beta = \dfrac{3}{2}$. Since $(\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2$,

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = 9 - 3 = \boxed{6}$$

YOUR TURN 4.6 The Discriminant

The nature of the roots

51. Evaluate the discriminant of each equation and state the nature of its roots.

(a) $x^2 - 4x + 4 = 0$
(b) $2x^2 + 3x + 5 = 0$
(c) $3x^2 - 2x - 1 = 0$
(d) $x^2 + 5x + 7 = 0$

Conditions on a coefficient

52. Find the value or values of k in each case.

(a) $x^2 + 6x + k = 0$ has equal roots
(b) $x^2 + (k-2)x + 4 = 0$ has equal roots
(c) $x^2 - 3x + k = 0$ has no real roots
(d) $(k-1)x^2 + 4x + 2 = 0$ has two distinct real roots. State any value of k that must be excluded.

53. The equation $x^2 + (k+1)x + 9 = 0$ has two distinct real roots. Find the set of values of k. *(Hint: the condition is a quadratic inequality, as in §4.5)*

A line meeting a parabola

54. Each part concerns a line and a parabola.

(a) Find the number of points at which the line $y = x + 3$ meets the parabola $y = x^2 - 3x + 5$.
(b) Show that the line $y = 2x - 1$ is a tangent to $y = x^2 - 4x + 8$, and find the point of contact.
(c) Find the value of c for which the line $y = 3x + c$ is a tangent to $y = x^2 + x + 2$.
(d) Find the set of values of m for which the line $y = mx - 1$ does not meet $y = x^2 + 3$. *(Hint: the inequality is solved as in §4.5)*

The discriminant with inequalities

55. The equation $x^2 + kx + (k+3) = 0$ has discriminant Δ.

(a) Show that $\Delta = (k-6)(k+2)$.
(b) Find the values of k for which the equation has two distinct real roots.
(c) Find the values of k for which the equation has no real roots.
(d) Find the values of k for which the roots are equal, and state the repeated root in each case.

56. Find the values of k for which $kx^2 - 4x + (k-3) > 0$ for every real x. *(Hint: two conditions are needed: one on the discriminant and one on the leading coefficient)*

57. Let $f(x) = \dfrac{x}{x^2+1}$.

(a) Show that $f(x) = k$ can be written as $kx^2 - x + k = 0$.

(b) Find the values of k for which this equation has a real solution, treating $k = 0$ separately.

(c) Hence state the range of f.

The sum and product of the roots

58. Answer each part without solving the equation.

(a) The roots of $3x^2 + 6x - 2 = 0$ are α and β. Write down $\alpha + \beta$ and $\alpha\beta$.

(b) The roots of $x^2 - 4x + 1 = 0$ are α and β. Find $\alpha^2 + \beta^2$.

(c) For the same equation as in part (b), find $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$.

(d) The roots of $2x^2 - 3x - 1 = 0$ are α and β. Find $(\alpha - \beta)^2$. *(Hint: expand $(\alpha - \beta)^2$ and compare it with $(\alpha + \beta)^2$)*

Concept check

59. A student says: "*the discriminant of $2x^2 - 3x + 5$ is negative, so its graph lies below the x-axis.*" Explain why this is false.

60. State the discriminant condition for "*real roots*" and for "*two distinct real roots*".

Chapter 4 · Lines & Quadratics

Reflections

TOK Mathematicians of the seventeenth century had a surprising idea: every line is an equation, and every linear equation is a line. A problem about shapes can then be solved with algebra, and a problem in algebra can be solved with a picture. Neither form is more real than the other. When you switch from one to the other, do you gain new knowledge, or only a new view of knowledge you already had?

IM Nearly four thousand years ago, scribes in Babylon pressed problems like this into clay tablets: "the area of a square added to its side is $\frac{3}{4}$; find the side." They solved them by a fixed sequence of steps that is completing the square, with no symbols at all. Over two thousand years later the Indian mathematician Brahmagupta stated a general rule for such equations in words. He also gave rules for calculating with zero and with negative numbers, which he called debts. The method of §4.4 is ancient; only the notation is new. Does a method belong to whoever first used it, or to whoever first wrote it down in general?

