NUMBER & ALGEBRA

Chapter 2

Sequences & Series

2

SYLLABUS 1.2 1.3 1.4 1.8 1.9 1.10

A sheet of paper is about 0.1 mm thick. Fold it once and it is 0.2 mm. Each fold doubles the thickness, so 42 folds would give 0.1×2^{42} mm, which is about $440\,000$ km. The Moon is about $384\,000$ km away. Nobody can fold a sheet of paper 42 times, but the arithmetic is correct.

One further fold would reach the Moon and return. The reason lies in how doubling accumulates. Each fold doubles a thickness that has already been doubled, so after 42 folds the paper is not 42 times its original thickness but 2^{42} times it.

Now stack the sheets instead of folding them. Place one sheet at a time, and after 42 sheets the pile is 4.2 mm high. The paper is the same and the number of steps is the same, yet the answer is smaller by a factor of about a hundred billion. Folding multiplies at each step; stacking adds.

Those two operations produce the two kinds of list studied in this chapter. A list built by repeatedly adding a fixed amount is *arithmetic*. A list built by repeatedly multiplying by a fixed factor is *geometric*. The two arise in the same situations, so choosing the wrong one gives a very different answer. A salary that rises by a fixed amount each year is arithmetic. A salary that rises by a fixed percentage is geometric, and the two grow far apart over a career.

These patterns let you model real situations, such as savings, loans and populations, and use the model to predict and interpret. A model is only as reliable as the data behind it. Counting methods then give the expansion of $(a+b)^{12}$ without multiplying out twelve brackets. By the end of this chapter you will be able to:

- find any term, and the sum of the first n terms, of an arithmetic or geometric sequence, and use sigma notation;
- find the sum of an infinite geometric series, and explain why it can be finite;
- apply geometric sequences to interest, depreciation and inflation;
- count arrangements and selections, and expand $(a+b)^n$ for any rational n.

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2.1 Arithmetic Sequences

A theatre has 20 seats in the front row, 23 in the second, and 26 in the third. Each row has 3 more seats than the one before it. To find how many seats are in row 50, counting up one row at a time would be slow. A regular pattern like this can be written as a formula, and the formula gives any term directly.

Label each row by its position n, and write u_n for the value in position n:

position n	1	2	3	4	⋯
value u_n	20	23	26	29	⋯

So $u_1 = 20$ is the first term, $u_2 = 23$ the second, and u_{50} the number of seats in row 50.

A sequence where each term is obtained by adding a fixed constant to the previous one is called **arithmetic** (also called an **arithmetic progression**, or **AP**). That constant is the **common difference** d.

DEFINITION 2.1 **Arithmetic sequence**

An **arithmetic sequence** has the form $u_1,\ u_1 + d,\ u_1 + 2d,\ \ldots$, where u_1 is the first term and d is the common difference.

For any two consecutive terms: $d = u_{n+1} - u_n$.

2.1.1 The general term

The n-th term, called the **general term**, is the first term plus $(n-1)$ steps of size d. Starting from u_1, you take one step to reach u_2, two steps to reach u_3, and so on. Five terms contain four gaps:

$$u_1 \xrightarrow{+d} u_2 \xrightarrow{+d} u_3 \xrightarrow{+d} u_4 \xrightarrow{+d} u_5,$$

so $u_5 = u_1 + 4d$, and in general u_n is $n-1$ steps from u_1.

FORMULA 2.1 **nth term of an arithmetic sequence** FORMULA BOOKLET · 1.2

$$u_n = u_1 + (n-1)d$$

If $d > 0$ the terms increase, and if $d < 0$ they decrease.

EXAMPLE 2.1

An arithmetic sequence has first term 7 and common difference -3. Find u_{20}.

SOLUTION

Take 19 steps of size -3 from $u_1 = 7$:

$$u_{20} = 7 + (20-1)(-3) = 7 - 57 = \boxed{-50}$$

A negative common difference means the sequence is decreasing. The general term can be negative even when the first term is positive.

EXAMPLE 2.2

In an arithmetic sequence, $u_5 = 11$ and $u_9 = 27$. Find u_1 and d.

SOLUTION

From u_5 to u_9 is exactly 4 steps:

$$4d = u_9 - u_5 = 27 - 11 = 16 \implies \boxed{d = 4}$$

Then $u_1 = u_5 - 4d = 11 - 16 = \boxed{-5}$

EXAMPLE 2.3

Find the number of terms in the arithmetic sequence $5, 9, 13, \ldots, 101$.

SOLUTION

Here $d = 4$. Set $u_n = 101$:

$$5 + (n-1)(4) = 101 \implies 4(n-1) = 96 \implies \boxed{n = 25}$$

CAUTION The formula is $u_1 + (n-1)d$, not $u_1 + nd$. For the first term ($n = 1$), you add zero differences, not one.

YOUR TURN 2.1 Arithmetic Sequences

Finding a term

1. Find the term indicated.

(a) u_7 of $3, 7, 11, \ldots$

(b) u_{15} of the AP with $u_1 = 12$, $d = -4$

(c) u_{10} of $-5, -1, 3, \ldots$

(d) u_{25} of the AP with $u_1 = 100$, $d = -7$

Finding the first term or the common difference

2. Find the quantity indicated.

(a) d, given $u_1 = 5$ and $u_9 = 37$
(b) u_1, given $u_6 = 23$ and $d = 3$
(c) d, given $u_6 = 20$ and $u_{14} = -4$
(d) u_1, given $u_{12} = 50$ and $d = 4$

3. In an arithmetic sequence, $u_2 + u_5 = 31$ and $u_8 = 38$. Find u_1 and d.

Number of terms

4. Find the number of terms in each AP.

(a) $4, 11, 18, \ldots, 144$
(b) $2, 9, 16, \ldots, 100$
(c) $100, 96, 92, \ldots, 4$
(d) $-7, -3, 1, \ldots, 61$

Testing whether a number is a term

5. Show that the sequence $5, 11, 17, \ldots$ has a term equal to 101. State which term it is.

6. Determine whether each number is a term of the sequence $3, 10, 17, \ldots$. If it is, state which term it is.

(a) 150
(b) 200

Concept check

7. A student finds the 20th term of $5, 8, 11, \ldots$ as $u_{20} = 5 + 20 \times 3 = 65$. Identify the error, and find the correct value.

2.2 Arithmetic Series

A **sequence** is a list of terms; a **series** is their sum. So $3, 6, 9, 12, \ldots$ is a sequence, and $3 + 6 + 9 + 12 + \cdots$ is a series. Write S_n for the sum of the first n terms,

$$S_n = u_1 + u_2 + \cdots + u_n,$$

so that here $S_3 = 3 + 6 + 9 = 18$.

When you need the total of the first n terms of an arithmetic sequence, adding them one at a time is possible but slow. Pairing the first term with the last, the second with the second-to-last, and so on, turns the whole sum into one short calculation.

Consider adding all integers from 1 to 100. Write the sum forwards and backwards:

$$1 \ + \ 2 \ + \ 3 \ + \ \cdots \ + \ 99 \ + \ 100$$

$$100 + 99 + 98 + \cdots + 2 + 1$$

Every vertical pair sums to 101, and there are 100 such pairs. So twice the sum equals 100×101, giving $S_{100} = 5050$.

The figure below shows the same idea for $3 + 6 + 9 + 12$.

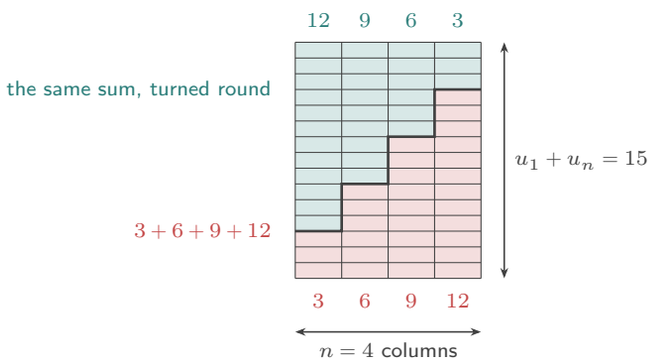


Figure 2.1 Two copies of $3 + 6 + 9 + 12$ fit together into a rectangle 4 columns wide and 15 high, so $2S_4 = 4 \times 15$.

The same pairing works for any arithmetic series: every pair, the first term with the last, the second with the second-to-last, and so on, has the same total $u_1 + u_n$. There are n such pairs when you write the series twice, so $2S_n = n(u_1 + u_n)$, and therefore:

FORMULA 2.2 **Sum of an arithmetic series** FORMULA BOOKLET · 1.2

$$S_n = \frac{n}{2}(u_1 + u_n) \qquad = \qquad \frac{n}{2}(2u_1 + (n-1)d)$$

In words, the first form is the **number of terms times the average of the first and last terms**, since $\frac{u_1+u_n}{2}$ is that average. The first form is faster when you know u_n. Substituting $u_n = u_1 + (n-1)d$ gives the second, which works when only u_1 and d are known.

EXAMPLE 2.4

Find the sum of the first 30 terms of the arithmetic sequence $2, 7, 12, \ldots$.

SOLUTION

Here $u_1 = 2$ and $d = 5$. The last term is not given, so use the second form:

$$S_{30} = \frac{30}{2}(2(2) + 29(5)) = 15(4 + 145) = 15 \times 149 = \boxed{2235}$$

When a formula for S_n is given, recover the terms from it: $u_1 = S_1$, and $u_n = S_n - S_{n-1}$ for $n \geq 2$, because the two sums differ only by their last term.

EXAMPLE 2.5

The sum of the first n terms of an arithmetic series is given by $S_n = 3n^2 + 5n$. Find u_1, d, and the general term u_n.

SOLUTION

$u_1 = S_1 = 3 + 5 = \boxed{8}$

$u_2 = S_2 - S_1 = (12 + 10) - 8 = 14$, so $d = u_2 - u_1 = \boxed{6}$

For the general term: $u_n = S_n - S_{n-1} = 3n^2 + 5n - 3(n-1)^2 - 5(n-1)$.

Expanding: $u_n = 3n^2 + 5n - 3n^2 + 6n - 3 - 5n + 5 = \boxed{6n + 2}$

Check: $u_1 = 8$ and $u_2 = 14$. Consistent.

CAUTION S_n is the total of the first n terms; u_n is just the n-th term. Confusing the two is a common error. A question asking for a "term" wants u_n; a question asking for a "sum" wants S_n.

2.2.1 Application: simple interest

A standard financial application of an arithmetic sequence is **simple interest**: the same fixed amount of interest is added each year, so the balance increases by the same amount each year.

Write the opening balance as B_0 and the yearly interest as d. After n years, n payments of interest have been added, so the balance is

$$B_n = B_0 + nd.$$

The subscript counts years, and B_0 is the balance before any interest has been paid, so no $n - 1$ appears here.

EXAMPLE 2.6

\$2 000 is deposited in an account paying 3% simple interest per year. Find the balance after 15 years.

SOLUTION

The interest is $3\% \times 2\,000 = \$60$ every year, so the balances form an arithmetic sequence with $d = 60$. After 15 years, 15 payments of interest have been added:

$$B_{15} = 2\,000 + 15 \times 60 = \boxed{\$2\,900}$$

2.2.2 When the data are not quite arithmetic

Real measurements rarely fit a sequence exactly. A model is still useful if the steps are *close* to constant, and then the common difference has to be estimated instead of read directly

from the data. A simple estimate is the average step: divide the total change by the number of steps.

For data that are approximately arithmetic over n terms, the following can be used:

$$d \approx \frac{u_n - u_1}{n-1}.$$

This is only an estimate. It is the mean of the successive differences, and it uses only the first and last readings. Fitting a least-squares line, as in Ch. 10, uses every reading instead.

A model fitted to data is reliable only within the range the data covered. A prediction outside that range is an estimate, and a model of real growth gives wrong values if you use it far enough beyond the data.

EXAMPLE 2.7

A tree's height is measured at the end of each year: 1.8, 2.5, 3.1, 3.9, 4.6 metres. Decide whether an arithmetic model is reasonable, estimate the common difference, and use the model to predict the height after 8 years.

SOLUTION

Look first at the successive differences: 0.7, 0.6, 0.8, 0.7. They are not equal, but they are close, so an arithmetic model is reasonable. Over four steps the height rose from 1.8 to 4.6, so

$$d \approx \frac{4.6 - 1.8}{5 - 1} = \frac{2.8}{4} = \boxed{0.7 \text{ metres per year}}$$

For the height after 8 years, take $u_1 = 1.8$ and $d = 0.7$; the eighth term is

$$u_8 = 1.8 + 7(0.7) = \boxed{6.7 \text{ metres}}$$

Treat that as an estimate, not a fact: year 8 lies outside the five years measured, and no tree grows at a constant rate forever.

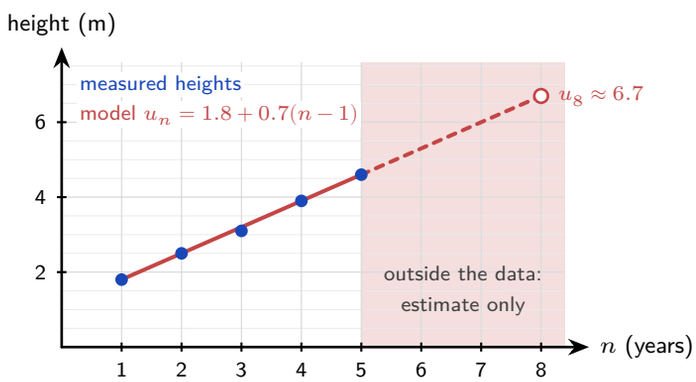


Figure 2.2 The measured heights (cobalt) and the arithmetic model (scarlet). The model is solid where it is fitted to data and dashed beyond year 5, where its values are estimates only.

