NUMBER & ALGEBRA

Chapter 1

Exponents & Logarithms

1

SYLLABUS 1.1 1.5 1.7 2.9

Japan lies in one of the world's most active seismic zones. Suppose a seismologist (a scientist who studies earthquakes) asks you to plot two recordings on one axis. The first, a tremor (a small earthquake), has an amplitude (the height of the wave) of 10^4. The second, a quake, is 10^9, a hundred thousand times larger.

An ordinary scale, with evenly spaced marks, cannot show both: if the scale is long enough for the second, the first lies so close to zero that you cannot see it.

So let equal spaces along the axis mean multiplication by 10 rather than addition of 1. The scale then reads $10^0, 10^1, 10^2, \ldots$, and the two recordings, with exponents 4 and 9, sit five equal steps apart. That exponent is the **logarithm** of the amplitude, and it is the central idea of this chapter.

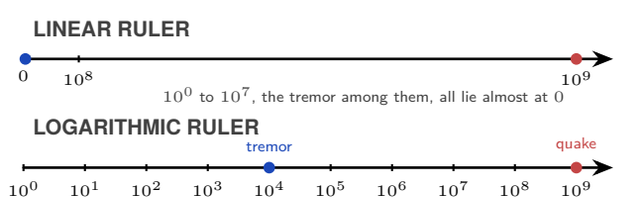


Figure 1.1 On a logarithmic scale each power of ten is one equal step.

Earthquake magnitude works this way, and so do pH, decibels and the brightness of stars. Writing a quantity as a power of ten turns multiplication into addition, so a range too wide to plot fits on an axis you can read. Exponents also model change, such as money in a bank or a decaying medicine, and logarithms find when such a model reaches a given value. By the end of this chapter you will be able to work with standard form, exponents and surds; move between exponential and logarithmic form in base 10 and base e; use the laws of logarithms; and solve exponential equations, including growth and decay models.

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1.1 What an Exponent Counts

An exponent, also called a *power* or an *index*, is shorthand for repeated multiplication. In a^n, a is the **base** and n is the **exponent**. The word *power* is also used for the whole expression a^n: 8 is a power of 2. When n is a positive integer,

$$a^n = \underbrace{a \cdot a \cdots a}_{n \text{ factors}}.$$

For example, $a^3 = a \cdot a \cdot a$.

When you multiply powers with the same base, a common mistake is to multiply the exponents. Writing out the factors shows what actually happens:

$$a^3 \cdot a^2 = (a \cdot a \cdot a)(a \cdot a) = a^5.$$

There are five factors of a, not six. Since the exponent counts how many times the base appears as a factor, the exponents are **added**:

$$a^m \cdot a^n = a^{m+n}.$$

Division cancels common factors, so the exponents are **subtracted**:

$$\frac{a^5}{a^2} = \frac{a \cdot a \cdot a \cdot a \cdot a}{a \cdot a} = a^3 = a^{5-2}.$$

Raising a power to another power repeats the whole power, so the exponents are **multiplied**:

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^6 = a^{2\times 3}.$$

A power outside a bracket applies to every factor inside:

$$(ab)^n = a^n b^n, \qquad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

Writing out the factors shows why: $(ab)^3 = ab \cdot ab \cdot ab = (a \cdot a \cdot a)(b \cdot b \cdot b) = a^3 b^3$, because the factors of a product can be multiplied in any order. A quotient works in the same way, with the factors of a in the numerator and the factors of b in the denominator.

The same laws extend naturally to zero and negative exponents. For $a \neq 0$, the subtraction rule gives $a^3/a^3 = a^{3-3} = a^0$; since any non-zero number divided by itself is 1, this forces $a^0 = 1$. The subtraction rule also gives $a^{-n} = 1/a^n$, because $a^0/a^n = a^{0-n} = a^{-n}$ and a^0/a^n is $1/a^n$. These values are **not chosen at random**. They are the only values that keep the laws consistent. The main laws are collected below.

KEY 1.1 **Laws of exponents**

For $a, b \neq 0$ and $m, n \in \mathbb{Z}$:

$a^m \cdot a^n = a^{m+n}$	product of powers: add the exponents
$\dfrac{a^m}{a^n} = a^{m-n}$	quotient of powers: subtract the exponents
$(a^m)^n = a^{mn}$	power of a power: multiply the exponents
$(ab)^n = a^n b^n$	power of a product: raise each factor
$\left(\dfrac{a}{b}\right)^n = \dfrac{a^n}{b^n}$	power of a quotient: raise top and bottom
$a^0 = 1$	zero exponent: gives 1
$a^{-n} = \dfrac{1}{a^n}$	negative exponent: take the reciprocal

The condition $m, n \in \mathbb{Z}$ in that card is read "m and n are integers".

CAUTION Do not confuse $(a^m)^n$ with a^{m^n}. In $(a^m)^n$, the power of a power, you multiply the exponents. In a^{m^n} you start at the top: first work out m^n, then raise a to that result. The two are usually not equal: $(2^3)^2 = 2^6 = 64$, but $2^{3^2} = 2^9 = 512$.

CAUTION A negative exponent means *reciprocal*, not *negative value*. $2^{-3} = \frac{1}{8}$, not -8.

EXAMPLE 1.1

Simplify $\dfrac{10c^7}{5c^3}$.

SOLUTION

Split the **coefficient** (the number that multiplies the variable) and the variable and handle them separately:

$$\frac{10c^7}{5c^3} = \frac{10}{5} \cdot c^{7-3} = \boxed{2c^4}$$

Apply one rule at a time.

EXAMPLE 1.2

Simplify $(2x^2y^{-4})^5$, giving your answer with positive exponents.

SOLUTION

The expression has two features: a bracket raised to a power, and a negative exponent. Raise each factor to the 5th power: the 2, x^2 and y^{-4}. Then use $y^{-20} = \dfrac{1}{y^{20}}$ to make the exponent positive.

$$(2x^2y^{-4})^5 = 32x^{10}y^{-20} = \boxed{\frac{32x^{10}}{y^{20}}}$$

CAUTION The bracket in $(2x^2)^3$ applies to everything inside it, including the 2. $(2x^2)^3 = 8x^6$, not $2x^6$.

1.1.1 Surds and fractional exponents

What is $9^{1/2}$? Since $\frac{1}{2}$ is not an integer, you cannot find it by repeated multiplication. But the power-of-a-power law gives

$$\left(9^{1/2}\right)^2 = 9^{(1/2)\times 2} = 9,$$

so $9^{1/2}$ is a number whose square is 9. Both 3 and -3 qualify, and $9^{1/2}$ is defined as the non-negative one:

$$9^{1/2} = \sqrt{9} = 3.$$

Notice the value is 3, not ± 3: the symbol $\sqrt{9}$ means the **principal** (non-negative) square root.

TIP In the real numbers, $(-4)^{1/2}$ is undefined: no real number squares to -4. In general, the **radicand** (the number inside the root) must be non-negative for an even root to have a real value. Square roots of negative numbers are complex numbers, which are introduced in Chapter 17.

KEY 1.2 Fractional exponents

For $a > 0$, $n \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$:

$a^{1/n} = \sqrt[n]{a}$	unit fraction: the denominator is the root
$a^{m/n} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$	in general: take the root and the power, in either order

Here $\mathbb{Z}^+$ is the set of positive integers, $1, 2, 3, \ldots$

A useful way to remember this:

denominator n $\longrightarrow$ root, numerator m $\longrightarrow$ power

These are not new laws; they are the meanings chosen so that the familiar exponent laws keep holding. The second line of the card is the power-of-a-power law used twice. Since $\frac{m}{n} = \frac{1}{n} \times m = m \times \frac{1}{n}$, it gives $a^{m/n} = \left(a^{1/n}\right)^m = \left(a^m\right)^{1/n}$.

Using fractional exponents

For $a^{m/n}$, it is usually easier to take the root first, then apply the power:

$$64^{2/3} = \left(\sqrt[3]{64}\right)^2 = 4^2 = 16.$$

Either order gives the same result, $64^{2/3} = \sqrt[3]{64^2} = 16$, but taking the root first keeps the numbers small.

The same laws can be used in reverse. To solve an equation in which the unknown has a fractional exponent:

1. raise both sides to the reciprocal of that exponent, since the two exponents multiply to 1, and by the power-of-a-power law the unknown is then on its own;
2. check the restriction on the unknown before counting solutions.

The definition above covers positive bases only, so step 1 is valid when the question restricts the unknown to positive values. Some questions allow negative values and read $x^{2/5}$ as $\left(\sqrt[5]{x}\right)^2$. Then the even power removes the sign: $x^{2/5} = 4$ gives $\sqrt[5]{x} = \pm 2$, so $x = \pm 32$. Raising both sides to $\frac{5}{2}$ gives only 32.

EXAMPLE 1.3

Solve $x^{3/4} = 125$ for $x > 0$.