RW A satellite dish is a parabola spun about its axis. Signals from a satellite far away arrive as parallel rays, and a parabola reflects every ray parallel to its axis to one point, the **focus**, where the receiver sits. For the parabola $y = x^2/(4p)$ the focus is at $(0, p)$. The proof of the reflection property is beyond this course. Used the other way round, the same shape makes a car headlight: a bulb at the focus sends its light out as a parallel beam. *Try it:* a dish is 60 cm across and 9 cm deep. How far above the centre of the dish should the receiver be?

EXT The discriminant divides quadratics into three cases. When $\Delta < 0$ the parabola does not meet the x-axis and there are no *real* roots, but the quadratic formula still returns answers, now containing $\sqrt{-1}$. If you keep these "impossible" numbers instead of rejecting them, you reach the **complex numbers** of Chapter 17. There every quadratic has exactly two roots, counting a repeated root twice, and the discriminant tells you only whether those roots are real.

TOK Did the parabola exist before anyone wrote $y = x^2$? Did the ball know its path?

End-of-Chapter Problems
Chapter 4: Lines & Quadratics

1. Consider $f(x) = \dfrac{3x-2}{x+1}$, $x \neq -1$.

(a) Find $f^{-1}(x)$. [3]

(b) State the domain of f^{-1}, and hence write down the range of f. [2]

2. GDC Consider $f(x) = x^3 - 4x + 1$.

(a) Using your GDC, sketch the graph of f for $-3 \leq x \leq 3$, labelling all key features. [2]

(b) Find the zeros of f, giving your answers to 3 significant figures. [2]

(c) Find the coordinates of the local maximum and local minimum points. [2]

(d) Solve $f(x) = 2x - 1$, giving your answers to 3 significant figures. [2]

3. Find the equation of the line through $(3, -1)$ parallel to $2x - y = 4$. [3]

4. The points $A(-2, 7)$ and $B(4, -1)$ lie on the line L_1.

(a) Find the gradient of L_1 and the distance AB. [3]

(b) Show that L_1 has equation $4x + 3y = 13$. [2]

(c) Write down the x- and y-intercepts of L_1. [2]

(d) The line L_2 has equation $x + y = 4$. Find the point of intersection of L_1 and L_2. [3]

(e) Show that this point is the midpoint of AB. [1]

5. A line passes through $A(1, 2)$ and $B(5, 4)$.

(a) Find the gradient of AB. [1]

(b) Find the equation of AB. [2]

(c) Find the equation of the perpendicular bisector of AB. [4]

6. Solve the system

$$x + 2y + z = 4, \qquad 2x + y - z = 2, \qquad x - y + 2z = 2.$$

[6]

7. Consider the system

$$x + y + z = 3, \qquad x + 2y + 3z = 4, \qquad 2x + 3y + 4z = k.$$

(a) Find the value of k for which the system has solutions. [3]
(b) For this value of k, find the general solution. [3]
(c) Describe the configuration of the three planes when $k = 10$. [1]

8. GDC A theatre sells adult tickets at \$14.50, student tickets at \$9.75 and child tickets at \$6.25. On one evening it sold 412 tickets for a total of \$4522. The number of adult tickets sold was 52 fewer than the number of student and child tickets together.

(a) Write down a system of three equations for the numbers a, s and c of adult, student and child tickets sold. [3]
(b) Use your GDC to solve the system. [2]

9. A quadratic function f has zeros at $x = -1$ and $x = 4$, and $f(0) = -8$.

(a) Write $f(x)$ in factorised form. [2]
(b) Write $f(x)$ in expanded form. [1]
(c) Find the coordinates of the vertex of the graph of f. [2]
(d) Hence write down the range of f. [1]

10. Consider $f(x) = x^2 + 2x - 3$.

(a) Factorise $f(x)$ and state its zeros. [2]
(b) Find the vertex. [2]
(c) Sketch the graph, showing all intercepts. [2]