YOUR TURN 2.2 Arithmetic Series

Finding a sum

8. Find the sum indicated.

(a) S_{12} for the AP $2, 5, 8, \ldots$

(b) S_{20} for the AP with $u_1 = 100$, $d = -5$

(c) S_{15} for the AP $7, 11, 15, \ldots$

(d) S_{25} for the AP with $u_1 = -12$, $d = 5$

9. An AP has $u_1 = 15$, $u_n = -9$ and $n = 13$. Find S_n.

10. Find each sum.

(a) $6 + 10 + 14 + \cdots + 90$

(b) $3 + 8 + 13 + \cdots + 98$

(c) $50 + 46 + 42 + \cdots + 2$

(d) $1 + 1.5 + 2 + \cdots + 20$

11. Find the sum of all integers from 1 to 200 that are multiples of 6.

12. Find the sum $4 + 7 + 10 + \cdots + 100$.

Working backwards from a sum

13. Find n if $S_n = 234$ for the AP $3, 6, 9, \ldots$

14. Given $S_n = 4n^2 - n$, find u_1, d and the general term u_n.

Simple interest

15. \$3 500 is deposited in an account paying 4% simple interest per year. Find the balance after 8 years.

16. An account pays simple interest. After 3 years the balance is \$2 360, and after 7 years it is \$2 840. Find the amount first deposited and the annual rate of interest.

Modelling with data

17. A plant's height is measured at the end of weeks 1 to 5, in centimetres: 4.2, 6.1, 7.8, 9.9, 11.8.

(a) Find the four successive differences, and explain why an arithmetic model is reasonable.

(b) Estimate the common difference from the first and last readings.

(c) Predict the height at the end of week 9, and give one reason to treat the prediction with caution.

Concept check

18. An arithmetic series has $S_n = n^2 + 3n$. Asked for the 10th term, a student writes $u_{10} = S_{10} = 130$. Identify the error, and find u_{10}.

2.3 Geometric Sequences

A geometric sequence multiplies by the same number at each step. Start at 1 and double: $1, 2, 4, 8, 16, \ldots$ Many real quantities change in this way. Bacteria double in every generation. A ball bounces to 60% of the height it fell from. Each extra filter that passes half the light reaching it halves the intensity again.

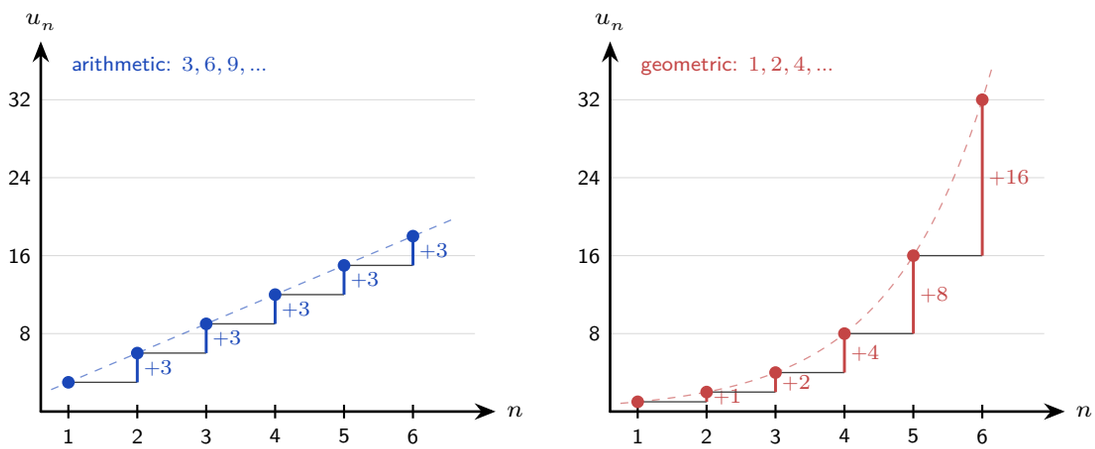


Figure 2.3 Adding a fixed amount gives equal rises, so the points lie on a straight line; multiplying by a fixed factor greater than 1 gives rises that themselves keep growing.

A sequence where each term is obtained by multiplying the previous term by a fixed constant is called **geometric** (also called a **geometric progression**, or **GP**). That constant is the **common ratio** r.

DEFINITION 2.2 **Geometric sequence**

A **geometric sequence** has the form $u_1,\ u_1r,\ u_1r^2,\ \ldots$, where u_1 is the first term and r is the common ratio.

For any two consecutive terms: $r = \dfrac{u_{n+1}}{u_n}$, with $u_n \neq 0$.

2.3.1 The general term

Starting from u_1, each step multiplies by r. After $(n-1)$ multiplications you reach u_n:

FORMULA 2.3 **nth term of a geometric sequence** FORMULA BOOKLET · 1.3

$$u_n = u_1 \cdot r^{n-1}$$

EXAMPLE 2.8

A geometric sequence has $u_1 = 5$ and $r = 3$. Find its sixth term.

SOLUTION

Five multiplications by 3 lead from u_1 to u_6:

$$u_6 = 5 \cdot 3^5 = 5 \times 243 = \boxed{1215}$$

EXAMPLE 2.9

A geometric sequence has $u_3 = 28$ and $u_6 = 224$. Find u_1 and r.

SOLUTION

From u_3 to u_6 is 3 multiplicative steps: $u_6 = u_3 \cdot r^3$, so

$$r^3 = \frac{224}{28} = 8 \implies \boxed{r = 2}$$

Then $u_1 = u_3/r^2 = 28/4 = \boxed{7}$

EXAMPLE 2.10

Find the number of terms in the geometric sequence $2, 6, 18, \ldots, 4374$.

SOLUTION

Here $r = 3$. Set $u_n = 4374$:

$$2 \cdot 3^{n-1} = 4374 \implies 3^{n-1} = 2187 = 3^7 \implies \boxed{n = 8}$$

EXAMPLE 2.11

Given that k, 8 and $k + 12$ are consecutive terms of a geometric sequence, find the possible values of k.

SOLUTION

Consecutive terms have a constant ratio, so the ratio across the first gap equals the ratio across the second:

$$\frac{8}{k} = \frac{k+12}{8} \implies 64 = k(k+12) \implies k^2 + 12k - 64 = 0.$$

Factorising, $(k + 16)(k - 4) = 0$, so $\boxed{k = -16 \text{ or } k = 4}$

Both values give a genuine sequence. With $k = 4$ the terms are $4, 8, 16$ and $r = 2$; with $k = -16$ they are $-16, 8, -4$ and $r = -\frac{1}{2}$. Do not discard the negative solution: a negative ratio makes the terms alternate in sign, which is still a geometric sequence.

EXT **Recursively defined sequences.** Every rule so far has been **explicit**: substitute n and you get the n-th term directly. A sequence can instead be given by a **recurrence**: a starting value and a rule that builds each term from the previous one. For example, $u_1 = 4$, $u_{n+1} = 3u_n - 4$ gives $4, 8, 20, 56, \ldots$ A recurrence produces terms one at a time, so a **closed form** (a formula for u_n in terms of n alone) is more useful. For $u_{n+1} = au_n + b$ with $a \neq 1$, first solve $c = ac + b$: this **fixed value** c is the value the rule does not change. Then $v_n = u_n - c$ satisfies $v_{n+1} = av_n$, so it is geometric with ratio a, and $u_n = c + (u_1 - c)\,a^{n-1}$. For the sequence above, $u_{n+1} = 3u_n - 4$, the equation $c = 3c - 4$ gives $c = 2$, so $v_n = u_n - 2$ runs $2, 6, 18, 54, \ldots$ and $u_n = 2 + 2 \cdot 3^{n-1}$.

2.3.2 Where geometric growth shows up

The pattern appears in three contexts, and the algebra is the same in each.

- **Population growth.** A population rising by a fixed *percentage* each year is geometric with $r = 1 + \frac{p}{100}$. A 2% annual rise gives $r = 1.02$.
- **Salary change.** An annual rise of 3% gives $r = 1.03$; a cut of 3% gives $r = 0.97$. Note that a rise then an equal cut does not return you to the start, because $1.03 \times 0.97 = 0.9991$.
- **The spread of a disease.** In a simplified model, if each infected person passes it to r others, the number of new cases in successive rounds is geometric. That r is what public health reporting calls the reproduction number. If it is above 1, the outbreak grows; if it is below 1, the outbreak becomes smaller and ends.

The behaviour depends on the sign and the size of r, as these sequences with positive first terms show.

Ratio	Example	Behaviour
$r > 1$	$r = 2$: $2, 4, 8, 16, \ldots$	grows without limit
$r = 1$	$5, 5, 5, 5, \ldots$	constant
$0 < r < 1$	$r = \frac{1}{2}$: $16, 8, 4, 2, \ldots$	decays towards 0
$-1 < r < 0$	$r = -\frac{1}{2}$: $16, -8, 4, -2, \ldots$	signs alternate; decays towards 0
$r = -1$	$3, -3, 3, -3, \ldots$	signs alternate; size stays the same
$r < -1$	$r = -2$: $2, -4, 8, -16, \ldots$	signs alternate; size grows

The disease model shows why the ratio matters more than the starting value. If $r > 1$ the terms grow without limit, however small u_1 is; if $0 < r < 1$ they shrink towards zero, however large u_1 is. In this model, a large outbreak with $r = 0.9$ shrinks towards zero, while a single case with $r = 1.1$ keeps growing. Real outbreaks are less regular: a small outbreak can end by chance even when $r > 1$.

When you read a question, check two things: whether it counts terms or periods, and whether n must be a whole number. A value n periods after the start has been multiplied n times, so it is $u_1 r^n$, not $u_n = u_1 r^{n-1}$. When n counts whole periods, as in "the first year in which the value exceeds 60 000", choose the least whole number that satisfies the inequality, and check it against the one before. This usually means rounding the solution up, but not when the solution is itself a whole number and the inequality is strict: $2^n > 8$ first holds at $n = 4$, not $n = 3$.

EXAMPLE 2.12

A town of $40\,000$ people grows by 2.5% each year. Find the population after 10 years, and the first year in which it exceeds $60\,000$.

SOLUTION

Take $u_1 = 40\,000$ and $r = 1.025$. Ten years later is ten multiplications, so the value is $u_1 r^{10}$, not u_{10}:

$$40\,000 \times 1.025^{10} \approx \boxed{51\,203}$$

For the second part, solve $40\,000 \times 1.025^n > 60\,000$, that is $1.025^n > 1.5$. Taking logarithms (Ch. 1),

$$n > \frac{\ln 1.5}{\ln 1.025} \approx 16.4,$$

so the population first exceeds $60\,000$ in year $\boxed{17}$

Round *up* here: year 16 is not yet enough.

TIP A GDC or a spreadsheet will generate and display a sequence for you. On a GDC, enter the rule for u_n as a function of n and show its table (a list of values of n with the terms beside them). If your calculator has a sequence mode (a setting in which you type u_1 and the rule that gives each term from the one before), you can use that instead. In a spreadsheet, fill a column so that each cell is the cell above plus d, or times r. If you use technology in an examination, still write down the first term and the common difference or common ratio you entered.

TIP To test whether a sequence is arithmetic, check that differences are constant. To test whether it is geometric, check that ratios are constant. A sequence passes both tests only when it is constant and non-zero, such as $5, 5, 5, \ldots$, which is arithmetic with $d = 0$ and geometric with $r = 1$.

YOUR TURN 2.3 Geometric Sequences

Finding a term

19. Find the term indicated.

(a) u_6 of the GP $2, 6, 18, \ldots$
(b) u_7 of the GP with $u_1 = 96$, $r = \frac{1}{2}$
(c) u_5 of $\frac{1}{4}, \frac{1}{2}, 1, \ldots$
(d) u_8 of the GP with $u_1 = 3$, $r = -2$

Finding the first term or the common ratio

20. Find the quantity indicated.

(a) r, given $u_1 = 5$ and $u_4 = 40$
(b) u_1, given $u_4 = 24$ and $r = 2$
(c) r, given $u_1 = 81$, $u_5 = 1$ and $r > 0$
(d) u_1, given $u_3 = 45$ and $r = 3$

21. Given $u_2 = 10$ and $u_5 = 1250$, find u_1 and r.

22. The 2nd term of a GP is 12 and the 4th term is 48. Find the two possible pairs of values of u_1 and r.

Number of terms

23. Find the number of terms in each GP.

(a) $3, 6, 12, \ldots, 768$
(b) $5, 15, 45, \ldots, 3645$
(c) $5, -10, 20, \ldots, -2560$
(d) $128, 64, 32, \ldots, \frac{1}{2}$

Three consecutive terms

24. Given that k, 10 and $k - 15$ are consecutive terms of a geometric sequence, find the possible values of k and the common ratio in each case.

25. Given that x, $x + 6$ and $9x$ are consecutive terms of a geometric sequence, where $x \neq 0$, find the possible values of x. For each value, write down the three terms and the common ratio.

Recurrences

26. A sequence is defined by $u_1 = 5$ and $u_{n+1} = 2u_n - 3$.

(a) Write down u_2, u_3 and u_4.
(b) Let $v_n = u_n - 3$. Show that $\{v_n\}$ is geometric.
(c) Hence find a closed form for u_n.

27. A sequence is defined by $u_1 = 10$ and $u_{n+1} = \frac{1}{2}u_n + 4$.

(a) Find the value of c for which $v_n = u_n - c$ is a geometric sequence.
(b) Find a closed form for u_n, and describe what happens to u_n as n increases.

Geometric growth in context

28. GDC Geometric growth in three contexts.

(a) A population of $25\,000$ grows by 1.8% each year. Find the population after 12 years.

(b) A salary of \$$48\,000$ rises by 4% each year. Find the salary in the sixth year.

(c) In an outbreak each infected person infects 1.2 others, and round 1 has 50 new cases. Find the number of new cases in round 6, and state what would happen in the long run if that figure fell to 0.8.

29. GDC A town of $32\,000$ people grows by 2.2% each year. Find the first year in which the population exceeds $45\,000$.

Concept check

30. A town of 2000 people grows by 5% each year. A student models the population after n years as $P = 2000 \times 0.05^n$. Identify the error, and write down the correct model.

2.4 Geometric Series

Adding the terms of a geometric sequence needs a different method. In an arithmetic series, the first and last terms have the same total as the second and second-to-last terms, so pairing works. In a geometric series these pair totals in general differ, so pairing does not help. Instead there is an algebraic method: multiply the sum by r, write the two sums one under the other, and subtract.