SOLUTION

The exponent $\frac{3}{4}$ is undone by $\frac{4}{3}$, since $\frac{3}{4} \times \frac{4}{3} = 1$. Raise both sides to that power:

$$\left(x^{3/4}\right)^{4/3} = 125^{4/3} \implies x^1 = \left(\sqrt[3]{125}\right)^4 = 5^4 = \boxed{625}$$

Check by substituting: $625^{3/4} = \left(\sqrt[4]{625}\right)^3 = 5^3 = 125$. The condition $x > 0$ is needed for the fractional exponent to have a meaning, since $x^{3/4}$ means $\left(\sqrt[4]{x}\right)^3$ and the fourth root needs a non-negative input.

Working with surds

A real number is **irrational** when it cannot be written as a ratio of two integers; its decimal expansion neither ends nor repeats. A **surd** is an irrational root left in exact form, such as $\sqrt{2}$ or $\sqrt[3]{5}$, rather than rounded to a decimal. On a non-calculator paper, surds should be kept exact and simplified using the root laws

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b}, \qquad \sqrt{\tfrac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

valid for $a, b \geq 0$ (with $b > 0$ in the quotient). To see why the first holds, square $\sqrt{a}\sqrt{b}$: the result is $(\sqrt{a})^2 (\sqrt{b})^2 = ab$. Since $\sqrt{a}\sqrt{b}$ is not negative, it is the principal square root of ab. The quotient law follows in the same way. Two techniques are used again and again:

1. **Simplify** a surd by extracting its largest perfect-square factor.
2. **Combine** only like surds, the terms that have the same value under the root.

Two surds that look unrelated often become like surds once each has been simplified, so always simplify before trying to combine.

CAUTION The root laws hold for products and quotients only. In general $\sqrt{a} + \sqrt{b}$ is not $\sqrt{a+b}$, and $\sqrt{a} - \sqrt{b}$ is not $\sqrt{a-b}$, so surds can be combined only when they are like surds.

EXAMPLE 1.4

Simplify $\sqrt{147} - \sqrt{48}$.

SOLUTION

Take the largest perfect-square factor out of each root:

$$\sqrt{147} = \sqrt{49 \cdot 3} = 7\sqrt{3}, \qquad \sqrt{48} = \sqrt{16 \cdot 3} = 4\sqrt{3}.$$

Both terms now carry $\sqrt{3}$, so they are like surds and only their coefficients subtract:

$$7\sqrt{3} - 4\sqrt{3} = \boxed{3\sqrt{3}}$$

Note that $\sqrt{147} - \sqrt{48}$ is not $\sqrt{99}$: one is $3\sqrt{3} \approx 5.20$, the other about 9.95.

1.1.2 Rationalising the denominator

An exact answer is usually written without a surd in the denominator; removing it is called **rationalising the denominator**.

- For a single surd, multiply top and bottom by that surd.
- For a sum or difference, multiply by its **conjugate**, formed by changing the sign between the terms.

Neither step changes the value of the fraction: multiplying the top and the bottom by the same non-zero expression is multiplying by 1. Multiplying by the conjugate works because $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$ is a difference of two squares, so the new denominator has no surd.

EXAMPLE 1.5

Rationalise the denominators of $\dfrac{1}{\sqrt{3}}$ and $\dfrac{6}{2+\sqrt{3}}$.

SOLUTION

For the first, multiply by $\frac{\sqrt{3}}{\sqrt{3}}$:

$$\frac{1}{\sqrt{3}} = \frac{1 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}} = \boxed{\frac{\sqrt{3}}{3}}$$

For the second, multiply by the conjugate $2 - \sqrt{3}$:

$$\begin{aligned} \frac{6}{2+\sqrt{3}} &= \frac{6(2-\sqrt{3})}{(2+\sqrt{3})(2-\sqrt{3})} \\ &= \frac{6(2-\sqrt{3})}{4-3} \\ &= \boxed{12-6\sqrt{3}} \end{aligned}$$

1.1.3 Exponents as continuous growth

There is another way to read a non-integer exponent. Suppose a quantity doubles once in every unit of time, so that 2^t is its growth factor after time t. Then 2^3 is three complete doublings, while

$$2^{1/2} = \sqrt{2}$$

is the growth factor halfway through one doubling period. Because the growth is exponential, "halfway" refers to the time, not to half the final increase: $\sqrt{2} \approx 1.41$, less than the 1.5 a straight-line halfway would give. This reading extends to every real value of t, whether negative, fractional, or irrational:

$$2^{-3}, \qquad 2^{0}, \qquad 2^{1/2}, \qquad 2^{\sqrt{2}}.$$

Read as growth over time t, each has a plain meaning. $2^0 = 1$ is no growth at all: at time zero the quantity is unchanged. $2^{-3} = \frac{1}{8}$ describes the past: three periods ago, the quantity was an eighth of its present size. $2^{1/2} = \sqrt{2}$ is the factor after half a period, and $2^{\sqrt{2}}$ the factor after an irrational length of time. For any positive base a, the expression a^x has a meaning for every real exponent x, which is where the exponential function of Chapter 5 begins.

1.1.4 The number *e*

Imagine a bacterial culture $(\text{bacteria grown in a laboratory})$ whose mass grows continuously. In this model, its growth rate is proportional to its current mass: a larger culture grows faster. Choose the time unit so that a culture of mass 1 gram is growing at 1 gram per time unit.

Start with 1 gram. How much will there be after one time unit?

A first estimate keeps the growth rate fixed at its starting value. The culture gains 1 gram, giving **2 grams**. But this underestimates the growth: as the culture grows, its growth rate increases.

You can improve the estimate by updating the rate halfway through.

- During the first half, the estimated gain is $1 \times \frac{1}{2} = 0.5$ grams.
- The mass is now 1.5 grams, so during the second half the estimated gain is $1.5 \times \frac{1}{2} = 0.75$ grams.

The new estimate is 2.25 grams, or

$$\left(1 + \frac{1}{2}\right)^2.$$

With n equal steps, each step adds $\frac{1}{n}$ of the current mass. Each step therefore multiplies the mass by $1 + \frac{1}{n}$, giving

$$\text{estimated final mass} = \left(1 + \frac{1}{n}\right)^n.$$

number of steps, n	estimated final mass (grams)
1	2.00000
2	2.25000
10	2.59374
100	2.70481
1000	2.71692

As the steps become shorter, the estimates approach a fixed number:

$$e = 2.71828\ldots$$

This is the mass predicted by the continuous-growth model after one time unit. Like π, the number e is irrational; the proof of that is beyond this course.

More generally, continuous growth at a rate proportional to the amount present is modelled by

$$A(t) = A_0 e^{kt},$$

where A_0 is the initial amount and k is the continuous growth rate: positive for growth, negative for decay. The base is e because the function e^x has a rate of change equal to its own value, a property Chapter 13 shows from its graph.

> EXT Formally, $e = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^n$. Letting the number of steps grow without bound is what "continuously" means. To see why a rate k gives the factor e^{kt}, take $k > 0$ and split the time t

into n equal steps, each of which multiplies the amount by $1+\frac{kt}{n}$. Writing $n=mkt$ gives the total factor as $\left(1+\frac{kt}{n}\right)^n=\left[\left(1+\frac{1}{m}\right)^m\right]^{kt}$, and the expression in the square bracket approaches e as the number of steps increases. The same result holds for a negative rate. Limits are defined properly in Ch. 13.

YOUR TURN 1.1 What an Exponent Counts

Laws of exponents

1. Simplify each of the following, giving your answers with positive exponents.

(a) $\dfrac{6c^9}{2c^4}$

(b) $(3x^2y^{-3})^4$

(c) $\dfrac{12a^5b^3}{4a^2b^{-1}}$

(d) $\dfrac{(2x^3)^2\cdot x^{-4}}{4x^2}$

2. Simplify each of the following.

(a) $\dfrac{4x^3-8x^2}{2x}$

(b) $\dfrac{16a^5b^2-12ab^4}{4ab^2}$

3. Show that $\dfrac{(a^2b^{-1})^3}{a^{-3}b^2}=\dfrac{a^9}{b^5}$.

4. Evaluate each pair of numbers without a calculator, and state whether the two numbers are equal.

(a) $(2^2)^3$ and 2^{2^3}

(b) $(3^2)^2$ and 3^{2^2}

Fractional exponents

5. Evaluate each of the following.

(a) $27^{2/3}$

(b) $81^{-3/4}$

(c) $\left(\dfrac{8}{27}\right)^{-2/3}$

(d) $32^{-2/5}$

6. For $x>0$, write each of the following in the form x^n.

(a) $\sqrt[4]{x^3}$

(b) $\dfrac{1}{\sqrt[3]{x^2}}$

(c) $\dfrac{\sqrt{x^5}}{x^2}$

(d) $\dfrac{x^2\cdot\sqrt{x}}{x^{3/2}}$

Equations with a fractional exponent

7. Solve each equation for $x>0$.

(a) $x^{3/2}=64$

(b) $x^{2/3}=9$

(c) $x^{-1/2}=\frac{1}{4}$

(d) $x^{5/2}=32$

8. In this question x may be any real number, and $x^{m/n}$ means $\left(\sqrt[n]{x}\right)^m$ with n odd. Solve each equation, giving every real solution.