11. Consider $f(x) = -x^2 + 4x + 5$, $x \in \mathbb{R}$.

(a) Write $f(x)$ in the form $a(x - h)^2 + k$. [2]
(b) Write down the vertex and the axis of symmetry of the graph of f. [1]
(c) State the range of f. [1]
(d) Find the x- and y-intercepts of the graph of f. [2]

12. A rectangle has length 3 more than its width w.

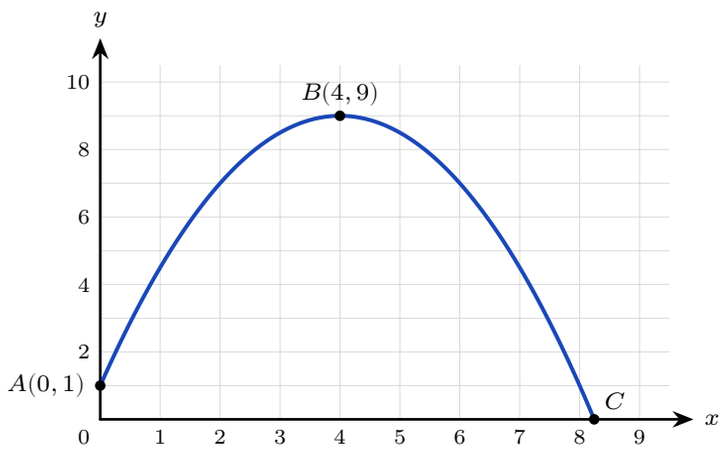
(a) Write its area A as a function of w. [2]
(b) Given the area is 40, find w. [3]

13. A rectangle has perimeter 20. Show that its area is $A = x(10 - x)$, where x is one side, and find the maximum possible area. [5]

14. Solve the inequality $x^2 - 2x - 8 \leq 0$. [3]

15. Solve the inequality $x^2 + x - 6 > 0$. [3]

16. A ball is thrown from a point 1 m above level ground. Its path is part of a parabola, shown in the diagram, where x m is the horizontal distance travelled and y m is the height above the ground. The ball is released at A, reaches its greatest height at B and lands at C.



(a) Find the equation of the path in the form $y = a(x - h)^2 + k$. [3]

(b) Find the exact horizontal distance from the point of release to C. [3]

(c) Find the values of x for which the ball is more than 7 m above the ground. [3]

17. The equation $x^2 - 4x + c = 0$ has no real roots. Find the set of values of c. [3]

18. The equation $x^2 + kx + 4 = 0$ has two distinct real roots. Find the set of values of k. [4]

19. Find the values of k for which $x^2 + kx + (k + 3) = 0$ has equal roots. [4]

20. Show that $x^2 + (k + 1)x + k = 0$ has real roots for every real k, and find the value of k for which the roots are equal. [4]

21. Find the points of intersection of $y = x^2$ and $y = 2x + 3$. [4]

22. Consider the line $y = 2x - 5$ and the parabola $y = x^2 - 2x + 2$.

(a) Show that they do not intersect. [3]

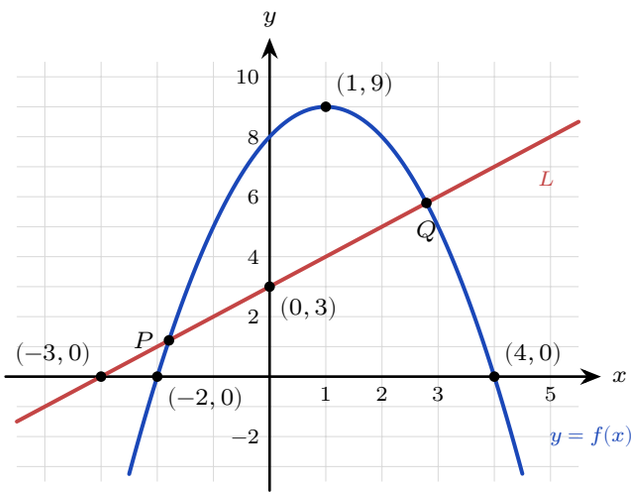
(b) Explain what this means geometrically. [1]

23. Consider the parabola $y = x^2$.

(a) Find the value of k for which the line $y = 3x + k$ is a tangent to the parabola. [2]

(b) Prove that the line $y = mx + c$ is a tangent to the parabola if and only if $c = -\frac{m^2}{4}$. [3]

24. The diagram shows the graph of a quadratic function f and a straight line L with equation $y = g(x)$. The line L meets the graph of f at P and Q.