Try the method on $S = 1 + 2 + 4 + 8$. Multiply by the ratio 2, and write the new line one place to the right so that matching terms sit under each other:

$$\begin{aligned} S &= 1 + 2 + 4 + 8 \\ 2S &= \quad 2 + 4 + 8 + 16 \end{aligned}$$

Subtract the first line from the second. Every term in the middle cancels, leaving $2S - S = 16 - 1$, so $S = 15$. The same three steps work in general: multiply by r, write one sum under the other, and subtract.

Let $S = u_1 + u_1 r + u_1 r^2 + \cdots + u_1 r^{n-1}$. Multiply by r:

$$rS = u_1 r + u_1 r^2 + \cdots + u_1 r^n.$$

Every term appears in both lines except u_1 in the first line and $u_1 r^n$ in the second. Subtract rS from S:

$$S(1 - r) = u_1 - u_1 r^n = u_1(1 - r^n).$$

Divide both sides by $(1-r)$ to get the formula.

FORMULA 2.4 **Sum of a geometric series** FORMULA BOOKLET · 1.3

$$S_n = \frac{u_1(1-r^n)}{1-r} \quad = \quad \frac{u_1(r^n-1)}{r-1}, \qquad r \neq 1$$

Both forms are equivalent: multiply the top and the bottom of the first by -1 to get the second. When $0 < r < 1$, use the first form: both $1-r^n$ and $1-r$ are then positive, so no negative signs appear. When $r > 1$ the second form is simpler, for the same reason. If r is negative, either form works, but r^n changes sign with n, so take care with the arithmetic.

EXAMPLE 2.13

Find the sum of the first 10 terms of $5 + 10 + 20 + \cdots$.

SOLUTION

Here $u_1 = 5$ and $r = 2$:

$$S_{10} = \frac{5(2^{10}-1)}{2-1} = 5 \times 1023 = \boxed{5115}$$

EXAMPLE 2.14

A geometric series has $u_1 = 80$ and $r = \frac{1}{2}$. Find the least value of n such that $S_n > 159$.

SOLUTION

Use the sum formula with $u_1 = 80$ and $r = \frac{1}{2}$:

$$S_n = \frac{80\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}} = 160\left(1-\frac{1}{2^n}\right).$$

Require $160\left(1-\frac{1}{2^n}\right) > 159$, so $\frac{1}{2^n} < \frac{1}{160}$, giving $2^n > 160$.
Since $2^7 = 128 < 160$ and $2^8 = 256 > 160$, the least value is $\boxed{n = 8}$

CAUTION The formula requires $r \neq 1$. If $r = 1$, every term equals u_1 and $S_n = nu_1$. Substituting $r = 1$ into the formula gives $0/0$, which is undefined.

Solving for n in geometric series often requires taking logarithms of both sides. See Ch. 1, §1.5.

Four words from this chapter are easy to confuse, so it helps to set them side by side.

Sequence	an ordered list of terms, $u_1, u_2, u_3, \ldots$
Series	the sum of the terms of a sequence, such as S_n
Arithmetic	each term is the one before plus a constant difference d
Geometric	each term is the one before times a constant ratio r

YOUR TURN 2.4 Geometric Series

Finding a sum

31. Find the sum indicated.

(a) S_8 for the GP $1, 3, 9, \ldots$
(b) S_{10} for the GP with $u_1 = 6$, $r = \frac{1}{3}$
(c) S_6 for the GP $128, 64, 32, \ldots$
(d) S_6 for the GP with $u_1 = 1$, $r = -2$

32. Find each sum.

(a) $3 + 6 + 12 + \cdots + 1536$
(b) $2 + 6 + 18 + \cdots + 486$
(c) $1 - 2 + 4 - \cdots + 256$
(d) $96 + 48 + 24 + \cdots + 3$

Working backwards from a sum

33. The sum of the first three terms of a GP is 21 and the common ratio is 2. Find the first term.

34. Find n such that $S_n = 63$ for the GP $1, 2, 4, \ldots$

35. Given $S_3 = 7$ and $S_6 = 63$, find u_1 and r.

36. A geometric sequence has $u_1 = 5$ and $S_3 = 15$.

(a) Find the two possible values of r.
(b) For each value of r, find S_{10}.

The least number of terms

37. For the geometric series $2 + 6 + 18 + \cdots$, find the least value of n such that $S_n > 10\,000$.

38. GDC A geometric series has $u_1 = 60$ and $r = 0.8$. Find the least value of n such that $S_n > 290$.

Concept check

39. A student finds S_6 for the series $2 + 6 + 18 + \cdots$ as $\dfrac{2(1-3)^6}{1-3} = \dfrac{128}{-2} = -64$. Identify the error, and find S_6.

2.5 Sigma Notation

By now you have written several sums in the form $u_1 + u_2 + \cdots + u_n$. There is a compact notation for this. For example,

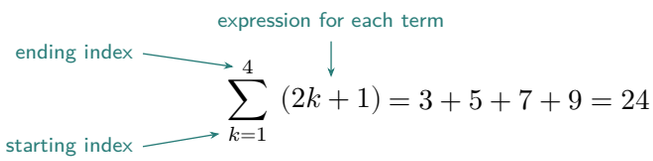


Figure 2.4 The parts of a sum in sigma notation.

To evaluate it, put $k = 1, 2, 3, 4$ into $2k + 1$, one after another, then add. The 1 under the sigma is the **starting index**, the 4 on top is the **ending index**, and $2k + 1$ is the **expression** for each term.

The Greek letter Σ (capital sigma) is used as shorthand for "sum". The expression

$$\sum_{k=1}^{n} u_k = u_1 + u_2 + u_3 + \cdots + u_n$$

is read "the sum of u_k from $k = 1$ to n." The variable k is the **index of summation**. It takes each integer value from the starting index to the ending index. Each value gives one term, and all the terms are added.

DEFINITION 2.3 **Sigma notation**

$$\sum_{k=m}^{n} f(k) \;=\; f(m) + f(m+1) + f(m+2) + \cdots + f(n), \quad \text{where } m \le n.$$

The index k is a **dummy variable**: the result depends only on f, m, and n, not on the name chosen for the index.

2.5.1 Properties of sigma notation

These three properties are familiar rules of arithmetic, written in sigma form. They are what you use to simplify or split a sum.

KEY 2.5 **Properties of sigma notation**

For a constant c and sequences $\{u_k\}$, $\{v_k\}$:

$$\sum_{k=1}^{n} c = nc$$ a constant summed n times

$$\sum_{k=1}^{n} c\,u_k = c\sum_{k=1}^{n} u_k$$ a constant factor comes outside the sum

$$\sum_{k=1}^{n}(u_k + v_k) = \sum_{k=1}^{n} u_k + \sum_{k=1}^{n} v_k$$ a sum splits over addition

Writing each sum out shows why. A constant c added n times is nc. In $cu_1 + cu_2 + \cdots + cu_n$ the common factor c can be taken outside a bracket. In $(u_1 + v_1) + (u_2 + v_2) + \cdots + (u_n + v_n)$ the terms can be added in any order, so add the u terms first and then the v terms.

When the lower limit is not 1, subtract the missing terms: for $m \geq 2$,

$$\sum_{k=m}^{n} f(k) = \sum_{k=1}^{n} f(k) - \sum_{k=1}^{m-1} f(k).$$

The sum has $n - m + 1$ terms.

EXAMPLE 2.15

Write $3 + 7 + 11 + 15 + 19$ in sigma notation.

SOLUTION

The terms form an arithmetic sequence with $u_1 = 3$, $d = 4$: the k-th term is $u_k = 4k - 1$. So

$$3 + 7 + 11 + 15 + 19 = \boxed{\sum_{k=1}^{5}(4k - 1)}$$

EXAMPLE 2.16

Evaluate $\sum_{k=3}^{10}(2k + 1)$.

SOLUTION

First list the terms. Putting $k = 3, 4, \ldots, 10$ into $2k + 1$ gives

$$7 + 9 + 11 + \cdots + 21,$$

which is $10 - 3 + 1 = 8$ terms of an arithmetic series. So the sum is $\frac{8}{2}(7 + 21) = \boxed{112}$

Alternative: subtract the missing terms. The full sum from $k=1$ minus the first two terms gives the same result:

$$\sum_{k=3}^{10}(2k+1) = \sum_{k=1}^{10}(2k+1) - \sum_{k=1}^{2}(2k+1) = \tfrac{10}{2}(3+21) - (3+5) = 120 - 8 = 112.$$

The index can also be renamed so that a sum starts at 1. This is called **re-indexing**: putting $k = j+2$ turns $\sum_{k=3}^{10}(2k+1)$ into $\sum_{j=1}^{8}(2j+5)$, which lists the same 8 terms and has the same total.

CAUTION In general, $\sum_{k=1}^{n}(u_k \cdot v_k) \neq \left(\sum u_k\right)\left(\sum v_k\right)$. You can split a sum over addition but not over multiplication.

YOUR TURN 2.5 Sigma Notation

Writing a sum in sigma notation

40. Write each sum in sigma notation.

(a) $5 + 10 + 15 + 20 + 25$

(b) $4 + 8 + 16 + 32 + 64$

(c) $3 + 6 + 9 + \cdots + 30$

(d) $1 + 4 + 9 + 16 + 25$

Evaluating a sum

41. Evaluate each sum.

(a) $\sum_{k=1}^{6}(3k-2)$

(b) $\sum_{k=0}^{4} 81\left(\frac{1}{3}\right)^k$

(c) $\sum_{k=5}^{12}(3k-4)$

42. Consider the sum $\sum_{k=4}^{15}(5-2k)$.

(a) State the number of terms in the sum.

(b) Evaluate the sum.

(c) Rewrite the sum using the index $j = k-3$, so that j starts at 1, and check that it gives the same value.

Using the sigma properties

43. This question uses the properties of sigma notation.

(a) By treating it as an arithmetic series, show that $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$.

(b) Hence find $\sum_{k=1}^{n} 5$ and $\sum_{k=1}^{n} 5k$.

44. Use the properties of sigma notation and the result of Question 43(a) to answer each part.

(a) Evaluate $\sum_{k=1}^{20}(3k+2)$.

(b) Evaluate $\sum_{k=11}^{30}(2k-1)$ by subtracting the terms that are missing.

(c) Evaluate $\sum_{k=1}^{3} k^2$ and $\left(\sum_{k=1}^{3} k\right)^2$. State what the two values show about sums of products.

Concept check

45. A student writes $\sum_{k=1}^{20}(3k+2) = 3\sum_{k=1}^{20} k + 2 = 3(210) + 2 = 632$. Identify the error, and find the correct value.

2.6 Infinite Geometric Series

Cut a pizza in half. Eat one half and keep the other. Cut what is left in half again, eat one piece, and keep the rest. If you could continue forever, how much pizza would you eat?

The pieces you eat are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, ... There are infinitely many of them, and every piece is larger than zero. Adding infinitely many positive numbers sounds like it must give an infinite total. But you began with one pizza, so you can never eat more than one pizza.

So *an addition that never ends can still have a finite answer.*

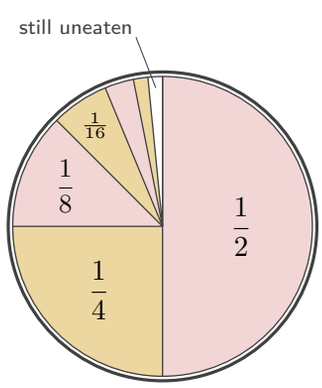


Figure 2.5 The pizza eaten as $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$. Each slice is half of what was left, so the total never reaches 1 whole pizza, yet it comes as close to 1 as you like.

Add the pieces one at a time and record the running total after each step. Each running total is a **partial sum**:

$$S_1 = 0.5, \quad S_2 = 0.75, \quad S_3 = 0.875, \quad S_4 = 0.9375, \quad \ldots$$

Every partial sum is larger than the one before it, and every partial sum is still smaller than 1. Look at the distance from 1: it is 0.5, then 0.25, then 0.125. The distance halves at every step, so it never becomes zero, but it becomes as small as you wish. This value that the partial sums approach is called the **limit**, and here the limit is 1. So the sum of an infinite series is not the last step of an endless addition: it is **the limit of the running totals**.

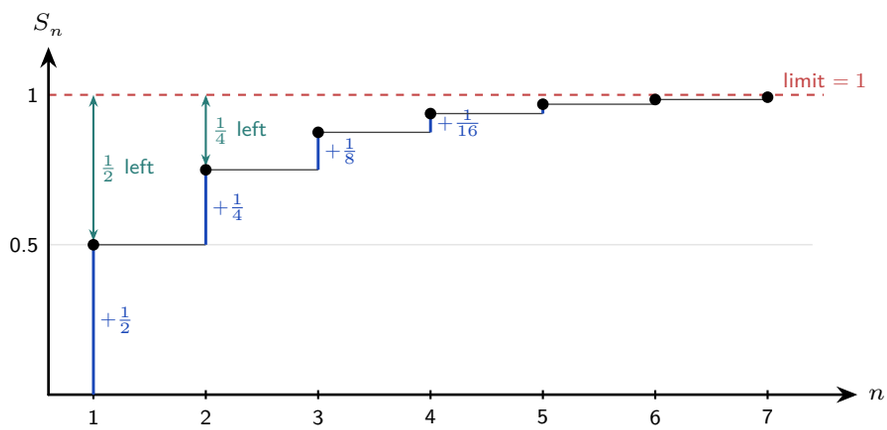


Figure 2.6 The partial sums of $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ increase towards 1 in steps that halve. The distance still to go (teal) is the uneaten part of the pizza: it halves too, so the sums never reach 1.

The pizza sum has a finite total because each piece is half the size of the piece before it. In general, $|r| < 1$ (the size of r, ignoring its sign) means that each term is smaller in size than the one before, by the fixed factor $|r|$. **Shrinking terms are not enough** on their own: the terms of $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ also shrink towards zero, but its partial sums grow without bound (see the enrichment at the end of the chapter). For a geometric series, the formula below shows that $|r| < 1$ is enough.

Now change the ratio to $r = 2$, as in $1 + 2 + 4 + 8 + \cdots$, and the behaviour changes completely. Each term is larger than the last, the partial sums are $1, 3, 7, 15, \ldots$, and they pass every value you can name. There is no finite total.