(a) $x^{2/5} = 4$ (b) $x^{4/3} = 81$

Surds

9. Simplify each of the following.

(a) $\sqrt{50} - \sqrt{8}$ (b) $\sqrt{12} + \sqrt{27}$

(c) $\sqrt{18} + \sqrt{32} - \sqrt{2}$ (d) $\sqrt{75} - 2\sqrt{12}$

Rationalising the denominator

10. Rationalise the denominator of each expression.

(a) $\dfrac{4}{3-\sqrt{5}}$, in the form $a + b\sqrt{5}$ (b) $\dfrac{6}{\sqrt{3}}$

(c) $\dfrac{1}{\sqrt{7}+\sqrt{5}}$ (d) $\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$

11. Rationalise $\dfrac{4+\sqrt{15}}{4-\sqrt{15}}$, giving your answer in the form $a + b\sqrt{15}$.

Concept check

12. A student writes $\sqrt{9+16} = \sqrt{9} + \sqrt{16} = 7$.

(a) Identify the error, and find the correct value.

(b) State the law for square roots that does hold, and use it to simplify $\sqrt{9 \times 16}$.

1.2 Standard Form

The mass of a proton, written out in full, is

$$0.00000000000000000000000000167 \text{ kg},$$

that is, 26 zeros after the decimal point before the first significant figure. The mass of the Sun is at the opposite extreme,

$$1\,990\,000\,000\,000\,000\,000\,000\,000\,000 \text{ kg},$$

a 1 followed by 30 digits. Both numbers are difficult to write, and even harder to compare or calculate with.

There is a shorter way to write both. Instead of all the digits, record *the power of 10* that places the decimal point where it belongs. Mathematicians call the result **standard form** (also known as scientific notation).

DEFINITION 1.1 **Standard form**

A number is in **standard form** when written as $a \times 10^k$, where $1 \leq a < 10$ and $k \in \mathbb{Z}$.

The proton becomes 1.67×10^{-27} kg. The Sun becomes 1.99×10^{30} kg. The exponent k shows at once whether a number is very small or very large.

1.2.1 Calculating with standard form

Multiplication and division follow straight from the exponent laws:

$$(a \times 10^m)(b \times 10^n) = (a \times b) \times 10^{m+n}, \qquad \frac{a \times 10^m}{b \times 10^n} = \frac{a}{b} \times 10^{m-n}.$$

The table sets out the method for each operation.

Operation	Method
$\times$	multiply the coefficients, add the powers of 10
$\div$	divide the coefficients, subtract the powers of 10
$+$, $-$	rewrite so both share one power of 10, then add or subtract the coefficients

Afterwards, check that the coefficient is still between 1 and 10. If it falls outside the range $1 \leq a < 10$, shift the decimal point and adjust the power. For example, $(5 \times 10^3)(4 \times 10^2) = 20 \times 10^5 = 2 \times 10^6$: the coefficient 20 becomes 2, and the exponent rises by one to account for the extra factor of 10.

Addition and subtraction need one extra step first. Just as you cannot add metres and kilometres without converting, you cannot add two numbers in standard form until they share the same power of 10. Rewrite the one with the smaller exponent so the powers match, then add the coefficients.

EXAMPLE 1.6

Calculate $(3.5 \times 10^4) \times (2 \times 10^{-7})$.

SOLUTION

Group and multiply:

$$(3.5 \times 2) \times 10^{4+(-7)} = \boxed{7 \times 10^{-3}}$$

EXAMPLE 1.7

Calculate $(4.8 \times 10^5) + (3.2 \times 10^4)$.

SOLUTION

Rewrite 3.2×10^4 as 0.32×10^5 so both terms share the power 10^5:

$$4.8 \times 10^5 + 0.32 \times 10^5 = \boxed{5.12 \times 10^5}$$

CAUTION In addition and subtraction, do **not** add the exponents. Only multiplication and division combine powers of 10.

CAUTION Calculator display notation is not acceptable in written answers. If your graphic display calculator (GDC) shows 5.2E30, you must write 5.2×10^{30}.

YOUR TURN 1.2 Standard Form

Converting between forms

13. Write 0.000047 in standard form.

14. Write each number in ordinary form.

(a) 3.2×10^5 (b) 8.06×10^{-4}

(c) 1.5×10^7 (d) 9×10^{-1}

Calculating in standard form

15. Calculate $(2.4 \times 10^3) \times (3 \times 10^{-5})$, giving your answer in standard form.

16. Calculate $\dfrac{6.6 \times 10^8}{2.2 \times 10^3}$, giving your answer in standard form.

17. Calculate each of the following, giving your answer in standard form.

(a) $(5.1 \times 10^4) + (3.9 \times 10^3)$ (b) $(7.2 \times 10^{-3}) - (5 \times 10^{-4})$

(c) $(2.5 \times 10^6) + (7.5 \times 10^6)$ (d) $(9.4 \times 10^{-2}) + (6 \times 10^{-2})$

Using a calculator

18. A GDC shows each result below in its display notation. Write each one in standard form, as it must appear in a written answer.

(a) 6.02E23 (b) 4.5E-8

19. GDC The mass of an electron is 9.11×10^{-31} kg. The mass of a proton is 1.67×10^{-27} kg. Find how many times heavier a proton is than an electron. Give your answer in standard form to 3 significant figures.

Concept check

20. A student writes $(3 \times 10^4) + (2 \times 10^3) = 5 \times 10^7$. Identify the error, and give the correct answer in standard form.

1.3 The Logarithm: an Exponent Reversed

Many familiar operations come in inverse pairs: addition with subtraction, multiplication with division. So what is the inverse of exponentiation (raising to a power)? Start with a question: 2 to what power gives 8? Since $2^3 = 8$, the answer is 3, and that answer is written $\log_2 8 = 3$. In general, if $a^x = b$ and you know a and b, the logarithm finds x. It is not a new idea: it is the exponential equation $a^x = b$ **rearranged to make the exponent x the subject**.

FORMULA 1.3 **Logarithm** FORMULA BOOKLET · 1.5

For $a > 0$, $a \neq 1$, and $b > 0$:

$$a^x = b \iff x = \log_a b$$

The **common logarithm** ($\log x$, also written $\log_{10} x$) uses base 10. The **natural logarithm** ($\ln x$) uses base $e \approx 2.718$, the continuous-growth constant of §1.1.4.

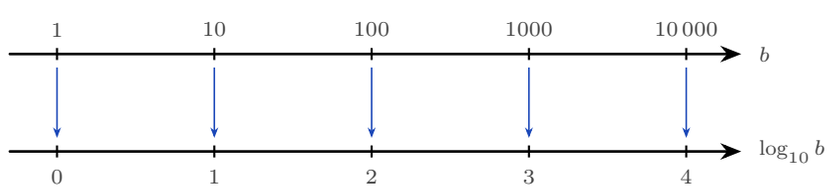


Figure 1.2 Equal steps along the upper scale multiply by 10. The logarithm counts the steps: each value of b sits directly above its exponent, so $\log_{10} 1000 = 3$.

The condition $b > 0$ in that definition is not a minor detail. A positive base raised to any real power gives a positive result: $10^2 = 100$, $10^0 = 1$, $10^{-3} = 0.001$. No exponent turns 10 into -5, and no exponent turns 10 into 0. So $\log(-5)$ and $\log 0$ do not exist. The expression inside a logarithm is called its **argument**. The argument must be positive: for every base, $\log_a x$ **is defined only when $x > 0$.** The value of the logarithm is an exponent, and it can be any real number, including a negative one: $\log 0.001 = -3$, because $10^{-3} = 0.001$. So the argument must be positive, but the value need not be.

The base has conditions of its own. It must be positive, because a negative base does not give a real power for every real exponent: $(-8)^{1/2}$ is not real. Base 0 gives only 0 or undefined powers. The base also cannot be 1, because every power of 1 is 1.

The word *logarithm* comes from the Greek *logos* (ratio) and *arithmos* (number).

The graph of $y = \log_a x$ is the mirror image of $y = a^x$ across the line $y = x$, with a vertical asymptote at $x = 0$. These graphs and their transformations are developed in Chapter 5 (Functions).

Because the exponential and logarithm are inverses, each undoes what the other does:

FORMULA 1.4 **Logarithm and exponential undo each other** FORMULA BOOKLET · 1.7

$$\log_a a^x = x = a^{\log_a x}$$

Here $a > 0$ and $a \neq 1$. The first equality holds for every real x; the second needs $x > 0$.

For base 10 and base e, the two bases used most often, this gives four results:

- $\log(10^x) = x$, true for every real x.
- $10^{\log x} = x$, the same pair reversed, but only for $x > 0$, since $\log x$ is undefined otherwise.
- $\ln(e^x) = x$, true for every real x.
- $e^{\ln x} = x$, reversed, and again only for $x > 0$.

Two special values follow at once from the definition, and both are used often:

KEY 1.5 **Two special logarithms**

For any base $a > 0$, $a \neq 1$:

$\log_a a = 1$ since $a^1 = a$

$\log_a 1 = 0$ since $a^0 = 1$

1.3.1 Working with logarithms

To evaluate a logarithm by hand, reverse the exponent question: $\log_a b$ is defined as the exponent that turns a into b. Read $\log_a b$ as **"a to what power gives b?"** and in many cases the answer is immediate. To solve an equation such as $\log_a X = c$, rewrite it in exponential form, $X = a^c$.