(a) Find $f(x)$, giving your answer in the form $ax^2 + bx + c$. [3]
(b) Find the equation of L. [2]
(c) Find the exact coordinates of P and Q. [5]
(d) Hence write down the values of x for which $f(x) > g(x)$. [1]

25. Two numbers have a sum of 10 and a product of 21. Find the numbers. [3]

26. The roots of $x^2 + bx + c = 0$ sum to 6 and differ by 4. Find b and c. [3]

27. The roots of $x^2 - px + q = 0$ are α and 2α. Show that $2p^2 = 9q$. [4]

28. GDC A ball is thrown upwards from a height of 1.6 m. Its height in metres t seconds later is $h(t) = 1.6 + 12.5t - 4.9t^2$, until it hits the ground.

(a) Find the maximum height of the ball, and the time at which it is reached. [2]
(b) Find the time at which the ball hits the ground. [2]
(c) Find the length of time for which the ball is more than 6 m above the ground. [3]

Give your answers to three significant figures.

29. GDC A bakery bakes x cakes a day. Its daily profit, in dollars, is modelled by $P(x) = -0.35x^2 + 42x - 600$.

(a) Find the number of cakes the bakery should bake each day to maximise its profit, and state the maximum daily profit. [2]
(b) Find the numbers of cakes for which the bakery makes a profit. [3]
(c) Find the numbers of cakes for which the daily profit is more than \$400. [2]

30. GDC The path of a javelin is modelled by $y = ax^2 + bx + c$, where x m is the horizontal distance from the point of release and y m is the height above level ground. The javelin is released at a height of 1.9 m, and it is 11.1 m high when $x = 20$ and 10.7 m high when $x = 40$.

(a) Write down three equations in a, b and c. [2]
(b) Use your GDC to find a, b and c. [2]
(c) Find the maximum height of the javelin. [2]
(d) Find the horizontal distance from the point of release to the point where the javelin lands. [2]
(e) Find the horizontal distance between the two points of the path at which the javelin is 8 m high. [2]

Give your answers to parts (c) to (e) to three significant figures.

31. A bridge has a parabolic arch. The arch is 40 m wide at road level and 10 m high at its centre. Take the road as the x-axis, with the centre of the arch at $x = 0$.

(a) Show that the arch is modelled by $y = 10 - \dfrac{x^2}{40}$. [4]
(b) Find the height of the arch 12 m from the centre. [2]
(c) A lorry 8 m wide and 9.5 m high drives along the centre of the road. Determine whether it passes under the arch. [4]
(d) Find the greatest width of a lorry 9 m high that can pass under the arch, giving your answer in the form $k\sqrt{10}$. [4]
(e) The model gives a negative value of y when $|x| > 20$. Explain what this means for the model. [2]

32. Consider the line $y = mx + 3$ and the parabola $y = x^2 + 5x + 7$.

(a) Show that at any point of intersection, $x^2 + (5 - m)x + 4 = 0$. [3]
(b) Find the values of m for which the line is a tangent to the parabola. [4]
(c) Find the values of m for which the line does not meet the parabola. [3]
(d) For the smaller value of m found in part (b), find the coordinates of the point of contact. [3]
(e) State the values of m for which the line meets the parabola twice. [2]

33. HL Consider the system of equations

$$x + 2y - z = 1, \qquad 2x + 5y + z = 5, \qquad x + 4y + az = b,$$

where a and b are real constants.