A series whose partial sums approach a finite limit is said to **converge**. A series that does not converge is said to **diverge**: its partial sums may grow without bound, as here, or keep oscillating (going up and down for ever), as they do for $1 - 1 + 1 - 1 + \cdots$. A geometric series with $u_1 \neq 0$ converges when $|r| < 1$ and diverges when $|r| \geq 1$.

To turn this into a formula, start from the finite sum and let n grow without bound. When $|r| < 1$, the power r^n shrinks towards 0:

$$S_n = \frac{u_1(1 - r^n)}{1 - r} \xrightarrow{n \to \infty} \frac{u_1(1 - 0)}{1 - r} = \frac{u_1}{1 - r}.$$

FORMULA 2.6 **Sum to infinity** FORMULA BOOKLET · 1.8

For a geometric series with $|r| < 1$:

$$S_\infty = \frac{u_1}{1 - r}$$

When $0 < r < 1$ every partial sum lies below the limit (for $u_1 > 0$) and they rise towards it. When $-1 < r < 0$ the terms alternate in sign, so the partial sums lie above and below the limit in turn, each one closer than the last.

A **recurring decimal** (a decimal in which a block of digits repeats for ever) that repeats from the decimal point is an infinite geometric series. The first term is the first copy of the repeating block, and $r = 10^{-m}$ for a block of m digits, so S_∞ gives the decimal as a fraction.

EXAMPLE 2.17

Find the sum to infinity of $6 + 4 + \frac{8}{3} + \cdots$.

SOLUTION

First find the common ratio, $r = 4/6 = \frac{2}{3}$. Since $|r| < 1$ the series converges:

$$S_\infty = \frac{6}{1 - \frac{2}{3}} = \frac{6}{\frac{1}{3}} = \boxed{18}$$

EXAMPLE 2.18

Find the sum to infinity of $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$, and describe how its partial sums approach it.

SOLUTION

Here $r = -\frac{1}{2}$, and a negative ratio makes the running totals approach the limit from alternating sides. The partial sums are

$$1, \qquad 0.5, \qquad 0.75, \qquad 0.625, \qquad 0.6875, \qquad \ldots$$

above, below, above, below, each closer than the last to the limit, 0.667 to three significant figures:

$$S_\infty = \frac{1}{1 - \left(-\frac{1}{2}\right)} = \boxed{\frac{2}{3}}$$

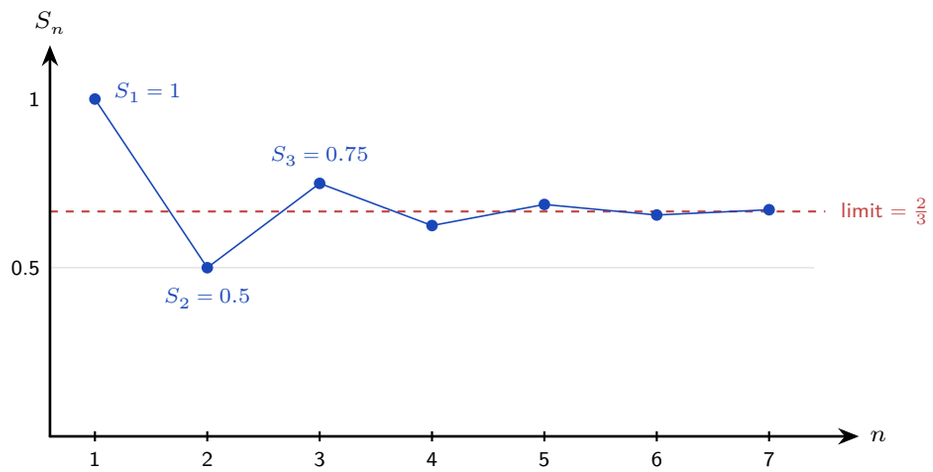


Figure 2.7 Partial sums of $1 - \frac{1}{2} + \frac{1}{4} - \cdots$. With a negative ratio they lie above and below the limit in turn, each closer than the last.

EXAMPLE 2.19

Express $0.\overline{27} = 0.272727\ldots$ as an exact fraction.

SOLUTION

Write as an infinite geometric series with $u_1 = \frac{27}{100}$ and $r = \frac{1}{100}$:

$$0.272727\ldots = \frac{27/100}{1 - 1/100} = \frac{27/100}{99/100} = \frac{27}{99} = \boxed{\frac{3}{11}}$$

EXAMPLE 2.20

A geometric series has first term u_1 and common ratio r. Given $S_\infty = 8$ and $u_2 = 2$, find u_1 and r.

SOLUTION

From S_∞: $\dfrac{u_1}{1-r} = 8$, so $u_1 = 8(1-r)$.
From $u_2 = u_1 r = 2$: substitute u_1:

$$8(1-r)r = 2 \implies 8r^2 - 8r + 2 = 0 \implies 4r^2 - 4r + 1 = 0 \implies (2r-1)^2 = 0.$$

So $\boxed{r = \frac{1}{2}}$, and $u_1 = 8\left(1 - \frac{1}{2}\right) = \boxed{4}$

CAUTION Always verify $|r| < 1$ before applying the infinite sum formula. If $|r| \geq 1$, no finite sum to infinity exists.

YOUR TURN 2.6 Infinite Geometric Series

Finding a sum to infinity

46. Find S_∞ for each series.

(a) $12 + 4 + \frac{4}{3} + \cdots$

(b) the GP with $u_1 = 10$, $r = \frac{3}{5}$

(c) $9 + 3 + 1 + \cdots$

(d) the GP with $u_1 = 8$, $r = -\frac{1}{4}$

47. A GP has $u_1 = 16$, $u_3 = 4$ and $r > 0$. Find S_∞.

Convergence

48. State whether each series converges, giving a reason. If it converges, find its sum to infinity.

(a) $6 + 9 + 13.5 + \cdots$

(b) $20 - 15 + 11.25 - \cdots$

(c) For (b), describe how the partial sums approach S_∞.

Recurring decimals

49. Express each recurring decimal as an exact fraction in its lowest terms.

(a) $0.\overline{153} = 0.153153\ldots$

(b) $2.1\overline{6} = 2.1666\ldots$

Finding the first term or the common ratio

50. Find the quantity indicated.

(a) u_1, given $S_\infty = 20$ and $r = \frac{1}{5}$

(b) r, given $u_1 = 6$ and $S_\infty = 9$

(c) u_1, given $S_\infty = 12$ and $r = \frac{1}{4}$

(d) r, given $u_1 = 15$ and $S_\infty = 12$

51. A geometric series has $S_\infty = 27$ and $u_2 = 6$. Find the two possible pairs of values of u_1 and r.

52. A geometric series has $S_\infty = 16$, and the sum of its first two terms is 12. Find the possible values of r and the corresponding values of u_1.

Concept check

53. A student finds the sum to infinity of $1 + \frac{3}{2} + \frac{9}{4} + \cdots$ as $\dfrac{1}{1 - \frac{3}{2}} = -2$. Explain why this answer is wrong.

2.7 Financial Applications

A bank pays you 4% per year. In the first year you earn interest on your deposit. In the second year you earn interest on the deposit *and* on the interest from the first year. The amount added is different every year, so the balances do not form an arithmetic sequence. They form a geometric one: each year the balance is multiplied by 1.04.

This is **compound interest**, and it is why money kept in an account for a long time grows far more than it would under simple interest.

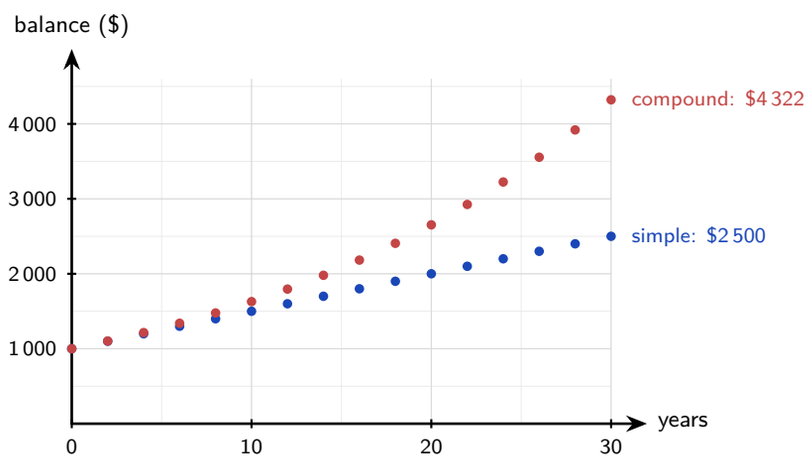


Figure 2.8 $1 000 at 5% per year for 30 years. Simple interest adds $50 each year, so its points lie on a straight line (arithmetic); compound interest multiplies by 1.05 each year, so its points lie on a curve that rises faster and faster (geometric).

Interest is often added more than once a year. If a **present value** PV (the amount now) earns interest at a **nominal** (stated) annual rate of $r\%$, compounded k times per year, then after n years its **future value** FV (the amount then) is:

FORMULA 2.7 **Compound interest** FORMULA BOOKLET · 1.4

$$FV = PV \times \left(1 + \frac{r}{100k}\right)^{kn}$$

$r\%$ is the nominal annual rate (enter $r = 4$ for 4%), k is the number of compounding periods per year and n the number of years.

A nominal rate of $r\%$ per year means that the interest for each of the k periods is $\frac{r}{k}\%$, so each period multiplies the balance by $1 + \frac{r}{100k}$. In words: **future value = starting amount $\times$ (multiplier per period)$^{\text{number of periods}}$**. The multiplier $\left(1 + \frac{r}{100k}\right)$ is the common ratio of the balances, and the number of periods is kn. **Note the change of letter meaning: here r is the interest rate in per cent, not the common ratio of §2.3.**

The letters used in this section are:

Letter	Meaning in §2.7
PV, FV	present value (now) and future value (later)
$r\%$	nominal annual interest rate (not the common ratio)
k	number of compounding periods per year
n	number of years
P, $i\%$	regular payment and its annual interest rate
$j\%$	annual inflation rate

EXAMPLE 2.21

\$5 000 is invested at 4% per year, compounded monthly. Find its value after 3 years.

SOLUTION

Here $PV = 5\,000$, $r = 4$, $k = 12$ and $n = 3$, so the value is

$$FV = 5\,000 \times \left(1 + \frac{4}{1\,200}\right)^{36} \approx \boxed{\$5\,636.36}$$

EXAMPLE 2.22

How many complete years does it take for an investment to double at 6% per year, compounded annually?

SOLUTION

Set $FV = 2 \cdot PV$:

$$(1.06)^n = 2 \implies n = \frac{\ln 2}{\ln 1.06} \approx 11.9.$$

The investment has more than doubled after $\boxed{12 \text{ complete years}}$

TIP The Rule of 72: dividing 72 by the annual interest rate gives an approximate doubling time in years. At 6%, that is $72/6 = 12$ years, close to the exact 11.9 above. For small rates $\ln\left(1 + \frac{r}{100}\right) \approx \frac{r}{100}$, so $n = \dfrac{\ln 2}{\ln\left(1 + \frac{r}{100}\right)} \approx \dfrac{100 \ln 2}{r} \approx \dfrac{69.3}{r}$. The 69.3 is rounded up to 72 because 72 is easier to divide: 2, 3, 4, 6, 8, 9 and 12 all divide it exactly.

EXT **Regular deposits.** A savings plan pays in the same amount P every year at $i\%$ compound interest, so the balance is a geometric *series*, not a single term. With payments at the **end** of each year for n years, the last payment earns no interest, the one before it earns one year's interest, and the first earns $n - 1$ years'. Read from the last payment backwards, the total is a geometric series with $u_1 = P$ and $r = 1 + \frac{i}{100}$, so the formula of §2.4 applies and there is nothing new to learn.

With payments at the **start** of each year, each earns one more year of interest, so $u_1 = P\left(1 + \frac{i}{100}\right)$. For example, \$800 paid at the end of each year for 12 years at 5% gives $S_{12} = \dfrac{800\left(1.05^{12} - 1\right)}{0.05} \approx \$12\,733.70$ just after the last payment; paid at the start of each year, the total is 1.05 times as much, about \$13 370.39.

2.7.1 Depreciation

Some things lose value instead of gaining it. A car, a laptop, or a machine may lose a fixed *percentage* of its value each year. This is called **depreciation**.

No new formula is needed. Use the same compound-interest formula with a **negative** rate r. A loss of 12% per year means $r = -12$, so the common ratio becomes

$$1 + \frac{r}{100} = 1 - \frac{12}{100} = 0.88,$$

a number less than 1. The value is still multiplied by the same number each year, so the sequence is still geometric; because that number is less than 1, the values shrink.

EXAMPLE 2.23

A car bought for \$24 000 loses 15% of its value each year. Find its value after 5 years.

SOLUTION

Take $PV = 24\,000$, $r = -15$, $k = 1$ and $n = 5$, so the value is multiplied by 0.85 five times:

$$FV = 24\,000 \times \left(1 - \frac{15}{100}\right)^5 = 24\,000 \times 0.85^5 \approx \boxed{\$10\,648.93}$$

The car has lost more than half of its value. The loss is largest in the first year, 15% of $24\,000$, which is \$3 600, and it becomes smaller each year, because each 15% is taken from a smaller value. A GDC finance solver gives the same value. Use 5 periods, a rate of -15%, a present value of $24\,000$, no regular payments ($PMT = 0$) and one period per year; on most models these are N, $I\%$, PV and PMT, but the labels may differ (see the tip at the end of this section). The solver may show the value with a negative sign.

2.7.2 Inflation and real value

A second geometric process reduces the value of your savings. **Inflation** is the yearly rise in prices. Interest increases the number in your account, while inflation reduces what that number can buy. If your account grows by 2% in a year and prices rise by 3% in the same year, you hold more money but you can buy less with it.

The **real value** of an investment is its future value adjusted for the price rises over the same period. It measures purchasing power: what the future amount would buy in today's money.

Prices also rise geometrically, so the correction is a second geometric factor. This time you divide by it instead of multiplying.

With interest producing a future value FV over n years, and prices rising at $j\%$ per year (a new letter, because $i\%$ was the interest rate in §2.7.1):

$$\text{real value} = \frac{FV}{\left(1+\frac{j}{100}\right)^{n}}.$$

Divide by the inflation factor, because the same goods cost $\left(1+\frac{j}{100}\right)^{n}$ times as much after n years.