EXAMPLE 1.8

Find $\log_3 81$.

SOLUTION

Ask the inverse question: 3 to what power gives 81? You know $3^4 = 81$, so

$\log_3 81 = 4$

EXAMPLE 1.9

Solve $\log_{10}(y+1) = 3$.

SOLUTION

The equation says: the exponent you need to get $(y+1)$ from base 10 is 3. So $y+1 = 10^3 = 1000$, giving $\boxed{y = 999}$

EXAMPLE 1.10

Without a calculator, find $e^{3\ln 2}$.

SOLUTION

Think of $e^{\ln 2}$ first: $e^{\ln 2} = 2$, because e^x and $\ln x$ undo each other. So $e^{3\ln 2} = (e^{\ln 2})^3 = 2^3 = \boxed{8}$

CAUTION Solving an equation with a logarithm in it can produce a value that makes an argument negative or zero. Such a value is an **extraneous solution**: it satisfies a later line of the working but not the original equation, so it is rejected, however correct the algebra was. Substitute every answer back into the original equation. Keep only those that leave every argument positive and every base positive and not equal to 1.

YOUR TURN 1.3 The Logarithm: an Exponent Reversed

Switching between forms

21. Rewrite each statement in the other form.

(a) Write $3^4 = 81$ in logarithmic form.

(b) Write $\log_5 x = 3$ in exponential form, and hence find x.

Evaluating logarithms

22. Evaluate each of the following without a calculator.

(a) $\log_2 32$

(b) $\log_3 \frac{1}{9}$

(c) $e^{2\ln 3}$

(d) $\ln e^5$

23. Evaluate each of the following without a calculator.

(a) $\log 10\,000$

(b) $\log 0.001$

(c) $\ln \frac{1}{\sqrt{e}}$

(d) $10^{\log 5+1}$

24. State whether each expression is defined. If it is, write down its value.

(a) $\log_5 125$ (b) $\log_5 1$
(c) $\log_5 0$ (d) $\log_5(-25)$
(e) $\log_1 7$ (f) $\log_{1/2} 8$

Logarithm and exponential as inverses

25. Each statement is either true for every value allowed by the definition, or not. State which, and correct those that are not.

(a) $\log_a(a^x) = x$ for every real x (b) $a^{\log_a x} = x$ for every real x
(c) $\log_a 1 = 0$ for every valid base a (d) $\log_a 0 = 0$ for every valid base a

Solving logarithmic equations

26. Solve each equation. Reject any value the definition does not allow.

(a) $\log_4(x+1) = 2$ (b) $\log_3(2x-1) = 4$
(c) $\log_2(x^2 - 3x) = 2$

Concept check

27. A student says: "$\log_{10} x = -2$ *has no solution, because a logarithm cannot be negative.*" Explain the error, and solve the equation.

1.4 The Laws of Logarithms

A single logarithm is just a number. What makes logarithms useful is that they turn harder arithmetic into easier arithmetic: multiplication becomes addition, division becomes subtraction, and a power becomes a coefficient. The last one matters most in this chapter: it turns an exponent into a coefficient, so an unknown in an exponent can be solved for.

The first three laws below are the three exponent laws restated for logarithms.

FORMULA 1.6 **Laws of logarithms** FORMULA BOOKLET · 1.7

$\log_a(xy) = \log_a x + \log_a y$	**product:** a product becomes a sum
$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$	**quotient:** a quotient becomes a difference
$\log_a(x^m) = m \log_a x$	**power:** an exponent becomes a coefficient
$\log_a x = \frac{\log_b x}{\log_b a}$	**change of base:** rewrite in any base you can compute

Valid for $a, b, x, y > 0$ with $a, b \neq 1$.

All three proofs start in the same way: write $x = a^p$ and $y = a^q$, so that $p = \log_a x$ and $q = \log_a y$ by definition. Each law is then a one-line consequence of the matching exponent law:

- **Product.** $xy = a^p \cdot a^q = a^{p+q}$, so $\log_a(xy) = p + q = \log_a x + \log_a y$.
- **Quotient.** $\dfrac{x}{y} = \dfrac{a^p}{a^q} = a^{p-q}$, so $\log_a\left(\dfrac{x}{y}\right) = p - q = \log_a x - \log_a y$.
- **Power.** $x^m = (a^p)^m = a^{mp}$, so $\log_a(x^m) = mp = m \log_a x$.

The third law is the one used to solve exponential equations. When the unknown is in an exponent, as in b^x, the power law gives $\log_a(b^x) = x \log_a b$, which moves the unknown into a coefficient.

1.4.1 Change of base

Your calculator finds logarithms in base 10 $(\log)$ and base e $(\ln)$. If it cannot find a logarithm in another base directly, use change of base. The law rewrites any logarithm as a ratio of two logarithms you can compute. Base 10 and base e both work. With base e, the law becomes $\log_a x = \dfrac{\ln x}{\ln a}$.

To prove it, let $\log_a x = y$, so $x = a^y$. Take $\ln$ of both sides: $\ln x = y \ln a$. Solve for y.

EXAMPLE 1.11

Evaluate $\log_3 5$ using logarithms your calculator computes.

SOLUTION

Convert to natural logarithms. (If your calculator has a logarithm to any base, it gives the same value directly.)

$$\log_3 5 = \frac{\ln 5}{\ln 3} \approx \frac{1.609}{1.099} \approx \boxed{1.46}$$

One special case is worth remembering: $\log_a b = 1/\log_b a$. To see it, take the argument to be b and compute in base b; since $\log_b b = 1$, the numerator is 1:

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}.$$

This holds for $a, b > 0$ with $a, b \neq 1$. The formula booklet does not print it, so quote it with the change-of-base step, or work from change of base directly.

When the unknown is in the base, swap the base and the argument with $\log_a b = 1/\log_b a$. The unknown is then in an argument, where the ordinary laws apply. Often no swap is needed at all: a logarithm with an unknown base is still an exponent question read backwards.

EXAMPLE 1.12

Solve $\log_x 81 = 4$.

SOLUTION

The statement says that x raised to the power 4 gives 81, so

$$x^4 = 81 \implies x = 3 \text{ or } x = -3.$$

A base must satisfy $x > 0$ and $x \neq 1$. That rules out $x = -3$, leaving $\boxed{x = 3}$
An unknown base is where those two conditions matter: the algebra gives two roots, and the conditions on a base reject one of them.

EXAMPLE 1.13

Solve $\log_2 x - 3\log_x 2 = 2$.

SOLUTION

The unknown is in the base of the second logarithm, so swap base and argument: $\log_x 2 = \dfrac{1}{\log_2 x}$. Write $t = \log_2 x$. The equation becomes

$$t - \frac{3}{t} = 2 \implies t^2 - 2t - 3 = 0 \implies (t-3)(t+1) = 0,$$

so $t = 3$ or $t = -1$, which gives $x = 2^3 = 8$ or $x = 2^{-1} = \frac{1}{2}$. Both are positive and neither is 1, so both are valid bases: $\boxed{x = 8 \text{ or } x = \frac{1}{2}}$
Check: $\log_2 8 - 3\log_8 2 = 3 - 1 = 2$ and $\log_2 \frac{1}{2} - 3\log_{1/2} 2 = -1 + 3 = 2$.

1.4.2 When one base is a power of another

Questions often mix bases that are powers of one another: $\log_2 x$ alongside $\log_4 x$, or $\log_2 x$ alongside $\log_8 x$. Change of base rewrites them all in one base, and for this particular case the result is short enough to remember.

Change $\log_{a^2} x$ to base a:

$$\log_{a^2} x = \frac{\log_a x}{\log_a a^2} = \frac{\log_a x}{2} = \tfrac{1}{2}\log_a x,$$

since $\log_a a^2 = 2\log_a a = 2$ by the power law. Squaring the base halves the logarithm. The same steps with a^n in place of a^2 give $\log_a a^n = n$, so for $a > 0$, $a \neq 1$, $x > 0$ and any $n \neq 0$,

$$\log_{a^n} x = \frac{1}{n}\log_a x.$$

This result is not in the formula booklet. Show the change-of-base step that gives it.

EXAMPLE 1.14

Solve $\log_2 x + \log_4 x = 6$.

SOLUTION

The bases are 2 and $4 = 2^2$, so rewrite the second one in base 2: $\log_4 x = \frac{1}{2}\log_2 x$. Writing $t = \log_2 x$, the equation becomes

$$t + \tfrac{1}{2}t = 6 \implies \tfrac{3}{2}t = 6 \implies t = 4,$$

so $x = 2^4$, giving $\boxed{x = 16}$; check: $\log_2 16 = 4$ and $\log_4 16 = 2$, and these sum to 6.

1.4.3 Applying the laws

These laws can be used in either direction: they split one logarithm into several, or gather several back into one. Both directions are needed, in simplifying expressions and in solving equations.