(a) Write the system as an augmented matrix and row reduce it. [4]
(b) Find the value of a for which the system does not have a unique solution. [3]

(c) For that value of a, find the value of b for which the system is consistent. [2]
(d) For those values of a and b, find the general solution of the system. [4]
(e) Describe how the three planes are arranged in each of the three cases: a not equal to the value in part (b); a equal to it and the system consistent; and a equal to it and the system inconsistent. [3]

34. Find all the values of k for which the equation $(k-2)x^2 + 4x + (k+1) = 0$ has exactly one real solution, and give the solution in each case. [6]

35. Solve each equation for real x, giving exact answers.

(a) $x - \sqrt{x} - 6 = 0$ [3]
(b) $3^{2x} - 3^{x+1} - 4 = 0$ [3]

36. Let $f(x) = x^2 + 4x + 1$, $x \leq -2$.

(a) Write $f(x)$ in the form $(x-h)^2 + k$. [2]
(b) State the range of f. [1]
(c) Find $f^{-1}(x)$, and state its domain. [4]
(d) Hence solve $f(x) = 6$. [2]

37.

(a) Solve $x(x-3) = 2(x-3)$. [2]
(b) Solve $x(x-3) < 2(x-3)$. [3]

Challenge Questions

C1. Tangents from a point. A non-vertical line is a tangent to the parabola $y = x^2$ when it meets the parabola exactly once. End-of-Chapter Question 23(b) proves that $y = mx + c$ is a tangent exactly when $c = -\dfrac{m^2}{4}$; take that result as given here. This investigation asks how many tangents pass through a given point (h, k), and where the points with a perpendicular pair of tangents lie.

(a) Show that the line through (h, k) with gradient m is a tangent to $y = x^2$ if and only if

$$m^2 - 4hm + 4k = 0.$$

[3]

(b) Find the two tangents to $y = x^2$ that pass through the point $(3, 5)$, and state the point where each one meets the parabola. [4]

(c) Hence show that the number of tangents to $y = x^2$ through (h, k) is 2 when $k < h^2$, 1 when $k = h^2$ and 0 when $k > h^2$. Describe where the point lies relative to the parabola in each case, and explain each count from the picture. [5]

(d) Suppose that $k < h^2$, and let m_1 and m_2 be the gradients of the two tangents through (h, k). Write down $m_1 + m_2$ and $m_1 m_2$ in terms of h and k. Show that the two tangents are perpendicular if and only if $k = -\frac{1}{4}$. (The line $y = -\frac{1}{4}$ is called the *directrix* of the parabola.) [4]

(e) The acute angle θ between two lines with gradients m_1 and m_2, where $m_1 m_2 \neq -1$, satisfies $\tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right|$. Show that the points from which the two tangents meet at 45° satisfy $16h^2 = (1 + 4k)^2 + 16k$, and describe the curve these points form. Suggest how the investigation could continue, for example for other angles or for other parabolas. [4]

C2. Three points fix a parabola. Two points fix a straight line. How many fix a parabola $y = A + Bx + Cx^2$? Each point on the curve gives one linear equation in A, B and C.

(a) Find the parabola $y = A + Bx + Cx^2$ that passes through $(1, 1)$, $(2, 3)$ and $(3, 7)$. Write the three equations as an augmented matrix and row reduce it. [4]

(b) The third point is now replaced by $(k, 7)$, where k is a real constant. Show that row reduction leads to the last row

$$(k - 1)(k - 2)\,C = 8 - 2k,$$

and hence find C in terms of k whenever the system has a unique solution. [5]

(c) State the values of k for which the system does not have a unique solution. For each one, decide whether the system has no solution or infinitely many, and explain the result geometrically. [3]

(d) Find the value of k for which $C = 0$. Explain what the three points then have in common, and state the curve that passes through all three. [3]

(e) Conjecture how many points, with distinct x-coordinates, are needed to fix a polynomial of degree at most n. Explain your count by comparing equations with unknowns, and prove that two polynomials of degree at most n that agree at $n + 1$ distinct values of x are the same polynomial. *(Hint: a nonzero polynomial of degree at most n has at most n zeros (Ch. 6))* [4]

(f) Without solving any equations, show that

$$p(x) = 1 \cdot \frac{(x-2)(x-3)}{(1-2)(1-3)} + 3 \cdot \frac{(x-1)(x-3)}{(2-1)(2-3)} + 7 \cdot \frac{(x-1)(x-2)}{(3-1)(3-2)}$$

passes through the three points and simplifies to your answer to part (a). Explain how this construction generalises to $n + 1$ points. [4]