EXAMPLE 2.24

$10 000 is invested for 5 years at 4.5% per year, compounded annually, while prices rise at 2.5% per year. Find the future value and the real value of the investment.

SOLUTION

The balance grows by a factor of 1.045 each year:

$$FV = 10\,000 \times (1.045)^{5} \approx \boxed{\$12\,461.82}$$

Now adjust for five years of price rises:

$$\text{real value} = \frac{12\,461.82}{(1.025)^{5}} \approx \boxed{\$11\,014.43}$$

The account shows a gain of about $2 460, but in purchasing power the investor is only about $1 014 better off. To judge an interest rate, compare it with the inflation rate.

TIP A GDC's finance solver, often labelled TVM (time value of money), handles compound interest, depreciation, and unknown rates or times. The labels below are the usual ones; they may differ on your model:

1. enter the known values of N, $I\%$, PV, FV and the number of compounding periods per year. On many models N is the total number of periods, so 3 years compounded monthly may need $N = 36$; check how your model counts them;
2. give money paid out and money received opposite signs (a positive PV returns a negative FV);
3. set $PMT = 0$ when there are no regular payments; otherwise choose start or end of period to match the question;
4. solve for the missing value, and in examinations state the values you entered.

YOUR TURN 2.7 Financial Applications

Compound interest

54. GDC Find the future value of each investment.

(a) \$3 000 at 4% p.a., compounded annually, for 5 years

(b) \$8 000 at 6% p.a., compounded monthly, for 3 years

(c) \$500 at 3.6% p.a., compounded monthly, for 10 years

(d) \$2 000 at 5% p.a., compounded quarterly, for 4 years

55. GDC Find the present value needed to accumulate \$15 000 in 8 years at 5% p.a., compounded quarterly.

Unknown rate or time

56. GDC Find the annual interest rate if \$5 000 grows to \$5 828.84 in 3 years, compounded annually.

57. GDC An investment earns 7% p.a.

(a) The interest is compounded monthly. How many complete years does it take the investment to triple?

(b) The interest is compounded annually instead. Use the Rule of 72 to estimate the time it takes the investment to double, and compare your estimate with the exact time.

Regular deposits

58. GDC \$1 200 is paid into an account every year for 10 years. The account pays 4.5% p.a., compounded annually.

(a) The payments are made at the end of each year. Find the value immediately after the tenth payment.

(b) The payments are made at the start of each year instead. Find the value at the end of the tenth year.

59. GDC A saver wants to have \$50 000 after 15 years. She pays equal amounts into an account at the end of each year. The account pays 5% p.a., compounded annually. Find the size of each payment.

Depreciation

60. GDC A car valued at \$25 000 depreciates at 15% per year. Find its value after 4 years.

61. GDC A machine bought for \$80 000 is worth \$41 000 after 5 years. It loses the same percentage of its value each year. Find that percentage.

62. GDC A car bought for \$28 000 loses 12% of its value each year.

(a) Find its value after 5 years.

(b) Explain why this model never gives the car a value of zero.

Inflation and real value

63. GDC \$6 000 is invested for 8 years at 3.8% p.a., compounded annually. Prices rise at 2.1% per year.

(a) Find the future value of the investment.

(b) Find its real value.

64. GDC Prices rise at 2.8% per year. A sum of money is kept without earning any interest. Find the number of complete years after which it has lost at least a quarter of its purchasing power.

Concept check

65. GDC A student finds the value of \$5 000 after 3 years at 6% p.a., compounded monthly, as $5\,000 \times 1.06^{36}$. Identify the error, and find the correct value.

2.8 Counting Principles HL

A four-digit PIN is built from the digits 0 to 9, and any digit may repeat. There are 10 choices for the first digit, 10 for the second, 10 for the third and 10 for the fourth, so the number of PINs is

$$10 \times 10 \times 10 \times 10 = 10\,000.$$

Writing the list out would take ten thousand lines. The total comes from the way the choices are made. Two principles give the count: **successive choices multiply; non-overlapping cases add**. The words *and* and *or* are a memory aid for these two principles.

And multiplies. When a result needs one choice *and* then another, the counts multiply. A menu offering 3 starters and 2 mains gives $3 \times 2 = 6$ meals.

A tree shows why. Each starter has the same number of branches below it, so the total is one count multiplied by the other rather than added.

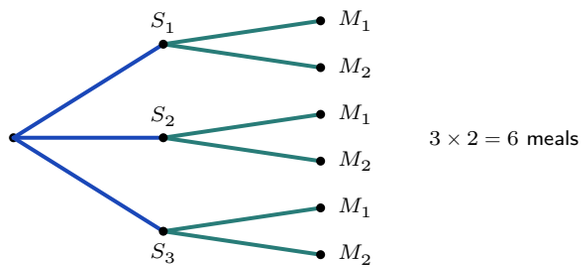


Figure 2.9 Three starters, two mains for each. Every path is a starter *and* a main, and there are $3 \times 2 = 6$ of them: that is what "and multiplies" looks like.

Add a third course and the same reasoning applies again: with 4 desserts the count becomes $3 \times 2 \times 4 = 24$. Every extra choice multiplies.

Or adds. Not every count is a 'chain' of choices (one choice made after another). Sometimes the answer is one thing *or* another, and the two cannot both happen: a password of length 3 *or* length 4, a team with three women *or* four. Count each possibility separately and add.

DEFINITION 2.4 **Addition and multiplication principles** HL

And multiplies (the **multiplication principle**). If a result needs a first choice *and* a second, made in m and n ways, there are $m \times n$ results. The same rule applies to any number of choices made one after another.

The condition: n must be the same number whatever the first choice was. It may differ from m (10 people, then 9), but it may not depend on which one you picked.

Or adds (the **addition principle**). If the result is one case *or* another, add the counts.

The condition: the cases must not overlap. Nothing may be counted twice.

CAUTION Everyday English uses *or* inclusively: "students who take French or Spanish" includes those taking both. That "or" does not simply add, because the students taking both would be counted twice. In this section *or* always means the exclusive one: this case or that case, never both. The overlapping kind is handled in Ch. 11, §11.3, where $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ subtracts the double count.

Arranging n different objects in a row is one long chain of *and*s: n choices for the first position, $n-1$ for the second, and so on down to 1. The product is n **factorial**.

KEY 2.8 **Factorial** HL

$$n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1, \qquad 0! = 1$$

Here n is a positive integer.

The value $0! = 1$ is set by definition, not derived. There is one way to arrange no objects at all, and with $0! = 1$ the formulas below also work at the ends of their range:

${}^nC_0 = \dfrac{n!}{0!\,n!} = 1$ and ${}^nP_n = \dfrac{n!}{0!} = n!$. The combinations ${}^nC_r$ counted in this section are also the coefficients of the binomial theorem §2.9.

2.8.1 Permutations and combinations

When you select r objects from n different ones, the standard question is whether the order of selection matters. Here r is a number of objects, not the common ratio of the earlier sections. If the *order* of selection matters, each ordered selection is a **permutation**. If only the final collection matters, each selection is a **combination**.

Start small. Take the letters A, B and C and pick two. Arranged in order, the results are AB, BA, AC, CA, BC and CB: six *permutations*. Chosen as a set, AB and BA are the same choice, so there are three *combinations*: $\{A, B\}$, $\{A, C\}$ and $\{B, C\}$. Each combination appears twice among the permutations, once for each order of its two letters.

To count permutations, stop the factorial product after r factors. There are n choices for the first object, $n-1$ for the second, and so on for r factors in total. Multiplying and dividing by the remaining $(n-r)$ factors turns this product into a ratio of factorials:

$$ {}^nP_r = n(n-1)\cdots(n-r+1) = \frac{n!}{(n-r)!}. $$

The combination count follows from the permutation count. Choose 3 letters from A, B, C, D, E. The selection $\{A, B, C\}$ can be written in $3! = 6$ orders: ABC, ACB, BAC, BCA, CAB, CBA. A permutation count records six results here; a combination count records one. Every combination is therefore counted $r!$ times among the permutations, so dividing by $r!$ removes the duplicates:

$$ {}^nC_r = \frac{{}^nP_r}{r!}. $$

FORMULA 2.9 **Permutations and combinations** FORMULA BOOKLET · 1.10 HL

$$ {}^nP_r = \frac{n!}{(n-r)!}, \qquad {}^nC_r = \frac{n!}{r!\,(n-r)!} $$

For selecting r from n distinct objects: ${}^nP_r$ counts ordered selections and ${}^nC_r$ unordered ones. Here r and n are whole numbers with $0 \le r \le n$, and no object is chosen twice.

The symbol ${}^nC_r$ is read "n choose r". Other books write the same number as $\binom{n}{r}$, so both forms should be recognised; the formula booklet uses ${}^nC_r$. Your GDC can evaluate both counts directly; the name of the function differs between models.

The question	The count	Example
Choose r times, repeats allowed	n^r	4-digit PIN: $10^4 = 10\,000$
Arrange all n objects	$n!$	6 books in a row: $6! = 720$
Arrange r of the n, order matters	${}^nP_r$	3 medals for 8 runners: ${}^8P_3 = 336$
Choose r of the n, order does not matter	${}^nC_r$	committee of 3 from 12: ${}^{12}C_3 = 220$

These three steps cover most counting questions in this course.

1. Decide whether the result is a chain of successive choices, which multiply, or a set of separate, non-overlapping cases, which add. The words *and* and *or* are a reminder, not the test. Check the condition each principle carries.
2. Within a single choice, ask whether the order matters. If it does, use ${}^nP_r$. If it does not, use ${}^nC_r$.
3. If the question has a restriction that is difficult to count directly, count every case and subtract the cases that do not satisfy it.

When one object or one place has a condition, deal with it first. The later counts are then the same whatever that first choice was, which is the condition the multiplication principle needs.

Three restrictions appear often: objects that must stay together, objects that must not be together, and *at least one* of something. When certain objects must stay together, treat the whole group as one object, arrange the objects, then arrange the group internally and multiply the two counts. When two objects must *not* be together, count the arrangements in which they are together and subtract them from the total. When a question asks for *at least one* of something, step 3 is usually shorter: subtract the arrangements that have none from the total.

EXAMPLE 2.25

Eight runners compete for gold, silver and bronze. Find the number of ways to award the three medals, and the number of ways if one particular runner must be among the medallists.

SOLUTION

The three medals are distinguishable, so order matters. The number of ways to award them is

$$ {}^8P_3 = 8 \times 7 \times 6 = \boxed{336} $$

Now add a restriction: one particular runner must be among the three medallists. Fill the restricted place first, so that the later counts are the same however it was filled, which is the condition the multiplication principle needs. That runner takes one of the 3 medals, and the other two medals go, in order, to two of the remaining 7 runners:

$$3 \times {}^{7}P_{2} = 3 \times 7 \times 6 = \boxed{126}$$

EXAMPLE 2.26

A committee of 5 is chosen from 6 boys and 5 girls. How many committees contain exactly 2 girls?

SOLUTION

The committee needs 2 girls *and* 3 boys, so the two counts multiply. A committee has no internal order, so each count is a combination:

$$^{5}C_{2} \times {}^{6}C_{3} = 10 \times 20 = \boxed{200}$$

EXAMPLE 2.27

Five people sit in a row. In how many ways can they sit if two of them refuse to sit next to each other?

SOLUTION

Count the seatings that place the two together, then subtract. Treat the pair as one object: four objects give $4! = 24$ arrangements, and the pair can sit in $2! = 2$ orders inside their block, so

$$24 \times 2 = 48$$

seatings place them together. All five can be seated in $5! = 120$ ways, so

$$120 - 48 = \boxed{72}$$

seatings keep them apart. Counting the cases that are not allowed and subtracting is shorter here than building the allowed ones directly.

EXAMPLE 2.28

A committee of 5 is chosen from 6 boys and 5 girls. How many committees contain at least 4 girls?

SOLUTION

Split the count into cases that cannot both happen: exactly 4 girls *or* exactly 5 girls. Count each case and add:

$$^5C_4 \times {}^6C_1 + {}^5C_5 = 5 \times 6 + 1 = \boxed{31}$$

EXAMPLE 2.29

A committee of 5 is chosen from 6 boys and 5 girls. How many committees contain at least one girl?

SOLUTION

Counting by cases would mean one, two, three, four or five girls. It is shorter to count every committee and subtract those with no girl:

$$^{11}C_5 - {}^6C_5 = 462 - 6 = \boxed{456}$$

"At least one" is everything except "none", and "none" is a single case.

EXAMPLE 2.30

Solve $^nC_2 = 66$ for the positive integer n.

SOLUTION

Write the count without factorials. Since $^nP_2 = n(n-1)$ and $^nC_2 = \dfrac{^nP_2}{2!}$,

$$\frac{n(n-1)}{2} = 66 \implies n^2 - n - 132 = 0 \implies (n-12)(n+11) = 0.$$

A number of objects cannot be negative, so $\boxed{n = 12}$, and the check $\frac{12\times11}{2} = 66$ confirms it.

CAUTION Ask two questions before every counting problem. *Can an object be chosen more than once?* If it can, as with the digits of a PIN, each position has all n choices and the count is n^r. *If not, does order matter?* Medals and rankings: yes, so use ^nP_r. Committees, hands of cards and subsets: no, so use ^nC_r.

TIP Two kinds of counting are not required in this course: arrangements in which some of the objects are identical, and arrangements in a circle.

YOUR TURN 2.8 Counting Principles HL

Arrangements and selections

66. Find each count.

(a) arrangements of 6 different books in a row

(b) 4-digit codes using the digits 1 to 9, no digit repeated

(c) committees of 3 chosen from 12 people

(d) groups of 2 men and 2 women chosen from 7 men and 6 women

Restrictions

67. Seven different people stand in a row. Find the number of arrangements in which

(a) three particular people stand next to each other

(b) two particular people stand at the two ends of the row

(c) two particular people do not stand next to each other

68. A 4-digit code uses the digits 1 to 9 with no digit repeated. Find the number of such codes that end in an even digit.