EXAMPLE 1.15

Given $p = \log_5 a$ and $q = \log_5 b$, express $\log_5\left(\dfrac{a^3}{b}\right)$ in terms of p and q.

SOLUTION

Apply the quotient law, then the power law:

$$\log_5\left(\frac{a^3}{b}\right) = \log_5 a^3 - \log_5 b = 3\log_5 a - \log_5 b = \boxed{3p - q}$$

EXAMPLE 1.16

Write $2\ln a + 6\ln b$ as a single logarithm.

SOLUTION

Use the power law in reverse first, then the product law:

$$2\ln a + 6\ln b = \ln a^2 + \ln b^6 = \boxed{\ln(a^2 b^6)}$$

EXAMPLE 1.17

Solve $\log_2 x + \log_2(x - 2) = 3$.

SOLUTION

Both logarithms need positive arguments, so $x > 2$ from the start. By the product law, the two logarithms on the left become one logarithm of a product:

$$\log_2[x(x-2)] = 3 \implies x(x-2) = 2^3 = 8.$$

$$x^2 - 2x - 8 = 0 \implies (x-4)(x+2) = 0.$$

So $x = 4$ or $x = -2$. But $x = -2$ fails the condition $x > 2$ (and $\log_2(-2)$ does not exist), so reject it. The answer is $\boxed{x = 4}$

Two logarithms with the same base are equal only when their arguments are equal. For $A, B > 0$,

$$\log_a A = \log_a B \implies A = B,$$

because raising a to the power of each side gives $a^{\log_a A} = a^{\log_a B}$, which is $A = B$. Use it after each side has been gathered into a single logarithm with the same base.

EXAMPLE 1.18

Solve $\log_3(2x+1) = \log_3 5 + \log_3(x-1)$.

SOLUTION

Gather the right side into one logarithm with the product law:

$$\log_3(2x+1) = \log_3(5(x-1)).$$

The bases are the same, so the arguments are equal:

$$2x + 1 = 5x - 5 \implies \boxed{x = 2}$$

Check: at $x = 2$ the arguments are 5, 5 and 1, all positive, and $\log_3 5 = \log_3 5 + \log_3 1$ is true because $\log_3 1 = 0$.

CAUTION In general, $\log(x+y) \neq \log x + \log y$. The product law is about $\log(xy)$, not a sum inside the log.

YOUR TURN 1.4 The Laws of Logarithms

Change of base

28. Use the change-of-base law in each of the following.

(a) Write $\log_4 9$ in terms of natural logarithms.

(b) Express $\log_8 x$ in terms of $\log_2 x$.

(c) Find the exact value of $\log_4 8$.

(d) Show that $\log_{1/2} x = -\log_2 x$.

(e) Show that $\log_a b \times \log_b(a^2) = 2$, for $a, b > 0$ and $a, b \neq 1$.

29. GDC Evaluate $\log_3 50$ to 3 significant figures.

An unknown base

30. Solve each equation for the base x. Reject any value a base does not allow.

(a) $\log_x 49 = 2$

(b) $\log_x 8 + \log_x 2 = 4$

(c) $\log_x(x+6) = 2$

(d) $\log_3 x - 2\log_x 3 = 1$

Bases that are powers of one another

31. Solve each equation by writing every logarithm in the same base.

(a) $\log_3 x - \log_9 x = 1$

(b) $\log_2 x + \log_8 x = 8$

(c) $\log_4 x = \log_2(x-2)$

Combining and expanding logarithms

32. Write each expression as a single logarithm.

(a) $\log_3 5 + \log_3 7$

(b) $\log_2 24 - \log_2 3$, and evaluate it

(c) $3\log_a x + \log_a y$

(d) $2\ln a - 3\ln b + \ln c$

33. Express each logarithm in terms of the quantities given.

(a) $\log_5\left(\frac{a^2}{b}\right)$, in terms of $\log_5 a$ and $\log_5 b$

(b) $\log_3 18$, given $p = \log_3 2$

(c) $\log_{10} 72$, given $\log_{10} 2 = p$ and $\log_{10} 3 = q$

(d) $\log_5 75$, given $\log_5 3 \approx 0.682$

34. Show that $\log_a(x^2-1) - \log_a(x+1) = \log_a(x-1)$, for $x > 1$.

Solving equations with the laws

35. Solve each equation, using a law of logarithms first. Reject any value the logarithms do not allow.

(a) $\log_3 x + \log_3(x+6) = 3$ (b) $\log_2(x+3) - \log_2(x-1) = 2$

(c) $2\log_5 x = \log_5 9 + \log_5 4$

Concept check

36. A student writes $\log_3(x+9) = \log_3 x + \log_3 9 = \log_3 x + 2$.

(a) Identify the law of logarithms the student has misremembered, and state it correctly.

(b) By taking $x = 18$, show that the two sides are not equal.

37. A student writes $\dfrac{\log_2 32}{\log_2 4} = \log_2 8 = 3$. Explain the error, and find the correct value.

1.5 Solving Exponential Equations

In an equation such as $5^x = 30$, adding, multiplying or factorising does not isolate x, because x is the exponent. Taking logarithms moves the exponent down into a coefficient, and the result is an equation you already know how to solve.

1.5.1 Same base first

Before anything else, look for a common base. If both sides can be written as powers of the same number, the exponents must be equal, provided the base is positive and not 1: $a^p = a^q$ forces $p = q$ for $a > 0,\ a \neq 1$. Taking $\log_a$ of both sides shows why, since $\log_a a^p = p$ and $\log_a a^q = q$. When it is available, this is usually quicker than a logarithm, and it suits non-calculator papers. Look for powers of 2 $(2, 4, 8, 16, 32)$, of 3 $(3, 9, 27, 81)$ and of 5 $(5, 25, 125)$, and for reciprocals such as $\frac{1}{3} = 3^{-1}$.

KEY 1.7 **Equal powers, equal logarithms**

For $a > 0$ and $a \neq 1$:

$$a^p = a^q \implies p = q, \qquad \log_a A = \log_a B \implies A = B \quad (A, B > 0).$$

EXAMPLE 1.19

Solve $\left(\frac{1}{3}\right)^x = 9^{x+1}$.

SOLUTION

Both sides are powers of 3: $\frac{1}{3} = 3^{-1}$ and $9 = 3^2$. Rewrite and equate exponents:

$$3^{-x} = 3^{2(x+1)} \implies -x = 2x + 2 \implies \boxed{x = -\tfrac{2}{3}}$$

1.5.2 The general strategy

When no common base exists, take a logarithm of both sides, use $\log(x^m) = m \log x$ to bring the exponent down, then solve what is now a linear or quadratic equation. The choice of which logarithm to use ($\log$ or $\ln$) does not matter: pick whichever gives simpler arithmetic.

EXAMPLE 1.20

Solve $5^x = 30$.

SOLUTION

Take $\ln$ of both sides:

$$x \ln 5 = \ln 30 \implies x = \frac{\ln 30}{\ln 5} \approx \boxed{2.11}$$

Or equivalently, $x = \log_5 30$.

EXAMPLE 1.21

Solve $7^{x-2} = 5^{x+3}$ exactly.

SOLUTION

With different bases on each side, take $\ln$ of both sides and use the power law:

$$(x-2)\ln 7 = (x+3)\ln 5.$$

Expand, then collect x on one side:

$$x \ln 7 - 2\ln 7 = x \ln 5 + 3 \ln 5,$$

$$x(\ln 7 - \ln 5) = 3\ln 5 + 2\ln 7 \implies x = \frac{\ln 5^3 + \ln 7^2}{\ln(7/5)} = \boxed{\frac{\ln 6\,125}{\ln(7/5)}}$$

TIP Follow the question's instruction. Give an exact answer, like $x = \dfrac{\ln 6\,125}{\ln(7/5)}$, whenever a question says *exact*, and round to the stated accuracy when a decimal is asked for. A non-calculator paper (Paper 1) usually calls for exact answers, but the wording of the question decides, not the paper. Keep full precision during the working, and round to **three significant**

figures unless told otherwise. Three significant figures is the IB default, and rounding too early or giving too few figures costs accuracy marks.

1.5.3 When the unknown appears twice

Taking logarithms does not help with $2^{2x} - 10 \cdot 2^x + 16 = 0$. The unknown is in two exponents, and no law of logarithms splits the logarithm of a sum. Two patterns are common: one power is the square of the other, or the two exponents differ by a constant.

When one power is the square of another, as $2^{2x} = (2^x)^2$, the equation is a quadratic in a^x. A **substitution** $u = a^x$ turns it into an ordinary quadratic. Solve the quadratic for u, then convert each root back. Reject any root with $u \leq 0$: for $a > 0$, $a^x > 0$ for every real x, so no exponent gives $u \leq 0$. Ch.4 calls this a 'quadratic in disguise', a quadratic in a new variable; see §4.4.5.

When the exponents differ by a constant, as in 3^{x+2} and 3^x, write the larger power as a number times the smaller one, $3^{x+2} = 3^2 \cdot 3^x$. Then take out the common factor 3^x and solve for it.

EXAMPLE 1.22

Solve $2^{2x} - 10 \cdot 2^x + 16 = 0$.

SOLUTION

The first term is the square of the second, since $2^{2x} = (2^x)^2$. Put $u = 2^x$:

$$u^2 - 10u + 16 = 0 \implies (u-2)(u-8) = 0,$$

so $u = 2$ or $u = 8$. Now convert each value back:

$$2^x = 2 \implies x = 1, \qquad 2^x = 8 \implies x = 3$$

Both values of u are positive, so both are valid: $\boxed{x = 1 \text{ or } x = 3}$

EXAMPLE 1.23

Solve $3^{x+2} - 3^x = 72$.

SOLUTION

Split the first exponent so that both terms carry 3^x: by the product-of-powers law $a^m a^n = a^{m+n}$, $3^{x+2} = 3^x \cdot 3^2 = 9 \cdot 3^x$. The equation becomes

$$9 \cdot 3^x - 3^x = 72 \implies 8 \cdot 3^x = 72 \implies 3^x = 9,$$

so $\boxed{x = 2}$, and the check $3^4 - 3^2 = 81 - 9 = 72$ confirms it. A constant added in the exponent is a constant factor outside it, and so the two terms can be gathered into one.

1.5.4 From words to an equation

Many growth and decay problems use one model,

$$A(t) = A_0 \times r^t,$$

where A_0 is the starting amount and r is the multiplier for each period. The wording gives you r. A rise of 6% per year means $r = 1.06$. A fall of 6% means $r = 0.94$. A quantity that doubles each period has $r = 2$.

When the change is continuous rather than in steps, the model uses base e:

$$A(t) = A_0\, e^{kt},$$

where k is the continuous growth rate met in §1.1.4: positive when the quantity grows, negative when it decays.

These are not two models but one, written twice. Since $r = e^{\ln r}$,

$$A_0\, r^t = A_0\, \left(e^{\ln r}\right)^t = A_0\, e^{(\ln r)\, t},$$

so the two forms agree exactly when

$$k = \ln r.$$

A quantity that doubles each period has $k = \ln 2 \approx 0.693$; one falling 6% a year has $k = \ln 0.94 \approx -0.0619$, the negative sign marking decay. Use whichever form the question gives: a percentage per period gives r, a rate described as *continuous* gives k.

	Periodic multiplier	**Continuous rate**
Model	$A(t) = A_0\, r^t$	$A(t) = A_0\, e^{kt}$
Meaning	the amount is multiplied by r once each period	the amount changes at every instant, at a rate k times the current amount
Wording	"rises by 6% per year": $r = 1.06$	"grows continuously at 6% per year": $k = 0.06$
Decay	$0 < r < 1$	$k < 0$
Link	$r = e^k$	$k = \ln r$

Decay is often described not by a rate but by a **half-life**: the time in which half of the quantity disappears. One half-life multiplies the amount by $\frac{1}{2}$, and a time t contains t/T half-lives, where T is the half-life itself. So

$$A(t) = A_0 \left(\tfrac{1}{2}\right)^{t/T}.$$

This is the same $A_0\, r^t$ model with $r = \frac{1}{2}$, and with time counted in half-lives instead of single periods. A **doubling time** T works in the same way: $A(t) = A_0 \cdot 2^{t/T}$.

EXAMPLE 1.24

Algae in a Finnish lake grow continuously at 12% per year, starting from 200 grams. When does the mass reach 500 grams?

SOLUTION

Continuous growth uses base e, so $A(t) = 200\, e^{0.12t}$. Set it equal to the target:

$$200\, e^{0.12t} = 500 \implies e^{0.12t} = 2.5 \implies t = \frac{\ln 2.5}{0.12} \approx \boxed{7.64 \text{ years}}$$

The 12% here is a continuous rate, so it sits in the exponent as 0.12 directly, with no conversion to a yearly factor.

EXAMPLE 1.25

A medical isotope has a half-life of 6 hours: every 6 hours, half of it decays. An 80 mg sample is prepared. How long until only 12 mg remains?

SOLUTION

Halving every 6 hours means $A(t) = 80\left(\frac{1}{2}\right)^{t/6}$, with t in hours. Set this equal to the target and take a logarithm:

$$80\left(\tfrac{1}{2}\right)^{t/6} = 12 \implies \left(\tfrac{1}{2}\right)^{t/6} = 0.15 \implies t = 6 \cdot \frac{\ln 0.15}{\ln 0.5} \approx \boxed{16.4 \text{ hours}}$$

The answer need not be a whole number of half-lives: 16.4 hours is a little over 2.7 of them, and the model gives a value at every time, not only at whole half-lives.

The graph of this model shows both facts: each half-life halves the amount, and the answer lies between whole half-lives.

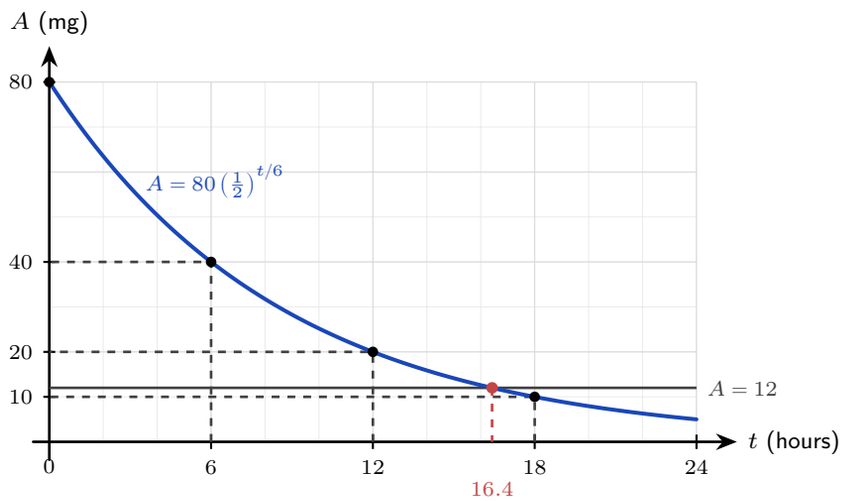


Figure 1.3 Each 6 hours halves the amount. The level 12 mg is reached at $t \approx 16.4$ hours, between the second and third half-lives.

The equations of this chapter can be sorted by their form, and the form tells you which method to use.

What the equation looks like	Method
Both sides are powers of one base	write with the common base, then equate exponents
One power on its own, no common base	take logarithms of both sides
Related powers, such as 2^{2x} and 2^{x}	substitute, then solve the quadratic
Two powers whose exponents differ by a constant, such as 3^{x+2} and 3^{x}	write the larger as a number times the smaller, then take out the common factor
Logarithms of the unknown	state the domain first, then combine, solve, and check against the domain
The unknown in the base of a logarithm	rewrite in exponential form, or swap base and argument; keep only bases > 0 and $\neq 1$

YOUR TURN 1.5 Solving Exponential Equations

Same base

38. Solve each equation, giving your answer as an exact value.

(a) $5^{2x} = 125$ (b) $4^x = 8^{x-1}$

(c) $9^{x+1} = 27^x$ (d) $2^x \cdot 4^{x+1} = 8$

Taking logarithms

39. GDC Solve each equation, giving your answer to 3 significant figures.

(a) $2^x = 10$ (b) $3^{x+1} = 20$

(c) $5^{2x} = 40$ (d) $7 \cdot 2^x = 200$

40. Solve each equation exactly. Give your answers to (a) and (b) in the form $\frac{\ln p}{\ln q}$.

(a) $3^{x+1} = 5^x$

(b) $2^{2x-1} = 7^{x+2}$

(c) $3 \cdot 2^x = 2 \cdot 3^x$

The unknown in two exponents

41. Solve each equation. Reject any value that no power can take. *(Hint: in (a), put $u = e^x$)*

(a) $e^{2x} - 5e^x + 6 = 0$

(b) $2^{2x} - 2 \cdot 2^x - 8 = 0$

(c) $3^{2x+1} - 8 \cdot 3^x - 3 = 0$

42. Solve each equation by taking out a common factor.

(a) $2^{x+3} - 2^x = 56$ (b) $5^{x+1} + 5^{x-1} = 130$

Growth and decay models

43. GDC The population of a town is $18\,000$ and rises by 4% each year.

(a) Write down a model for the population P after t years.

(b) Find the number of complete years after which the population first exceeds $30\,000$.

44. GDC A radioactive isotope has a half-life of 5.3 years. A sample has a mass of 200 g.

(a) Find the mass that remains after 12 years.

(b) Find how long it takes until only 15 g remains.

45. GDC A bacterial culture grows continuously. After t hours the number of bacteria is $N = 500\,e^{0.35t}$.

(a) Find the time taken for the number to reach 5000.

(b) Write the model in the form $N = 500\,r^{t}$, and hence find the percentage by which the culture grows each hour.

46. GDC Sea ice in a Finnish bay covers 1200 km^2 and shrinks by 6% each year. Find the number of years after which it has shrunk to 800 km^2.

Concept check

47. A student solves $3 \cdot 2^x = 48$ by writing $6^x = 48$, so $x = \dfrac{\ln 48}{\ln 6} \approx 2.16$. Identify the error, and solve the equation correctly.