69. A committee of 3 is chosen from 12 people, one of whom is Ana. Find the number of committees that

(a) include Ana

(b) do not include Ana

Adding separate cases

70. Each of these is an *or*: split the count into parts that cannot both happen, then add.

(a) A group of 4 is chosen from 7 men and 6 women. Find the number of groups containing at least 3 women.

(b) A password is 3 or 4 letters long, chosen from the 26 lowercase letters with no letter repeated. Find the number of possible passwords.

At least one

71. A committee of 4 is chosen from 5 men and 7 women. By subtracting the committees with no men from the total, find the number of committees with at least one man.

72. A 4-digit code uses the digits 0 to 9, and digits may be repeated. Find the number of codes that contain at least one 7.

Solving for n

73. Solve each equation for the positive integer n.

(a) ${}^nC_2 = 45$ (b) ${}^nP_2 = 56$

Concept check

74. A student says the number of committees of 3 that can be chosen from 10 people is ${}^{10}P_3 = 720$. Explain the error, and find the correct number.

75. A committee of 4 is chosen from 5 men and 7 women. To count committees with at least one man, a student chooses one man (5 ways) and then any 3 of the remaining 11 people, giving $5 \times {}^{11}C_3 = 825$.

(a) Explain why this overcounts.

(b) Find the correct number.

2.9 The Binomial Theorem

Expanding $(a+b)^2$ is quick. Expanding $(a+b)^3$ takes longer. Expanding $(a+b)^{12}$ by multiplying out twelve brackets is not realistic. The coefficients follow a pattern, the rows of Pascal's triangle, so you do not need to multiply out the brackets.

Look at the small cases:

$$(a+b)^1 = a+b, \qquad (a+b)^2 = a^2+2ab+b^2, \qquad (a+b)^3 = a^3+3a^2b+3ab^2+b^3.$$

The powers are regular: the power of a falls from n to 0 while the power of b rises from 0 to n, and in every term the two powers add to n. The coefficients form a row of **Pascal's triangle**, in which each row begins and ends with 1 and each of the other entries is the sum of the two entries above it.

Number the rows from zero. The top row is row 0, the next is row 1, and so on, so that **row n holds the coefficients of** $(a+b)^n$. The top row has a meaning: $(a+b)^0 = 1$, which is the single entry 1. Number the entries along a row from zero as well, so that the entry in position r of row n multiplies $a^{n-r}b^r$.

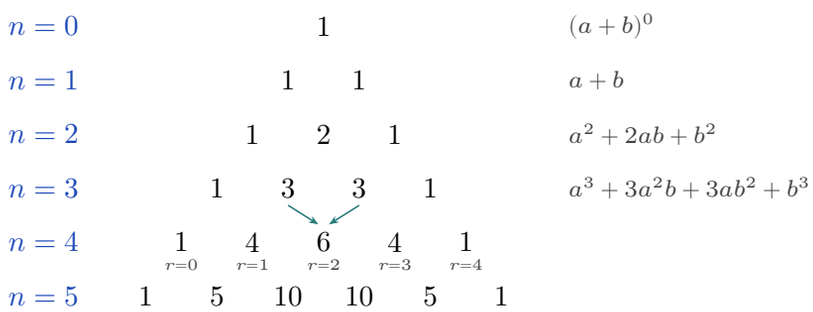


Figure 2.10 Rows 0 to 5 of Pascal's triangle. Each entry is the sum of the two above it (teal arrows: $3+3=6$), and the entry in position r of row n multiplies $a^{n-r}b^r$.

Row n has $n+1$ entries, which is the number of terms in the expansion.

EXAMPLE 2.31

Expand $(x+2)^4$.

SOLUTION

Take $n=4$ and read row 4 of the triangle, the fifth row down: $1, 4, 6, 4, 1$, with the powers of 2 increasing as the powers of x decrease:

$$(x+2)^4 = x^4 + 4x^3(2) + 6x^2(4) + 4x(8) + 16 = \boxed{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Number the terms by $r = 0, 1, 2, 3, 4$, the power of 2 in each. The term with $r=2$ is $6x^2(2^2) = 24x^2$, and it is the third term: term number $r+1$, because the counting starts at $r=0$.

2.9.1 Why the entries are the combinations

Expanding $(a+b)^n$ means going through the n brackets and choosing either a or b from each one, then adding every product so formed. A product equals $a^{n-r}b^r$ exactly when b was chosen from r of the brackets and a from the other $n-r$. So the coefficient of $a^{n-r}b^r$ counts one thing: **how many ways there are to choose which r of the n brackets supply the b.** Choosing r objects from n, order not mattering, is what ${}^nC_r$ counts §2.8, so the coefficient *is* ${}^nC_r$.

The case $n=3$ is small enough to check by hand. Write the three brackets as $(a+b)(a+b)(a+b)$. The term a^2b appears when exactly one bracket gives a b. There are three ways for that to happen: the first bracket, the second, or the third. So the coefficient is 3, and ${}^3C_1 = 3$. Likewise ab^2 needs two brackets to give a b, which can be done in ${}^3C_2 = 3$ ways, while a^3 and b^3 each arise in only one way: ${}^3C_0 = {}^3C_3 = 1$.

This also explains the addition rule used to build the triangle. Pick one bracket and ask whether it supplies a b. If it does, the other $r-1$ copies of b come from the remaining $n-1$ brackets, in ${}^{n-1}C_{r-1}$ ways; if it does not, all r come from those $n-1$ brackets, in ${}^{n-1}C_r$

ways. The two cases cover every selection once, so

$$^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r,$$

which is exactly "each entry is the sum of the two entries above it". Since both descriptions start each row with 1 and end it with 1 ($^nC_0 = {}^nC_n = 1$) and then grow by the same rule, the triangle and the combinations agree row for row.

FORMULA 2.10 **Binomial theorem** FORMULA BOOKLET · 1.9

For $n \in \mathbb{N}$:

$$(a+b)^n = a^n + {}^nC_1\, a^{n-1}b + \cdots + {}^nC_r\, a^{n-r}b^r + \cdots + b^n,$$

$$\text{where} \quad {}^nC_r = \frac{n!}{r!\,(n-r)!}.$$

In sigma notation §2.5 the theorem reads $(a+b)^n = \sum_{r=0}^{n} {}^nC_r\, a^{n-r}b^r$.

Pascal's triangle is fastest for small n. When n is large, or when a question asks for one coefficient only, use the single term $^nC_r\, a^{n-r}b^r$ shown in the middle of the expansion above and choose the r you need. When the second term is negative, as in $(2x-3)^5$, take $b = -3$ with its sign: the powers $(-3)^r$ then give the alternating signs.

TIP The booklet shows that single term inside the expansion without naming it. The general term of the expansion is

$$T_{r+1} = {}^nC_r\, a^{n-r}b^r,$$

so $r = 0$ gives the first term, $r = 1$ the second, and r gives term number $r+1$. The subscript is $r+1$ because r counts how many factors of b the term contains, and the first term contains none. In an examination, write down this term, find the r that produces the power you want, then substitute.

A question asking for the **constant term**, or for the term *independent of* x, is asking for the one in which **the power of x is 0**, since $x^0 = 1$ leaves a number. Simplify the general term until x appears once, as a single power, then set that power equal to 0 and solve for r.

EXAMPLE 2.32

Find the coefficient of x^3 in $(2x-3)^5$.

SOLUTION

The general term of the expansion is $^5C_r\,(2x)^{5-r}(-3)^r$. For x^3, take $5-r=3$, so $r=2$:

$$^5C_2\,(2x)^3(-3)^2 = 10 \times 8x^3 \times 9 = 720x^3.$$

The coefficient is $\boxed{720}$

With $b = -3$, the factor $(-3)^2 = 9$ is positive.

EXAMPLE 2.33

Find the term independent of x in $\left(x^2 + \frac{2}{x}\right)^6$.

SOLUTION

Track the power of x in the general term:

$$T_{r+1} = {}^6C_r\,(x^2)^{6-r}\left(\frac{2}{x}\right)^r = {}^6C_r\,2^r\,x^{12-3r}.$$

Independent of x means $12 - 3r = 0$, so $r = 4$:

$${}^6C_4\,2^4 = 15 \times 16 = \boxed{240}$$

TIP ${}^nC_r$ can also be read from a GDC table. To solve ${}^6C_r = 20$, generate ${}^6C_r$ for $r = 0, \ldots, 6$ and read $r = 3$ from the table. Because of the symmetry ${}^nC_r = {}^nC_{n-r}$, you need to check only half of the values. The symmetry holds because choosing the r objects to take also chooses the $n - r$ objects to leave.

YOUR TURN 2.9 The Binomial Theorem

Expanding

76. Expand or write down the terms requested.

(a) Expand $(2 + y)^5$, using row 5 of Pascal's triangle.

(b) Write down the first three terms, in ascending powers of x, of $(1 + 2x)^7$.

(c) Expand $(2x - 1)^4$.

(d) Write down the first three terms, in ascending powers of x, of $(1 - 3x)^6$.

(e) Write $(1 + x)^5$ in sigma notation.

A single coefficient

77. Find the coefficient requested.

(a) x^4 in $\left(3 - \frac{x}{3}\right)^6$

(b) x^3 in $(2 + x)^6$

(c) x^4 in $(1 - 2x)^7$

(d) x^5 in $(x + 3)^8$

78. Find the middle term in the expansion of $(x - 2y)^6$.

Term numbers and symmetry

79. Answer each part without expanding in full.

(a) Write down the 5th term of the expansion of $(1+x)^{10}$.

(b) Given that ${}^{12}C_5 = 792$, write down ${}^{12}C_7$, giving a reason.

(c) In the expansion of $(1+x)^n$, the coefficients of x^3 and x^7 are equal. Find n.

Terms with a negative power of x

80. Find the term requested.

(a) the term independent of x in $\left(x+\dfrac{1}{x^2}\right)^9$

(b) the term independent of x in $\left(2x-\dfrac{1}{x}\right)^6$

(c) the coefficient of x^3 in $\left(x-\dfrac{2}{x}\right)^7$

(d) the term independent of x in $\left(3x^2-\dfrac{1}{x}\right)^6$

Concept check

81. A student says the coefficient of x^3 in $(1-2x)^5$ is ${}^5C_3 \times 2^3 = 80$. Identify the error, and find the correct coefficient.

2.10 The Binomial Series HL

So far n has been a non-negative whole number, because $(a+b)^n$ meant n brackets multiplied together. But expressions such as $\sqrt{1+x} = (1+x)^{1/2}$ and $\dfrac{1}{1+x} = (1+x)^{-1}$ also have exponents. Can they be expanded too?

They can, and the same coefficient pattern still gives a correct expansion for *any* rational n. When n is not a non-negative whole number, two things change: the expansion never ends, and it is valid only when $|x| < 1$.

In practice the series is used in the form

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots, \qquad |x| < 1 \text{ when } n \notin \mathbb{N},$$

and a question of this kind is usually solved in four steps:

1. **Factorise** into the form $a^n(1 + \text{something})^n$.
2. **Identify** what replaces x.
3. **Expand**, using the coefficients above.
4. **State the validity**: $|\text{something}| < 1$.

EXAMPLE 2.34

Use the binomial series to approximate $\sqrt{1.02}$.

SOLUTION

Write $\sqrt{1.02}$ as $(1+x)^{1/2}$ with $x = 0.02$, which satisfies $|x| < 1$. With $n = \frac{1}{2}$,

$$(1+x)^{1/2} = 1 + \tfrac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \cdots = 1 + \frac{x}{2} - \frac{x^2}{8} + \cdots,$$

so $\sqrt{1.02} \approx 1 + 0.01 - 0.00005 = \boxed{1.00995}$, against the true value $1.0099505\ldots$.

Where the extension comes from. Write the coefficient of §2.9 without factorials. In

$$^nC_r = \frac{n!}{r!\,(n-r)!},$$

the $(n-r)!$ cancels all but the first r factors of $n!$, leaving

$$^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r!},$$

which is r factors counting down from n, divided by $r!$. Now compare the two forms. The factorial $n!$ makes sense only for a non-negative whole number. The product of r descending factors makes sense for *any* n. With $n = \frac{1}{2}$ and $r = 2$ it is $\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!} = -\frac{1}{8}$, and with $n = -1$ and $r = 3$ it is $\frac{(-1)(-2)(-3)}{3!} = -1$. Keeping the coefficients in this form, and letting n be any rational number, is all that the extension does. For a whole n nothing has changed: at $n = 3$, $r = 2$ the formula gives $\frac{3\times 2}{2!} = 3 = {}^3C_2$.

The counting meaning is lost. For a whole n the coefficient counts brackets. For $n = \frac{1}{2}$ it counts nothing, so the equality with the function needs a different argument. That argument is where the condition on x comes from. The derivation uses repeated differentiation. See Ch. 16, §16.4. The case $n = \frac{1}{2}$ can be checked now: its first three terms are $1 + \frac{x}{2} - \frac{x^2}{8}$, and

$$\left(1 + \frac{x}{2} - \frac{x^2}{8}\right)^2 = 1 + x - \frac{x^3}{8} + \frac{x^4}{64},$$

which agrees with $1 + x$ up to the term in x^2, as the square of $\sqrt{1+x}$ should.

FORMULA 2.11 **Extended binomial theorem** FORMULA BOOKLET · 1.10 HL

For $n \in \mathbb{Q}$:

$$(a+b)^n = a^n\left(1 + n\frac{b}{a} + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots\right)$$

The booklet omits the validity condition. When n is not a non-negative integer the series never ends, so it is used only when $\left|\frac{b}{a}\right| < 1$, with $a \neq 0$: factor out the term of larger size to make this hold. When $n \in \mathbb{N}$ the series stops and needs no condition.

The booklet states the series for $(a+b)^n$. Factoring out a^n turns it into the $(1+x)^n$ form above, with $\frac{b}{a}$ in place of x.

Why does the expansion stop for whole numbers but not otherwise? Look at those descending factors $n(n-1)(n-2)\cdots$ again. If $n = 3$, the fourth factor is $n - 3 = 0$, so every later coefficient is zero and the series ends: that is the finite theorem of §2.9. If $n = \frac{1}{2}$ or $n = -1$, no factor is ever zero, so the terms continue forever.