48. A student solves $e^{2x} + e^x = 12$ by taking $\ln$ of each term, to get $2x + x = \ln 12$. Explain why this step is not valid, and solve the equation correctly.

Chapter 1 · Exponents & Logarithms

Reflections

TOK John Napier spent about twenty years computing logarithms by hand, to spare astronomers and navigators their long multiplications. He published them in 1614. Henry Briggs travelled to meet him the following year, and the two agreed on base 10, which then served scientists for over three hundred years. Does mathematics develop because we need it, or do we find uses for what mathematicians have already built?

TOK The exponent laws force $a^0 = 1$ for every $a \neq 0$. But what is 0^0? As x gets closer and closer to 0 from above (written $x \to 0^+$, see Ch. 13), $x^0 \to 1$ along one route and $0^x \to 0$ along another. These are limits of two different expressions, and neither is a value of 0^0 itself. So analysis $(\text{the branch of mathematics that studies limits})$ often leaves 0^0 undefined, while combinatorics $(\text{the mathematics of counting})$ sets $0^0 = 1$ because its formulas need it. Is leaving 0^0 undefined a limitation of mathematics, or a sign of its precision?

IM Base-10 logarithms feel natural because we count in tens, on ten fingers. The choice is cultural, not mathematical. The Babylonians counted in sixties, and their system survives in your clock: 60 seconds in a minute, 60 minutes in an hour. Notation is cultural too: the word *algebra* comes from *al-jabr*, a method in a ninth-century book by al-Khwarizmi of Baghdad, written entirely in words. What does it mean for a mathematical tool to be "natural"?

RW Your sense of loudness works roughly on a logarithmic scale. Sound level in decibels is $10 \log_{10}(I/I_0)$, where I_0 is a fixed reference intensity, so every step of 10 dB multiplies the intensity by 10. A quiet conversation is about 60 dB and a loud concert about 110 dB: a difference of 50 dB, which is 10^5, a hundred thousand times the intensity. Yet it does not sound a hundred thousand times louder: perceived loudness roughly doubles with each 10 dB, so the concert sounds about $2^5 = 32$ times as loud.

EXT Look at the first digit of each country's population. You might expect each digit to lead one time in nine, but 1 leads about 30% of the time and 9 only about 5%. This is **Benford's law**, and it appears in many data sets that span several powers of ten. The proportion starting with the digit d is $\log_{10}(1 + \frac{1}{d})$, so $d = 1$ gives $\log_{10} 2 \approx 0.301$. On a logarithmic scale, 1 to 2 is a far longer stretch than 9 to 10. Auditors $(\text{people who check a company's accounts})$ use it as a first check: for suitable data, a departure from the pattern is a reason to look more closely, not proof that figures were invented. *Try it:* check that the nine proportions add up to 1.

End-of-Chapter Problems
Chapter 1: Exponents & Logarithms

1.

(a) Write $\sqrt{50}+\sqrt{18}-\sqrt{8}$ in the form $k\sqrt{2}$, where k is an integer. [2]

(b) Express $\dfrac{4+\sqrt{3}}{2-\sqrt{3}}$ in the form $a+b\sqrt{3}$, where $a, b \in \mathbb{Z}$. [3]

(c) Given that $27^x = 3\sqrt{3}$, find the exact value of x. [2]

2.

(a) Simplify $\left(2x^{-2}y^3\right)^3 \times \left(4x^4y^{-1}\right)^{-1}$, giving your answer with positive exponents. [3]

(b) Solve $8^x = 2^{x+6}$. [2]

(c) Show that $\left(a^{1/2}+a^{-1/2}\right)^2 = a+2+a^{-1}$, for $a>0$. [2]

3. GDC The mass of the Earth is 5.97×10^{24} kg and the mass of the Moon is 7.35×10^{22} kg.

(a) Find how many times heavier the Earth is than the Moon, to 3 significant figures. [2]

(b) Find their combined mass, giving your answer in standard form to 3 significant figures. [2]

4. Without a calculator, show that $2<\log_2 5<3$. Hence find the integer part of $\log_2 40$. [4]

5. Given that $\log_a x = 3$ and $\log_a y = 5$, find each of the following.

(a) $\log_a(x^2y)$ [1]

(b) $\log_a \sqrt{xy}$ [2]

(c) $\log_a\left(\dfrac{x}{y^2}\right)$ [1]

6. Let $\log_a 2 = p$ and $\log_a 3 = q$.

(a) Write $\log_a 6$ in terms of p and q. [1]

(b) Write $\log_a \sqrt{12}$ in terms of p and q. [2]

(c) Write $\log_a \dfrac{4}{9}$ in terms of p and q. [2]

7.

(a) Show that $\log_a \sqrt[n]{x} = \dfrac{1}{n}\log_a x$. [2]

(b) Hence solve $\log_5 \sqrt[3]{x} + \log_{125} x = 2$. [4]

8. Prove that $\log_a b \cdot \log_b c \cdot \log_c a = 1$ for all valid bases and arguments. Hence evaluate $\log_2 3 \cdot \log_3 5 \cdot \log_5 8$ without a calculator. [4]

9. Given that $\log_2 3 = a$, express each of the following in terms of a:

(a) $\log_2 9$ [1]
(b) $\log_4 3$ [2]
(c) $\log_8 27$ [2]
(d) $\log_3 8$ [2]

10. The reciprocal of a logarithm is a logarithm with its base and argument swapped.

(a) Let $N = a^k$, where $a > 0$, $a \neq 1$, $N > 0$ and $N \neq 1$. Write $\log_a N$ and $\log_N a$ in terms of k, and hence show that $\dfrac{1}{\log_a N} = \log_N a$. Explain why $N = 1$ must be excluded. [3]
(b) Hence evaluate $\dfrac{1}{\log_2 36} + \dfrac{1}{\log_3 36}$ without a calculator. [4]

11. Solve $\log_3(x+5) - \log_3(x-1) = 2$. [4]

12. Solve $\log_6(x+2) + \log_6(x-3) = 1$, justifying any rejection of solutions. [4]

13. Solve the simultaneous equations:

$$\log_2 x + \log_2 y = 5, \qquad \log_2(x-y) = 3.$$

[5]

14. Solve $\log_x 8 = \log_4 x$. [5]

15. GDC Solve $\log_2(x^2 - 2x) = \log_4(x+4)$. [5]

16. GDC

(a) Express $\log_3 x - 2 = \log_9(x+6)$ as a quadratic equation in x, using change of base. [3]
(b) Hence find the value of x. [2]

17. Solve $\log_5(x^2 - 2x + 1) = 2\log_5(x-1) + 1$, or show that no solution exists. [3]

18. Solve $3^{2x+1} = 27^{x-1}$. [3]

19. Solve $3^{2x-1} = 4^{x+2}$, giving your answer in exact logarithmic form. [4]

20. Show that the equation $3^{x+2} + 3^x = k$ has a real solution only when $k > 0$, and find this solution in terms of k. [4]

21. Find the exact solution to $e^{2x} = 3e^x + 10$. *(Hint: let $u = e^x$)* [4]

22. Solve the equation $9^x - 4 \cdot 3^{x+1} + 27 = 0$. *(Hint: let $u = 3^x$; note $9^x = u^2$)* [5]

23. GDC The number of bacteria in a culture doubles every 3 hours. Initially there are 500 bacteria.

(a) Write an expression for the number of bacteria after t hours. [2]
(b) Find the time, to the nearest minute, when there are 50 000 bacteria. [3]

24. GDC The population of a city is modelled by $P(t) = 250\,000 \times e^{0.032t}$, where t is the number of years after the start of 2020.

(a) Write down the population in 2020. [1]
(b) Find the population in 2030, to the nearest thousand. [2]
(c) Find the year in which the population first exceeds 400 000. [3]

25. GDC The value V dollars of a delivery van t years after it is bought is modelled by $V = 45\,000 \times (0.82)^t$.

(a) Write down the value of the van when it is bought. [1]
(b) Find the value after 5 years, to the nearest dollar. [2]
(c) Find the least number of complete years after which the value is below \$10 000. [3]

26. GDC A radioactive substance decays so that its mass M grams after t years satisfies $M = M_0 e^{-kt}$, where M_0 and k are positive constants (so the continuous growth rate of §1.5 is $-k$). After 10 years the mass is halved.

(a) Show that $k = \dfrac{\ln 2}{10}$. [2]
(b) Find the percentage of the original mass remaining after 25 years. [3]
(c) Find the time for the mass to reduce to 10% of its original value. [3]

27. Solve $\log_2 x + \log_x 2 = \dfrac{5}{2}$. *(Hint: let $t = \log_2 x$ and use question 10)* [6]

28. Consider the equation $(\log_3 x)^2 = \log_3 x^2 + 3$.

(a) Explain why the right-hand side equals $2\log_3 x + 3$, and state the restriction on x this requires. [2]
(b) Hence solve the equation. [4]

(c) The expression $\log_3 x^2$ is defined for every non-zero x, yet negative values of x cannot be solutions. Explain why. [2]

29. GDC The pH of a dilute solution is given by $\text{pH} = -\log_{10}[\text{H}^+]$, where $[\text{H}^+]$ is the hydrogen ion concentration in mol/L.

(a) A solution has $[\text{H}^+] = 3.5 \times 10^{-4}$ mol/L. Find its pH to 2 decimal places. [2]
(b) A solution has $\text{pH} = 8.3$. Find its hydrogen ion concentration in standard form. [2]
(c) Solution A has $\text{pH} = 3$ and solution B has $\text{pH} = 7$. Find how many times greater the hydrogen ion concentration is in solution A than in solution B. [2]

30. GDC This question is about writing very large and very small quantities in standard form.

(a) A human cell has diameter about 2×10^{-5} m, and a water molecule about 2.8×10^{-10} m. Find how many times wider the cell is, giving your answer in standard form to three significant figures. [2]
(b) The observable universe is about 8.8×10^{26} m across, and the Planck length, the scale at which current physical theories are expected to break down, is about 1.6×10^{-35} m. Find the ratio of the two, in standard form. [2]
(c) A sheet of paper is 0.1 mm thick. It is folded in half, then in half again, and so on. Find the thickness after 20 folds, in metres and in standard form. [2]
(d) Find the least number of folds needed for the thickness to exceed the distance to the Moon, 3.84×10^8 m. [4]
(e) Your answer to part (d) may be smaller than you expected. Explain why, referring to how the thickness grows. [2]

31. Consider the equation $\log_2(\log_3(\log_2 x)) = 0$.

(a) Find x. [4]
(b) Working outwards from the innermost logarithm, state the condition each layer imposes on x, and hence give the values of x for which the left-hand side is defined at all. [4]

32. Evaluate

$$\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdot \cdots \cdot \log_{31} 32,$$

the product of all thirty factors, without a calculator. Justify your method. [5]

33. Let a, b and c be positive, with a, b, c and ab all different from 1.

(a) Show that $\log_{ab} c = \dfrac{\log_a c \cdot \log_b c}{\log_a c + \log_b c}$. [5]
(b) Verify your identity with $a = 2$, $b = 5$ and $c = 7$, giving both sides to four decimal places. [2]

34. Solve the simultaneous equations

$$\log_x y + \log_y x = \frac{10}{3}, \qquad xy = 81,$$

where x and y are positive and neither is 1. [7]

35.

(a) Prove the change-of-base law $\log_a b = \dfrac{\log_c b}{\log_c a}$, for valid bases a and c. *(Hint: let $x = \log_a b$, so $a^x = b$, then take $\log_c$ of both sides)* [3]

(b) Hence show that $\log_{27} x = \frac{1}{3}\log_3 x$. [2]

(c) Solve $\log_3 x + \log_9 x + \log_{27} x = 11$. [4]

(d) Solve $\log_x 2 + \log_{x^2} 2 = 3$, giving an exact answer. [5]

36.

(a) Solve $\log_3(x^2) = 4$, giving every real solution. [3]

(b) A student writes $\log_3(x^2) = 2\log_3 x$, solves $2\log_3 x = 4$, and finds only one solution. State which solution is lost, and explain why the student's first step loses it. [2]

37. GDC Two savings accounts each start with \$1000. Account A grows by 5% each year. Account B grows continuously at 5% per year.

(a) Write down a model for the value of each account after t years. [2]

(b) Find how much more Account B holds than Account A after 10 years, to the nearest cent. [2]

(c) Find the continuous rate, as a percentage to 3 significant figures, at which Account B would have to grow to match Account A exactly. [2]

38. In this question x may be any real number, and $y^{m/n}$ means $\left(\sqrt[n]{y}\right)^m$ with n odd. Solve $(x-1)^{2/3} = 4$, giving every real solution. [4]

Challenge Questions

C1. GDC **Decibels.** The sound intensity level L of a sound, in decibels, is $L = 10\log_{10}\left(\dfrac{I}{I_0}\right)$, where I is the intensity in W/m^2 and $I_0 = 10^{-12}$ W/m^2 is the threshold of hearing. The scale is logarithmic because human hearing spans a range of intensities far too wide to put on a linear axis.

(a) A concert has level 110 dB. Find the intensity I. [2]

(b) A whisper has level 30 dB. Show that the concert is 10^8 times as intense as the whisper. [2]

(c) Show that doubling the intensity of *any* sound raises its level by $10\log_{10} 2$ dB, whatever the original level was. Give this value to two decimal places. [4]

(d) Assume that each source gives the same intensity at the listener and that the intensities add. Hence find how many identical sources, playing together, are needed to raise the level by 10 dB. [3]

(e) Intensity obeys an inverse-square law: at distance r from a source, $I \propto \frac{1}{r^2}$. Show that doubling your distance reduces the level by a fixed number of decibels, independent of where you started, and find it. [4]

(f) A workplace safety rule allows 8 hours of exposure at 85 dB, and halves the permitted time for every 3 dB above that. Find the permitted time at 100 dB, and explain what part (c) has to do with the choice of 3 dB. [4]

(g) Two sources, of levels L_1 and L_2 dB, play together, and their intensities add. Show that the combined level is

$$L = 10\log_{10}\left(10^{L_1/10} + 10^{L_2/10}\right),$$

and find the combined level of a 90 dB source and an 80 dB source. [3]

(h) Let $L_1 \geq L_2$ and $D = L_1 - L_2$. Show that $L - L_1 = 10\log_{10}\left(1 + 10^{-D/10}\right)$, tabulate this rise for $D = 0, 3, 6, 10, 20$, and decide, with a reason, how much quieter a second source must be before it can be ignored. Suggest how this could be extended into an exploration. [3]

C2. GDC **Counting digits, and the first digit.** Logarithms count digits, and explain a pattern that auditors (people who check a company's accounts) use to check reported figures. Throughout, $\lfloor t \rfloor$ is the greatest integer not exceeding t, and $t - \lfloor t \rfloor$ is the fractional part of t.

(a) Let N be a positive integer. Show that N has exactly $\lfloor \log_{10} N \rfloor + 1$ digits. [3]

(b) Hence find the number of digits in 2^{1000}. [2]

(c) Let d be one of the digits $1, 2, \ldots, 9$. Prove that N has first digit d exactly when the fractional part of $\log_{10} N$ lies in $[\log_{10} d,\ \log_{10}(d+1))$. Verify this for $N = 2^7$. [4]

(d) Prove that $\log_{10} 2$ is irrational. Hence show that no two of the numbers $\log_{10}(2^n)$, for $n = 1, 2, 3, \ldots$, have the same fractional part. [4]

(e) Assume the fractional parts of $\log_{10}(2^n)$ are spread evenly over $[0, 1)$ as n increases. Show that the proportion of powers of 2 whose first digit is d is $\log_{10}\left(1 + \frac{1}{d}\right)$. [4]

(f) Evaluate this proportion for $d = 1$ and for $d = 9$. Then count the first digits of 2^1, 2^2, …, 2^{30} on your GDC and compare. [4]

(g) The pattern in part (e) is Benford's law, also reported for data such as town populations. Comment on how well your counts in (f) support it, and why 30 values might mislead. Name one kind of real data you would expect to follow the law and one you would not, with a reason for each. [3]

C3. GDC **Trapping e.** For each integer $n \geq 1$ let

$$a_n = \left(1 + \frac{1}{n}\right)^n \qquad \text{and} \qquad b_n = \left(1 + \frac{1}{n}\right)^{n+1}.$$

You may use *Bernoulli's inequality*: for every integer $m \geq 2$ and every real $x \geq -1$ with $x \neq 0$, $(1+x)^m > 1 + mx$. (It can be proved by induction, the method of Ch. 3.) You may also use the fact that an increasing sequence that is bounded above has a limit, and so does a decreasing sequence that is bounded below.

(a) Find a_n and b_n for $n = 1, 2, 5, 10, 100$, giving values to four decimal places where they are not exact. Make a conjecture about how each sequence behaves. [3]

(b) Show that $\dfrac{a_{n+1}}{a_n} = \left(1 + \dfrac{1}{n}\right)\left(1 - \dfrac{1}{(n+1)^2}\right)^{n+1}$, and use Bernoulli's inequality to prove that $a_{n+1} > a_n$ for all $n \geq 1$. [5]

(c) Show that, for $n \geq 2$, $\dfrac{b_{n-1}}{b_n} = \dfrac{n}{n+1}\left(1 + \dfrac{1}{n^2 - 1}\right)^n$, and prove that $b_n < b_{n-1}$. [5]

(d) Show that $b_n - a_n = \dfrac{a_n}{n}$. Explain why $\{a_n\}$ and $\{b_n\}$ both have limits, and why the two limits are equal. This common limit is e; deduce that $a_n < e < b_n$ for every $n \geq 1$. [4]

(e) Use $n = 1000$ to write down an interval that contains e, and state its width. [2]

(f) Compute $n(e - a_n)$ for $n = 10$, 100 and 1000, and conjecture its limit. Explain what your conjecture says about how many correct decimal places of e are gained each time n is multiplied by 10, and suggest a way to obtain e more quickly that could be investigated further. [3]