The condition $|x| < 1$ is the convergence condition for a geometric series. Put $n = -1$:

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

This is a geometric series with first term 1 and common ratio $-x$, and by §2.6 it converges to $\frac{1}{1+x}$ exactly when $|-x| < 1$. So in this case the condition is not an extra assumption, and the same bound holds for every rational n. The proof for a general n is beyond this course. So for $n = -1$ the binomial series and the geometric series are the same series.

EXAMPLE 2.35

Expand $(1-3x)^{1/3}$ up to and including the term in x^2, stating the validity.

SOLUTION

Apply the series with $n = \frac{1}{3}$ and x replaced by $-3x$:

$$(1-3x)^{1/3} = 1 + \tfrac{1}{3}(-3x) + \frac{\frac{1}{3}\left(-\frac{2}{3}\right)}{2}(-3x)^2 + \cdots = \boxed{1 - x - x^2} - \cdots$$

Validity requires $|-3x| < 1$, that is $\boxed{|x| < \frac{1}{3}}$

EXAMPLE 2.36

Expand $\dfrac{1}{2+x}$ in ascending powers of x up to x^2, stating the validity.

SOLUTION

Factor out the 2:

$$(2+x)^{-1} = \frac{1}{2}\left(1+\frac{x}{2}\right)^{-1} = \frac{1}{2}\left(1-\frac{x}{2}+\frac{x^2}{4}-\cdots\right) = \boxed{\frac{1}{2}-\frac{x}{4}+\frac{x^2}{8}-\cdots}$$

valid for $\left|\frac{x}{2}\right| < 1$, that is $\boxed{|x| < 2}$

A **truncated** expansion (one cut off after a few terms) gives an approximation. Choose x inside the interval of validity so that the bracket becomes the value you want, or a value it can be found from. The closer x is to 0, the faster the later terms shrink and the better the approximation.

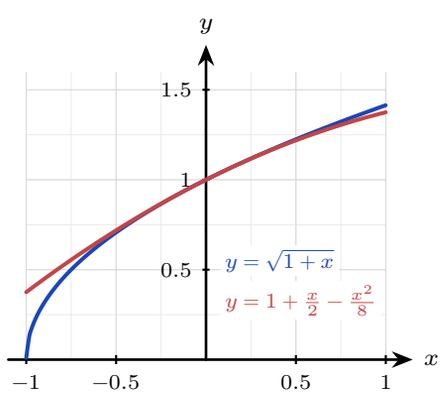


Figure 2.11 $\sqrt{1+x}$ and its first three terms. The two graphs are nearly the same near $x = 0$ and separate towards the ends of $-1 < x < 1$.

EXAMPLE 2.37

By expanding $(1-x)^{1/2}$ up to x^2 and choosing a suitable value of x, find an approximation to $\sqrt{2}$.

SOLUTION

The expansion is

$$(1-x)^{1/2} = 1 - \frac{x}{2} - \frac{x^2}{8} - \cdots$$

Now choose x so that $1-x$ is related to 2 in a simple way. Putting $1-x=2$ needs $x=-1$, which is outside $|x|<1$, and in any case x must be close to 0 for three terms to be enough. So look for a number close to 1 whose square root is a simple multiple of $\sqrt{2}$. Since $0.98 = \frac{49}{50} = 2\times\left(\frac{7}{10}\right)^2$, the choice is $x = 0.02$, giving $1-x = 0.98$:

$$\sqrt{0.98} = \frac{7}{\sqrt{50}} = \frac{7\sqrt{2}}{10} \implies \sqrt{2} = \frac{10}{7}\sqrt{0.98}.$$

The series gives $\sqrt{0.98} \approx 1 - 0.01 - 0.00005 = 0.98995$, hence

$$\sqrt{2} \approx \frac{10}{7}(0.98995) \approx \boxed{1.41421},$$

agreeing with the true value to five decimal places, from just three terms.

CAUTION Always state the interval of validity alongside an HL binomial expansion; the infinite series must not be used outside it. And the expansion needs the form $(1+\bullet)^n$: factor first, then expand.

At HL these expansions appear again as Maclaurin series in Ch. 16, where the same coefficients arise from repeated differentiation.

YOUR TURN 2.10 The Binomial Series HL

Expanding $(1+x)^n$

82. Expand each up to and including the term in x^3, stating the validity.

(a) $(1+x)^{-1}$ (b) $(1-x)^{-2}$

83. Expand each up to and including the term in x^2, stating the validity.

(a) $(1+4x)^{1/2}$ (b) $(1+x)^{1/3}$

Factorising first

84. Expand $\dfrac{1}{3-x}$ in ascending powers of x up to x^2, stating the validity.

85. Expand $(8+x)^{1/3}$ in ascending powers of x up to x^2, stating the validity.

Approximations

86. Use the expansion of $(1+x)^{1/2}$ up to the term in x^2, with $x = 0.1$, to estimate $\sqrt{1.1}$ to four decimal places.

87. This question estimates $\sqrt{3}$.

(a) Expand $(4-x)^{1/2}$ in ascending powers of x up to x^2, stating the validity.

(b) By choosing a suitable value of x, use your expansion to estimate $\sqrt{3}$ to three decimal places.

The link with geometric series

88. This question compares the binomial series with a geometric series.

(a) Use the binomial series to expand $(1-3x)^{-1}$ up to and including the term in x^3.

(b) Show that these are the first four terms of a geometric series, and state its common ratio.

(c) Explain why the binomial expansion and the geometric series both need $|x| < \frac{1}{3}$.

Concept check

89. A student expands $(1-3x)^{-2}$ as a binomial series and states that it is valid for $|x| < 1$. Identify the error, and state the correct condition.

Chapter 2 · Sequences & Series

Reflections

TOK About 2500 years ago the Greek thinker Zeno argued that a runner can never finish a race. First the runner must reach the halfway point, then half of what remains, and so on without end. The sum $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$ seems to answer him, since infinitely many steps can have a finite total. The times taken shrink in the same way, so the runner covers all the steps in a finite time. Yet no finite number of steps ever reaches 1: we accept the answer because we *define* the sum to be the limit of the partial sums. Is that a discovery about infinity, or a decision we made so that the addition has an answer?

IM The triangle of binomial coefficients has different names in different places. Indian scholars met these numbers when counting the patterns of short and long syllables in poetry. Persian mathematicians used them to expand powers, and Iran still calls the triangle the Khayyam triangle. China calls it Yang Hui's triangle and Italy calls it Tartaglia's. Much of Europe names it after Pascal, who wrote about it centuries after most of these. Each name records who made it well known in that place, not always who reached it first. What does that tell you about who a mathematical result belongs to?

RW A patient takes 100 mg of a drug every 12 hours, and in each 12-hour interval the body removes half of the drug present. Just after each dose the amount in the body is 100, then $100+50$, then $100+50+25$ mg, and so on: the partial sums of a geometric series with $u_1=100$ and $r=\frac{1}{2}$. The amount rises towards $S_\infty=\dfrac{100}{1-\frac{1}{2}}=200$ mg and never reaches it. Just *before* each dose, half has gone: 50, then 75, then 87.5 mg, approaching 100 mg. Doctors use these limiting amounts: a peak of 200 mg just after a dose and a trough (lowest level) of 100 mg just before the next. They keep the drug above the smallest amount that has an effect and below the amount that causes harm.

EXT Stack identical cards at the edge of a table, each sticking out beyond the one below. Let the top card overhang the second by $\frac{1}{2}$ a card length, the second overhang the third by $\frac{1}{4}$, the next by $\frac{1}{6}$, and so on. This stack balances, and n cards reach $\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)$ beyond the edge: half of the **harmonic series**. Its terms shrink to zero, yet, unlike a geometric series with $|r|<1$, it has no finite sum, because

$$1+\tfrac{1}{2}+\left(\tfrac{1}{3}+\tfrac{1}{4}\right)+\left(\tfrac{1}{5}+\cdots+\tfrac{1}{8}\right)+\cdots>1+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\cdots.$$

So with enough cards the top one can reach as far beyond the table as you like.

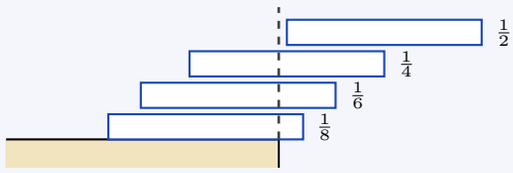


Figure 2.12 Four cards: each overhangs the one below by $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$ and $\frac{1}{8}$ of a card length, from the top down. The top card ends $\frac{25}{24}$ of a card beyond the table edge (dashed).

Try it: show that four cards already reach $\frac{25}{24}$ of a card length out, so the top card lies completely beyond the edge. Then build it.

End-of-Chapter Problems
Chapter 2: Sequences & Series

1. Consider the two arithmetic sequences $4, 7, 10, \ldots$ and $3, 7, 11, \ldots$ Both sequences contain the value 7.

(a) Find the general term of each sequence. [2]
(b) Find the next three values that appear in both sequences. [4]
(c) Show that the common terms form an arithmetic sequence and state its common difference. [2]

2. An arithmetic sequence has first term u_1 and common difference d. The 5th term is 17 and the sum of the first 10 terms is 155.

(a) Write down two equations in u_1 and d. [2]
(b) Find u_1 and d. [2]
(c) Find the first negative term of the sequence. [2]

3. An arithmetic sequence has $u_3 = 7$ and $u_9 = 25$.

(a) Find u_1 and d. [3]
(b) Find S_{20}. [2]
(c) Find the least value of n such that $S_n > 600$. [3]

4. An arithmetic sequence has 20 terms. The sum of the odd-positioned terms $u_1 + u_3 + u_5 + \cdots + u_{19}$ is 160, and the sum of the even-positioned terms $u_2 + u_4 + u_6 + \cdots + u_{20}$ is 140. Find the first term u_1 and the common difference d. [6]

5. The sum of the first n terms of a series is given by $S_n = 2n^2 + 5n$.

(a) Find u_1 and u_2. [2]
(b) Show that the series is arithmetic and state the common difference. [3]
(c) Find the value of n such that $u_n = 83$. [2]

6. An AP has $u_1 = 120$ and $d = -3$.

(a) Find the first term that is negative. [3]
(b) Find the maximum value of S_n and the value(s) of n for which it occurs. [4]

7. A geometric sequence has first term u_1 and common ratio r, where $r > 0$. Given $u_3 + u_4 = 12$ and $u_5 + u_6 = 3$, find u_1 and r. [5]

8. A sequence is defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 1$ for $n \geq 1$.

(a) Write down u_2, u_3, and u_4. [2]
(b) Let $v_n = u_n + 1$. Show that v_n is a geometric sequence. [3]
(c) Find an explicit formula for u_n. [2]

9.

(a) By expanding $(1-r)(1+r+r^2+\cdots+r^{n-1})$, show that $\sum_{k=1}^{n} r^{k-1} = \frac{1-r^n}{1-r}$ for $r \neq 1$. [3]
(b) Hence find $\sum_{k=1}^{10} \left(\frac{2}{3}\right)^{k-1}$, giving your answer as an exact fraction. [3]

10. An arithmetic series has $u_1 = 2$ and $S_{10} = 155$.

(a) Find the common difference. [2]
(b) Find the sum $\sum_{k=11}^{20} u_k$. [4]

11.

(a) Find $\sum_{k=1}^{n}(4k-3)$ in terms of n. [3]
(b) Find the value of n such that $\sum_{k=1}^{n}(4k-3) = 780$. [3]

12. GDC A geometric series has $u_1 = 3$ and $S_\infty = 12$.

(a) Find the common ratio. [2]
(b) Show that $S_n = 12\left(1 - \left(\frac{3}{4}\right)^n\right)$. [2]
(c) Find the least value of n such that $S_n > 11.9$. [3]

13. Consider the geometric series $1 + 2x + 4x^2 + 8x^3 + \cdots$

(a) Write down the common ratio. [1]
(b) Find the values of x for which the sum to infinity exists. [2]
(c) For these values, find S_∞ as a function of x. [2]

14. A geometric sequence has all positive terms. The sum of the first two terms is 10 and the sum to infinity is 25. Find u_1 and r. [5]

15. The sum to infinity of a geometric series is $\frac{3}{2}$ times the sum of the first 4 terms. The common ratio is positive. Find the common ratio. [5]

16. The repeating decimal $0.\overline{142857} = 0.142857142857\ldots$ can be written as an infinite geometric series.

(a) Write down u_1 and the common ratio r. [2]
(b) Find the exact fractional value of $0.\overline{142857}$. [3]

17. GDC A ball is dropped from a height of 10 m. After each bounce it reaches 70% of the height it fell from.

(a) Find the height reached after the 3rd bounce. [2]
(b) Find the total vertical distance travelled by the ball before it comes to rest. [4]
(c) Find the least number of bounces after which the height reached is less than 1 cm. [3]

18. GDC A machine is bought for \$60 000 and depreciates at 18% per year.

(a) Find its value after 4 years. [2]
(b) Find the least number of complete years until the value falls below one quarter of the purchase price. [3]

19. GDC \$8 000 is invested for 6 years at 5% per year, compounded quarterly. Over the same period, inflation runs at 3% per year.

(a) Find the future value of the investment. [2]
(b) Find the real value of the investment after 6 years. [2]
(c) Comment on what your answers say about the investment's true growth. [1]

20. GDC An investment account pays 5% per year, compounded annually. A sum P is invested.

(a) Write an expression for the value of the account after n years. [1]
(b) Find the value of P if the account is worth \$20 000 after 10 years. [3]
(c) Find the number of complete years for the account to triple in value. [3]

21. GDC The value of a car decreases by the same percentage each year. A new car costs \$32 000, and after 3 years it is worth \$16 384.

(a) Show that the value falls by 20% each year. [2]
(b) Find the year in which the value first falls below \$10 000. [3]
(c) Find the total loss in value over the first 5 years. [3]

22. GDC A student can invest \$3 000 in one of two accounts. Account A pays 5% per year *simple* interest; account B pays 4% per year, compounded annually.

(a) Write down an expression for the value of each account after n years. [3]

(b) Find the value of each account after 10 years. [2]

(c) Using a table or graph on your GDC, find the least whole number of years after which account B is worth more than account A. [3]

23. GDC A savings plan pays \$500 into an account at the start of each year. The account earns 4% per year, compounded annually.

(a) Show that after 3 years the total value is $500(1.04)^3 + 500(1.04)^2 + 500(1.04)$. [2]

(b) Write this as a geometric series and find the total after 10 years. [4]

(c) Find the number of complete years until the total exceeds \$8 000. [3]

24. GDC An amount P is invested. Bank A pays 4% interest compounded annually. Bank B pays 3.9% compounded monthly, so its value after t years is $P\left(1+\frac{0.039}{12}\right)^{12t}$.

(a) Write down an expression for the value in Bank A after t years. [1]

(b) Show that Bank B's value can be written as PR^t, where R is its yearly growth factor, and find R to five decimal places. [3]

(c) Determine which bank gives more after 10 years, and by how much as a percentage of P. [3]

(d) Use logarithms to find how long an investment in Bank A takes to double. [2]

(e) Bank C pays 4% compounded continuously, giving $Pe^{0.04t}$. Show that Bank C gives more than Bank A for every $t > 0$, and find the difference after 10 years as a percentage of P. [3]

25. The seven letters of the word FACTORS are all different. Two of them, A and O, are vowels.

(a) Find the number of arrangements of all seven letters. [1]

(b) Find the number of these arrangements that begin with a vowel and end with a consonant. [2]

(c) Find the number of these arrangements in which the two vowels are not next to each other. [3]

(d) A code is formed from 4 of the seven letters, in order, with no letter repeated. Find the number of codes that contain the letter T. [2]

26. A school committee of 6 is to be chosen from 9 teachers and 7 students.

(a) Find the number of committees containing exactly 4 students. [2]

(b) Find the number of committees containing at least 4 students. [3]

27. In the expansion of $(1+kx)^8$, the coefficient of x^2 is 112.

(a) Find the possible values of k. [3]

(b) For $k > 0$, find the coefficient of x^3. [2]

28. Consider the expansion of $\left(x + \frac{2}{x^2}\right)^{12}$.

(a) Find the term independent of x. [4]
(b) Find the coefficient of x^3. [2]

29. Consider the expansion of $\left(2x - \frac{3}{x}\right)^{10}$.

(a) Write down an expression for the general term of the expansion. [2]
(b) Find the term that is independent of x. [4]
(c) Find the coefficient of x^4. [3]
(d) Explain why the expansion contains no term in x^3. [2]
(e) By substituting a suitable value of x, find the sum of all the coefficients. [2]
(f) State how many terms the expansion has, and which powers of x occur. [2]

30.

(a) Expand $(1 + 2x)^{-2}$ in ascending powers of x up to and including the term in x^3, and state the values of x for which the expansion is valid. [4]
(b) Use your expansion with $x = 0.01$ to estimate $\frac{1}{1.02^2}$ to 5 decimal places. [2]

31.

(a) Show that $(4 + x)^{1/2} = 2\left(1 + \frac{x}{4}\right)^{1/2}$, and expand it in ascending powers of x up to the term in x^2, stating the validity. [4]
(b) Hence find an approximation to $\sqrt{4.2}$, giving four decimal places. [2]

32. The first three terms of a geometric sequence are k, 6, and $k + 5$.

(a) Show that $k^2 + 5k - 36 = 0$, and hence find the possible values of k. [4]
(b) For the value of k that gives a convergent series, find S_∞. [3]

33. The first, second, and fifth terms of an arithmetic sequence with first term 3 are, in that order, the first, second, and third terms of a geometric sequence.

(a) Given the arithmetic sequence is non-constant, show that its common difference is $d = 6$ and the geometric sequence has common ratio $r = 3$. [4]
(b) Determine whether the geometric series $3 + 9 + 27 + \cdots$ has a sum to infinity, giving a reason. [2]

34. GDC Let $S_n = \sum_{k=1}^{n} (3k + 2(0.8)^k)$.

(a) Show that $S_n = \frac{3n(n+1)}{2} + 8\,(1 - 0.8^n)$. [3]

(b) Find the least value of n for which $S_n > 1000$. [2]

35.

(a) Show that ${}^nC_2 = \frac{n(n-1)}{2}$, and explain the connection with the sum $1 + 2 + \cdots + (n-1)$. [3]

(b) Hence find n such that ${}^nC_2 = 190$. [2]

36. An arithmetic sequence has first term u_1 and common difference d, where $u_1 > 0$ and $d \neq 0$. Its 1st, 4th and 13th terms form a geometric sequence.

(a) Show that $3d = 2u_1$. [4]

(b) Given that $u_1 = 6$, find d and write down the first four terms of the arithmetic sequence. [3]

(c) Find the common ratio of the geometric sequence. [2]

(d) Find the sum of the first 20 terms of the arithmetic sequence. [3]

(e) Find the sum of the first 8 terms of the geometric sequence. [3]

37. GDC Two cyclists train for the same race. In week 1 Ana rides 40 km, and each week after that she rides 8% further than in the week before. In week 1 Ben rides 60 km, and each week after that he rides 5 km further than in the week before.

(a) Find the distance each cyclist rides in week 10. Give Ana's distance to 3 significant figures. [3]

(b) Find the total distance each cyclist rides in the first 10 weeks, to the nearest km. [4]

(c) Find the first week in which Ana rides further than Ben. [3]

(d) Find the least number of weeks after which Ana's total distance is greater than Ben's total distance. [3]

38. A geometric sequence has first term 3, and the sum of its first three terms is 9.

(a) Find the two possible values of the common ratio r. [3]

(b) For each value of r, find S_{10}. [3]

(c) Take $r > 0$. Find the least value of n for which $S_n > 3069$. [3]

39. The digits 0 to 9 are used to make codes and numbers.

(a) A four-digit number is formed with no digit repeated. A four-digit number cannot begin with 0. Find how many such numbers there are. [2]

(b) Find how many of the numbers in part (a) are even. [4]

(c) A PIN is any string of four digits; digits may repeat and the first digit may be 0. Find the number of PINs in which at least one digit is repeated. [3]

40. Consider the expansion of $\left(2-\frac{x}{2}\right)^8$ in ascending powers of x.

(a) Find the 4th term. [3]

(b) Hence, or otherwise, find the coefficient of x^4 in the expansion of $(1+2x)\left(2-\frac{x}{2}\right)^8$. [4]

Challenge Questions

C1. GDC **When does a sequence settle?** A sequence is defined by a starting value u_1 and the rule $u_{n+1} = au_n + b$, where a and b are constants. This investigation finds, for every value of a, whether the terms approach a limit.

(a) Take $a = \frac{1}{2}$, $b = 3$ and $u_1 = 2$. Tabulate $u_1, u_2, \ldots, u_6$, and state the value the terms appear to approach. [2]

(b) Let $a \neq 1$. Show that if the sequence has a limit L, then $L = aL + b$, and hence $L = \dfrac{b}{1-a}$. Check this against part (a). [2]

(c) With $a \neq 1$ and $L = \dfrac{b}{1-a}$, let $v_n = u_n - L$. Show that $v_{n+1} = av_n$, and deduce that $u_n = L + (u_1 - L)\,a^{n-1}$. [4]

(d) Suppose $u_1 \neq L$. Use part (c) to describe the long-run behaviour of $\{u_n\}$ when $0 < |a| < 1$, when $a = -1$ and when $|a| > 1$, saying in each case whether the sequence converges, and whether the terms stay on one side of L or alternate about it. Describe also what happens when $a = 1$. [4]

(e) A patient takes 100 mg of a drug every 12 hours, and by the time of the next dose the body has removed 60% of the drug present. Let u_n be the amount, in mg, just after the nth dose, so $u_1 = 100$ and $u_{n+1} = 0.4u_n + 100$. Find the long-run level, and the first dose after which the amount is within 1 mg of it. Find also the largest dose that keeps the long-run level at or below 200 mg. [4]

(f) The rule $u_{n+1} = u_n^2 + c$, with $u_1 = 0$, is not of the form above. Show that a limit L can exist only if $c \leq \frac{1}{4}$. Explore the sequence for $c = -0.5,\ 0.3,\ -1$ and -2.1, describe what you see, and state a question about this rule that could be investigated further. [3]

C2. **A sum that is not a sum.** You may have seen the claim that $1 + 2 + 3 + 4 + \cdots = -\frac{1}{12}$. Every term is positive, so the statement cannot mean what addition normally means. This question takes the popular argument apart.

Write $G = 1 - 1 + 1 - 1 + \cdots$ and $T = 1 + 2 + 3 + 4 + \cdots$

(a) Write down the first six partial sums of G. Explain why G has no sum in the usual sense. [3]

(b) Bracketing the terms as $(1 - 1) + (1 - 1) + \cdots$ gives 0; bracketing them as $1 - (1 - 1) - (1 - 1) - \cdots$ gives 1. State what this shows about applying the ordinary rules of algebra to such a series. [3]

(c) Applying $S_\infty = \dfrac{u_1}{1 - r}$ to G with $u_1 = 1$ and $r = -1$ gives $\frac{1}{2}$. State the condition under which that formula was derived, and explain why it does not hold here. [3]

(d) Write down the nth partial sum of T and show that it increases without bound. [3]

(e) The popular argument runs: assume $G = \frac{1}{2}$; deduce $1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$; subtract this from T term by term to obtain $4 + 8 + 12 + \cdots = 4T$; conclude $T - \frac{1}{4} = 4T$, so $T = -\frac{1}{12}$. Identify the step that is not valid, and say why. [4]

(f) Take the partial sums of G from part (a) and average the first n of them. Compute these averages for $n = 1$ to 6, and state what value they appear to approach. [3]

(g) Part (f) shows that $\frac{1}{2}$ is not arbitrary: it is the value G receives when "sum" is defined as the limit of these averages. Show that this averaging definition gives T no value. Then explain what a statement such as $1 + 2 + 3 + \cdots = -\frac{1}{12}$ actually asserts, and why writing it with an equals sign is misleading. [3]

(h) Another definition (Abel's) replaces a series $c_0 + c_1 + c_2 + \cdots$ by $c_0 + c_1 x + c_2 x^2 + \cdots$ for $0 < x < 1$, and lets x approach 1 from below. Find the value this gives G. Given that $1 - 2x + 3x^2 - 4x^3 + \cdots = \dfrac{1}{(1 + x)^2}$ for $0 < x < 1$, find the value it gives $1 - 2 + 3 - 4 + \cdots$, and comment on the result in the light of part (e). Suggest how comparing definitions of "sum" could be taken further. [3]

C3. GDC **Counting multiples.** Each set of multiples below is an arithmetic sequence, so its total is an arithmetic series. Throughout, $\lfloor t \rfloor$ is the greatest integer not exceeding t; in parts (a) to (d), work with the integers from 1 to 1000.

(a) Find the sum of those divisible by 3, and the sum of those divisible by 5. [3]

(b) Explain why adding your two answers counts some integers twice, and find the sum of those divisible by both. [3]

(c) Hence find the sum of those divisible by 3 or by 5. [2]

(d) Find the sum of those divisible by 3 or by 5, but not by both. [3]

(e) Let p and q be coprime integers greater than 1 (their only common positive factor is 1), and let N be a positive integer. Show that the sum of the integers from 1 to N divisible by p or q but not both is

$$S_p + S_q - 2S_{pq}, \qquad \text{where } S_m = m\,\frac{t(t+1)}{2}, \quad t = \left\lfloor \frac{N}{m} \right\rfloor.$$

[4]

(f) Use the result of part (e) with $p = 7$, $q = 11$ and $N = 1000$. Verify your answer by summing a list on your GDC. [3]

(g) Let $T(N)$ be the sum in part (e). Show that $\dfrac{T(N)}{N^2} \to \dfrac{1}{2p} + \dfrac{1}{2q} - \dfrac{1}{pq}$ as $N \to \infty$, and evaluate this for $p = 3$, $q = 5$. Suggest how the investigation could be extended. [3]

C4. GDC **How many terms for $\sqrt{2}$?** The binomial series for $(1-x)^{1/2}$ converges for $|x| < 1$, but how quickly it converges depends on x. This investigation measures the error made by stopping the series early, and uses the result to choose x well. Write

$$(1-x)^{1/2} = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots .$$

(a) Show that each of the following is true:

$$\sqrt{2} = 2\left(1 - \tfrac{1}{2}\right)^{1/2}, \qquad \sqrt{2} = \tfrac{10}{7}\left(1 - \tfrac{1}{50}\right)^{1/2}, \qquad \sqrt{2} = \tfrac{17}{12}\left(1 - \tfrac{1}{289}\right)^{1/2}.$$

[3]

(b) Find $c_0, c_1, \ldots, c_4$. Show that $\dfrac{c_{k+1}}{c_k} = \dfrac{2k-1}{2k+2}$ for $k \geq 1$, and deduce that every coefficient after c_0 is negative and that $|c_{k+1}| < |c_k|$ for $k \geq 1$. [4]

(c) Let $0 < x < 1$ and $m \geq 1$, and let E_m be the amount by which $c_0 + c_1 x + \cdots + c_m x^m$ exceeds $(1-x)^{1/2}$. Show that

$$0 < E_m < \frac{|c_{m+1}|\, x^{m+1}}{1-x}.$$

(Hint: compare the omitted terms with a geometric series) [4]

(d) For each expression in part (a), use the series up to the x^3 term to approximate $\sqrt{2}$. Give the actual error in each approximation and the bound from part (c), each to two significant figures. [4]

(e) Using $\sqrt{2} = \tfrac{17}{12}\left(1 - \tfrac{1}{289}\right)^{1/2}$, find the least m for which the bound in part (c) guarantees an error in $\sqrt{2}$ below 10^{-12}. [3]

(f) The fraction $\tfrac{17}{12}$ satisfies $p^2 - 2q^2 = 1$. Show that whenever positive integers p and q satisfy $p^2 - 2q^2 = 1$, $\sqrt{2} = \dfrac{p}{q}\left(1 - \dfrac{1}{p^2}\right)^{1/2}$. Find the next such fraction after $\tfrac{17}{12}$, and estimate how many correct decimal places each extra term of the series gives for it, compared with $x = \tfrac{1}{2}$ and $x = \tfrac{1}{289}$. Suggest a direction in which this could be investigated further. [3]